Structural vector autoregressions 2

A. Problem statement

Reduced-form (can easily estimate):
\[ y_t = \mathbf{c} + \Phi_1 y_{t-1} + \cdots + \Phi_p y_{t-p} + \varepsilon_t \]
\[ E(\varepsilon_t \varepsilon_t') = \Omega \]
\[ \frac{\partial y_t}{\partial \varepsilon_t} = \mathbf{\Psi} \]

Structural model of interest:
\[ \mathbf{B}_0 y_t = \lambda + \mathbf{B}_1 y_{t-1} + \cdots + \mathbf{B}_p y_{t-p} + \mathbf{u}_t \]
\[ \varepsilon_t = \mathbf{B}_0^{-1} \mathbf{u}_t \]
\[ \frac{\partial y_{ts}}{\partial \mathbf{u}_t} = \frac{\partial y_{ts}}{\partial \varepsilon_t} \frac{\partial \varepsilon_t}{\partial \mathbf{u}_t} = \mathbf{\Psi}_s \mathbf{B}_0^{-1} \]

Problem: How to estimate \( \mathbf{B}_0^{-1} \)
(or at least one column of \( \mathbf{B}_0^{-1} \))
Example: if
\[
B_0 = \begin{bmatrix}
    b^{(1,1)}_0 & b^{(1,2)}_0 & 0 & 0 & 0 \\
    b^{(2,1)}_0 & b^{(2,2)}_0 & 0 & 0 & 0 \\
    b^{(3,1)}_0 & b^{(3,2)}_0 & 1 & 0 & 0 \\
    b^{(4,1)}_0 & b^{(4,2)}_0 & b^{(4,3)}_0 & b^{(4,4)}_0 & b^{(4,5)}_0 \\
    b^{(5,1)}_0 & b^{(5,2)}_0 & b^{(5,3)}_0 & b^{(5,4)}_0 & b^{(5,5)}_0
\end{bmatrix}
\]

Then \( \frac{\partial y_t}{\partial \alpha_0} = \hat{\Psi} \hat{p}^{-1} \hat{p}_3 \)

for \( \hat{p}_3 \) col 3 and \( \hat{p}^{-1} \) row 3 col 3 of

Cholesky factor \( \hat{\Omega} = \hat{P} \hat{P}' \)

Alternatively, with zero or other restrictions solve
\( \hat{\Omega} = \hat{B}_0^{-1} \hat{D} (\hat{B}_0^{-1})' \)

Theme today: what alternative strategies are available for identifying \( \Psi, B_0^{-1} \)?

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A. Problem statement
B. Identification using long-run restrictions
$x$ log of productivity  
(log GDP minus log civilian labor force)  

$n$ log of civilian labor force  

\[ y_t = \begin{bmatrix} \Delta x_t \\ \Delta n_t \end{bmatrix} \sim I(0) \]

VAR (reduced-form)  
\[ y_t = c + \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \cdots + \Phi_p y_{t-p} + \varepsilon_t \]
\[ E(\varepsilon_t) = 0 \]
Structural model:
\[ B_0 y_t = b_0 + B_1 y_{t-1} + B_2 y_{t-2} + \cdots + B_p y_{t-p} + u_t \]
\[ E(u_t u_t') = I_2 \quad \text{(normalization)} \]

Relation between representations:
\[ u_t = B_0 e_t \]
\[ \Omega = B_0' (B_0')' \]

Premultiply structural model, \[ B(L) y_t = b_0 + u_t \]
by \( C(L) = B(L)^{-1} \):
\[ y_t = \mu + C_0 u_t + C_1 u_{t-1} + C_2 u_{t-2} + \cdots \]
which gives structural MA representation
$u_t = \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}$

