III. Linear state-space models

A. State-space representation of a dynamic system
B. Kalman filter
C. Using the Kalman filter
D. Bayesian analysis of linear state-space models
E. Solutions to linear rational expectations models
F. Estimating DSGE models

Instantaneous utility function:

\[ U_t = a_t^b \left[ \frac{1}{1-\lambda_c} (C_t - hC_{t-1})^{1-\lambda_c} - \frac{\alpha_t^L}{1+\lambda_c} \right] \]

\( h > 0 \) \( \Rightarrow \) habit persistence

\( a_t^b = \rho^b a_{t-1}^b + \eta_t^b \quad \eta_t^b \sim \text{i.i.d. } N(0, \sigma^2_b) \)

\( \Rightarrow \) shock to intertemporal subs

\( a_t^b = \rho_L a_{t-1}^L + \eta_t^L \)

\( \Rightarrow \) shock to intratemporal subs

Let \( \hat{\tilde{C}_t} \) denote deviation of \( \log(C_t) \) from its steady-state value

\[
\hat{\tilde{C}_t} = \left( \frac{h}{1+h} \right) \hat{\tilde{C}_{t-1}} + \left( \frac{1}{1+h} \right) E_t \hat{\tilde{C}}_{t+1}
\]

\[
- \frac{(1-h)}{(1+h)\lambda_c} \left( \hat{R}_t - E_t \hat{\tilde{R}}_{t+1} \right)
\]

\[
+ \frac{(1-h)}{(1+h)\lambda_c} \left( \hat{\alpha}_t^b - E_t \hat{\tilde{\alpha}}_{t+1}^b \right)
\]
capital evolution:
\[ K_t = K_{t-1}(1 - \delta) + [1 - S(a_t^I I_t)I_t]I_t \]
\[ S(\cdot) = \text{adjustment costs} \]
\[ a_t^I = \rho a_{t-1}^I + \eta_t^I \]

(2) \[ \hat{K}_t = (1 - \delta)\hat{K}_{t-1} + \delta\hat{I}_{t-1} \]
(3) \[ \hat{I}_t = \left(\frac{1}{1+\beta}\right)\hat{I}_{t-1} + \left(\frac{\beta}{1+\beta}\right)E_t\hat{I}_{t+1} + \frac{\varphi}{1+\beta} \hat{Q}_t - \frac{\beta E_t a_{t+1} - \delta_{t+1}}{1+\beta} \]
\[ \varphi = 1/S^n \]
\[ Q_t = \text{value of capital stock} \]

(4) \[ \hat{Q}_t = -\left(\hat{R}_t - \hat{r}_{t+1}\right) + \frac{1-\delta}{1-\delta + \tau_k