- $u_{1t}$: technology shock
- $u_{2t}$: demand disturbances

Assumption: demand shocks can not have a permanent effect on productivity

$$\lim_{s \to \infty} \frac{\partial X_{t+s}}{\partial u_{2t}} = 0$$

Notice

$$\frac{\partial X_{t+s}}{\partial u_{2t}} = \frac{\partial (x_{t+s} - x_{t+s-1})}{\partial u_{2t}} + \frac{\partial (x_{t+s-1} - x_{t+s-2})}{\partial u_{2t}} + \cdots + \frac{\partial (x_s - x_{s-1})}{\partial u_{2t}}$$
\[
\begin{align*}
y_t &= \begin{bmatrix} x_t - x_{t-1} \\ n_t - n_{t-1} \end{bmatrix} \\
\frac{\partial(x_t - x_{t-1})}{\partial u_{2t}} &= \frac{\partial y_{1t}}{\partial u_{2t}} \\
y_{it} &= \mu + C_0 u_i + C_1 u_{i-1} \\
&\quad + C_2 u_{i-2} + \cdots \\
\frac{\partial y_{1+m}}{\partial u_i} &= C_m \\
\frac{\partial x_{1+s}}{\partial u_{2t}} &= \frac{\partial(x_{1+s} - x_{1+s-1})}{\partial u_{2t}} + \frac{\partial(x_{1+s-1} - x_{1+s-2})}{\partial u_{2t}} \\
&\quad + \cdots + \frac{\partial(x_t - x_{t-1})}{\partial u_{2t}} \\
is given by the row 1 column 2 element of 
C_0 + C_1 + C_2 + \cdots + C_t \\
\lim_{s \to \infty} \frac{\partial x_{1+s}}{\partial u_{2t}} &= 0 \\
requires that the following matrix is lower triangular:
C_0 + C_1 + C_2 + \cdots = C(1)
Goal: find structural disturbances $u_t$ that are a linear combination of the VAR innovations, $u_t = H_{6t}$, such that:

(1) $E(u_t, u_t') = I_2$
   $\Rightarrow H\Omega H' = I_2$
   $\Rightarrow \Omega = (H^{-1})(H^{-1})'$

(2) $y_t = \mu + C(L)u_t$

(3) $C(1)$ is lower triangular
\[
\Phi(L)y_t = c + \varepsilon_t \\
\varepsilon_t = H^{-1}u_t \\
\Rightarrow \Phi(L)y_t = c + H^{-1}u_t \\
\Rightarrow y_t = \mu + [\Phi(L)]^{-1}H^{-1}u_t \\
y_t = \mu + C(L)u_t \\
\Rightarrow C(1) = [\Phi(1)]^{-1}H^{-1}
\]

\[
C(1) = [\Phi(1)]^{-1}H^{-1} \\
C(1)[C(1)]' = \\
[\Phi(1)]^{-1}H^{-1}(H^{-1})'[\Phi(1)]^{-1}'
\]

\[
C(1)[C(1)]' = \\
[\Phi(1)]^{-1}\Omega[\Phi(1)]^{-1}'
\]

Can estimate: \(\Phi(1)\) and \(\Omega\) from VAR
Want: Lower triangular matrix $\mathbf{C}(1)$ such that
\[ \mathbf{C}(1)[\mathbf{C}(1)]' = [\Phi(1)]^{-1}\Omega[\Phi(1)]^{-1}'. \]

Conclusion: $\mathbf{C}(1)$ is Cholesky factor of $[\Phi(1)]^{-1}\Omega[\Phi(1)]^{-1}'.

To get $\mathbf{H}$ we then use fact that
\[ \mathbf{C}(1) = [\Phi(1)]^{-1}\mathbf{H}^{-1}, \]
\[ \mathbf{H} = [\mathbf{C}(1)]^{-1}[\Phi(1)]^{-1}. \]
Summary:

(1) Estimate VAR’s by OLS
\[ y_t = \begin{bmatrix} \Delta x_t \\ \Delta n_t \end{bmatrix} \]
\[ y_t = c + \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \cdots + \Phi_p y_{t-p} + \varepsilon_t \]
\[ \hat{\Omega} = T^{-1} \sum_{t=1}^T \varepsilon_t \varepsilon_t' \]

(2) Find Cholesky factor or lower triangular matrix \( \hat{C} \) such that
\[ \hat{C} \hat{C}' = \hat{\Omega} \hat{\Omega}' \]
\[ \hat{Q} = (I_2 - \hat{\Phi}_1 - \hat{\Phi}_2 - \cdots - \hat{\Phi}_p)^{-1} \]

(3) Technology shock and demand shock for date \( t \) are first and second elements of
\[ \hat{u}_t = \hat{B}_0 \hat{\varepsilon}_t \]
where
\[ \hat{B}_0 = \hat{C}^{-1} \hat{Q} \]
(4) Effect of tech shock or demand shock at date $t$ on $y_{i,t}$ are given by first and second columns, respectively, of $\frac{\partial y_{i,t}}{\partial u_t} = \Psi_i B_0^{-1}$.

More generally, if $y_i$ is $n$-dimensional vector of differences, long-run effect of structural shock $j$ on level of $y_i$ is given by row $i$, col $j$ of $[\Phi(1)]^{-1}B_0^{-1}$.

If this is postulated to be zero for some subset of $i$ and $j$ can use this as set of restrictions on $B_0$ along with zero or other restrictions to maximize $(T/2) \log |B_0|^2 - (T/2) \log |D| - (T/2) \text{trace} \left\{ B_0' D^{-1} B_0 \hat{\Omega} \right\}$.
Drawbacks:
(1) \( \hat{\Phi} = (I_2 - \hat{\Phi}_1 - \hat{\Phi}_2 - \cdots - \hat{\Phi}_p)^{-1} \)
is estimated poorly, sensitive to \( p \)

(2) technology shock could be temporary (e.g., delay in adoption of discovered technology)
(3) demand shock could be permanent (e.g., lost human capital)

Structural vector autoregressions 2

A. Problem statement
B. Identification using long-run restrictions
C. Identification using high-frequency data
Faust, Swanson, and Wright (JME, 2004)

Observe: some financial variables move dramatically after Fed announces target change

Inference: these changes reflect the effects of policy

Goal: can we somehow use this to identify VAR?

*d* particular day in sample

*t(d)* month associated with day *d*

*f^h_d* h-month fed funds futures rate on day *d*

*r_t* avg. fed funds rate for month *t*

assumption: *f^h_d* = *E_d(r_{t(d)+h})*

Implications:

\[
\frac{f^0_d - f^0_{d-1}}{f^1_d - f^1_{d-1}} = \frac{\partial E_{t(d)}r_{t+h}}{\partial u_f}
\]

where *u_f* is change in fed policy in month *t*
\[
\frac{f^h_d - f^h_{d-1}}{f^0_d - f^0_{d-1}} = \frac{\partial E_{r,t+h}}{\partial u_t}
\]

Average value for all days \(d\) on which there is a target change gives estimate of

\[\gamma_h = \frac{\partial E_{r,t+h}}{\partial u_t}\]

VAR (reduced-form)

\[
y_t = c + \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \cdots + \Phi_{p} y_{t-p} + \varepsilon_t
\]

Structural model:

\[
B_0 y_t = b_0 + B_1 y_{t-1} + B_2 y_{t-2} + \cdots + B_{p} y_{t-p} + u_t
\]

Relation:

\[u_t = B_0 \varepsilon_t\]

Suppose shock to fed policy is represented by

\[
u_t = e_t^1 u_t = e_t^1 B_0 \varepsilon_t
\]
Reduced-form MA representation:

\[ y_t = \mu + \varepsilon_t + \Psi_1 \varepsilon_{t-1} + \Psi_2 \varepsilon_{t-2} + \cdots \]

\[ \varepsilon_t = y_t - \hat{y}_{t|t-1} \]

\[ y_t = \mu + B_0^{-1} B_0 \varepsilon_t + \Psi_1 B_0^{-1} B_0 \varepsilon_{t-1} + \Psi_2 B_0^{-1} B_0 \varepsilon_{t-2} + \cdots \]

\[ y_t = \mu + B_0^{-1} u_t + \Psi_1 B_0^{-1} u_{t-1} + \Psi_2 B_0^{-1} u_{t-2} + \cdots \]

\[ \frac{\hat{C}_{r_{1h}}}{\hat{u}_R} = e_4' \Psi_h b^{(4)} = \gamma_h \]

where \( b^{(4)} \) is fourth column of \( B_0^{-1} \)
\[
\frac{\partial r_{12h}}{\partial u_{ht}} = e_4 \Psi_h b^{(4)} = \gamma_h
\]

Can estimate:
- \( \Psi_h \) from estimated monthly VAR
- \( \gamma_h \) from daily target change data

\[
\hat{\gamma}_h = \text{average} \frac{f_{d}^{h} - f_{d-1}^{h}}{f_{d}^{h} - f_{d-1}^{h}}
\]

\[
e_4 \Psi_h b^{(4)} = \gamma_h
\]

Let \( \psi_{4h} = e_4 \Psi_h \)

Then:
- \( \psi_{3h} b^{(4)} = \gamma_0 \)
- \( \psi_{2h} b^{(4)} = \gamma_1 \)
- \( \vdots \)
- \( \psi_{4h,n-1} b^{(4)} = \gamma_{n-1} \)
Let \( H = \begin{bmatrix} \Psi_{40} \\ \Psi_{41} \\ \vdots \\ \Psi_{4,n-1} \end{bmatrix} \) and \( \gamma = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{n-1} \end{bmatrix} \).

Both \( H \) and \( \gamma \) can be estimated.

If rows of \( \hat{H} \) are linearly independent, then

\[ \hat{b}^{(4)} = \hat{H}^{-1} \hat{\gamma} \]
Summary:
We assumed that we can use daily interest rate data to infer effect of policy shock on future interest rates:

\[
\frac{\hat{r}_{t+h}}{\hat{u}_{\beta}}
\]

But now we can calculate effect of policy shock on any variable:

\[
\frac{\hat{Y}_{t+\gamma}}{\hat{u}_{\beta}} = \Psi \, b^{(4)}
\]

Problem: the matrix \( \hat{H} \) does not appear to have full rank.
Solution: Calculate confidence sets under partial identification rather than point estimates.
\[ \begin{bmatrix} \hat{\Psi}_{40} \\ \hat{\Psi}_{41} \\ \vdots \\ \hat{\Psi}_{4,n-1} \end{bmatrix} \]

\[ T^{1/2} \left[ \text{vec}(\hat{H} - H_0) \right] \overset{L}{\rightarrow} N(0, R) \]

* \( R \) can be consistently estimated from VAR distribution
  (e.g., simulate draws from asymptotic distribution of \( \{ \hat{\phi}_t \} \) and calculate \( \{ \hat{\Psi}_t \} \) for each draw)

\[ T^{1/2} (\hat{\gamma} - \gamma_0) \overset{L}{\rightarrow} N(0, G) \]

* \( G \) can be consistently estimated from covariance of futures observations
$H_0: \quad \mathbf{H} \mathbf{b}^{(4)}_{(p\times(n+1))} = \mathbf{y}_{(p\times1)}$

If $\mathbf{b}^{(4)}$ is the true value, then

$S(\mathbf{b}^{(4)}) = T(\hat{\mathbf{H}}\mathbf{b}^{(4)} - \hat{\mathbf{y}})' \times$

$\left[ (\mathbf{b}^{(4)'} \otimes \mathbf{I}_g) \hat{\mathbf{R}}(\mathbf{b}^{(4)} \otimes \mathbf{I}_g) + \hat{\mathbf{G}} \right]^{-1} \times (\hat{\mathbf{H}}\mathbf{b}^{(4)} - \hat{\mathbf{y}}) \sim \chi^2(g)$

The set $\mathcal{A}$ of all values $\mathbf{b}^{(4)}$ such that

$S(\mathbf{b}^{(4)}) \leq c$

where $c$ is 95% critical value for $\chi^2(g)$ then is a 95% confidence set for $\mathbf{b}^{(4)}$

For statistic such as structural impulse-response coefficients, use Bonferroni to find outer bounds on 90% confidence interval.
e.g., let \( h_{is} = \text{row } i \) element of \( \Psi_s b^{(4)} \)

(1) For any given \( b^{(4)} \) use distribution of \( \Psi_s \) to find 95% upper and lower bounds \( h_{is}^{(u)}(b^{(4)}) \) and \( h_{is}^{(l)}(b^{(4)}) \)

(2) Find the value \( b^{(u)} \in A \) for which \( h_{is}^{(u)}(b^{(4)}) \) is largest and the value \( b^{(l)} \in A \) for which \( h_{is}^{(l)}(b^{(4)}) \) is smallest.

(3) 90% confidence interval for \( h_{is} \) is \([h_{is}^{(l)}(b^{(4)}), h_{is}^{(u)}(b^{(4)})]\)

Structural vector autoregressions 2

A. Problem statement
B. Identification using long-run restrictions
C. Identification using high-frequency data
D. Identification using external instruments
Stock and Watson (BPEA, 2012)

Suppose:
(1) structural shocks \( u_{it}, \ldots, u_{nt} \) are mutually uncorrelated
(2) have instrument \( z_{it} \) that is relevant
\[
E(z_{it}u_{it}) = \alpha_i \neq 0 \text{ and valid}\]
\[
E(z_{it}u_{jt}) = 0 \quad \text{for } i \neq j
\]

Under the above assumptions,
\[
E(\varepsilon_i z_{it}) = B_0^{-1} E(u_i z_{it}) = B_0^{-1} \alpha e_i
\]
so can estimate \( i \)th column of \( B_0 \) (up to unknown constant) by
\[
\tilde{b}^{(i)} = T^{-1} \sum_{t=1}^{T} \hat{\varepsilon}_i z_{it}
\]

Can normalize by defining shock \( u_{it} \) to be something that increases \( y_{it} \) by one unit:
\[
\hat{b}^{(i)} = \check{b}^{(i)} \hat{B}_i
\]

\[
\frac{\partial y_{it}}{\partial u_a} = \hat{\Psi}_i \check{b}^{(i)}
\]
Can also estimate $\hat{u}_{it}$ as follows.
Suppose we observed $u_t$ and regressed $z_{it}$ on $u_t$:

$$z_{it} = \pi' u_t + v_{it}$$
$$\text{plim} \, \hat{\pi}_i = (a/d_{ii})e_i$$

If instead we regressed $z_{it}$ on $\varepsilon_t$,

$$z_{it} = \lambda' \varepsilon_t + v_{it}$$
this would just be rotation of above regression since $\varepsilon_t = B_0^{-1} u_t$.

Hence fitted values from regression
of $z_{it}$ on $\hat{\varepsilon}$, give consistent estimate
of $(a/d_{ii})u_{it}$.
Illustration:
Using high-frequency market response to Fed announcements to identify effects of unconventional monetary policy (Gertler and Karadi, 2013)

Event study methodology

- Nov 25, 2008: LSAP announced
- Dec 1, 2008: Bernanke: “could purchase longer-term Treasury… in substantial quantities”
- Dec 16, 2008: FOMC “stands ready to expand its purchases of agency debt and mortgage-backed securities”
- Mar 18, 2009: Announced new purchases of MBS and agency debt

10-year yield fell 170 bp Nov 3 - Dec 31

• fell 61 bp on 3 indicated dates
Oil price declined 30% Nov 3 - Dec 31

- fell 19% on 3 indicated dates
$$z_{it} = \text{change in 1-year yield within 30-minute window of key Fed announcement in month } i (= 0 \text{ if no event in month } i)$$

Source: Gertler and Karadi (2013)

Structural vector autoregressions 2

A. Problem statement
B. Identification using long-run restrictions
C. Identification using high-frequency data
D. Identification using external instruments
E. Identification using heteroskedasticity
Suppose $y_t$ consists of high-frequency observations (e.g., daily changes in interest rates, exchange rates, stock prices, commodity prices)

$$y_t = c + \Phi_1 y_{t-1} + \cdots + \Phi_p y_{t-p} + \varepsilon_t$$

$\varepsilon_t = B_0^{-1} u_t$

$u_{1t} = \text{monetary policy shock}$

want to estimate $b^{(1)}$ (first column of $B_0^{-1}$)

Suppose we believed that:

1. monetary policy shocks have higher variance on particular days

$$E(u_{1t}^2) = \begin{cases} d^{(0)}_{11} + \lambda & \text{if } t \in S \\ d^{(0)}_{11} & \text{if } t \notin S \end{cases}$$

Set $S$ is known (e.g., FOMC dates)
(2) A monetary policy shock of given size would have the same effects on these dates as others.

(3) Variance and effects of other shocks same on these dates as others.

Then

\[
E(u_t, u'_t) = \begin{cases} 
D + \lambda e_1 e_1' & \text{if } t \in S \\
D & \text{if } t \notin S 
\end{cases}
\]

\[e_1 = \text{col } 1 \text{ of } I_n\]

\[
e_t = B_0^{-1} u_t = \sum_{i=1}^{n} b^{(i)} u_{it} 
\]

\[
E(\varepsilon_t, \varepsilon'_t) = \begin{cases} 
B_0'^{-1} D(B_0^{-1})' + \lambda b^{(1)}(b^{(1)})' & \text{if } t \in S \\
B_0'^{-1} D(B_0^{-1})' & \text{if } t \notin S 
\end{cases}
\]
\[ \hat{\Omega}_1 = T_1^{-1} \sum_{t=1}^{T} \hat{\delta}_t \hat{\varepsilon}_t \delta(t \in S) \]
\[ T_1 = \sum_{t=1}^{T} \delta(t \in S) \]
\[ \hat{\Omega}_0 = T_0^{-1} \sum_{t=1}^{T} \hat{\delta}_t \hat{\varepsilon}_t \delta(t \notin S) \]
\[ T_0 = \sum_{t=1}^{T} \delta(t \notin S) \]
\[ \hat{\Omega}_1 - \hat{\Omega}_0 \xrightarrow{L} \lambda \text{b}^{(1)}(\text{b}^{(1)})' \]
so we can estimate \text{b}^{(1)} up to an unknown scale, e.g.: normalize \( \lambda = 1 \)

\[ \sqrt{T_1} [ \text{vech}(\hat{\Omega}_1) - \text{vech}(\Omega_1)] \]
\[ \xrightarrow{L} \mathcal{N}(0, \text{V}_1) \]
element of \text{V}_1 corresponding to covariance between \( \hat{\sigma}_{ij} \) and \( \hat{\sigma}_{im} \)
given by \( (\sigma_{ij} \sigma_{jm} + \sigma_{im} \sigma_{jm}) \)
(Hamilton, TSA, p. 301).

(1) Test null hypothesis that \( \Omega_0 = \Omega_1 \)
\[ \hat{q}^* [\hat{\text{V}}_1/T_1 + \hat{\text{V}}_0/T_0]^{-1} \hat{q} \xrightarrow{L} \chi^2(n(n+1)/2) \]
\[ \hat{q} = \text{vech}(\hat{\Omega}_1) - \text{vech}(\hat{\Omega}_0) \]
or bootstrap critical value
(should reject \( H_0 \) if assumptions correct)
(2) Estimate $\mathbf{b}^{(1)}$ by minimum chi square:

$$
\hat{\mathbf{b}}^{(1)} = \arg \min_{\mathbf{b}^{(1)}} \tilde{q} \left[ \tilde{V}_1/T_1 + \tilde{V}_0/T_0 \right]^{-1} \tilde{q}
$$

$$
\tilde{q} = \tilde{q} - \text{vech}[\mathbf{b}^{(1)}(\mathbf{b}^{(1)})']
$$

$$
\frac{\partial \tilde{q} \tilde{\nu}^T}{\partial \tilde{\nu}} = \tilde{\Psi} \hat{\mathbf{b}}^{(1)}
$$

(3) Test null hypothesis restriction valid:

value of objective function asymptotically

$$
\chi^2(n(n - 1)/2) \text{ or bootstrap critical value}
$$

(should not reject $H_0$ if assumptions correct)

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**Structural vector autoregressions 2**

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B. Identification using long-run restrictions
C. Identification using high-frequency data
D. Identification using external instruments
E. Identification using heteroskedasticity
F. Identification using sign restrictions
Rubio-Ramirez, Waggoner, and Zha, Rev Econ Studies, 2010. We can achieve partial identification with sign restrictions such as: monetary policy shock raises short-term rate and lowers output and inflation.

Even if true $\Omega$ is known, we could only infer that $B_0^{-1} \in S(\Omega)$. $\Rightarrow B_0$ is set-identified, not point-identified.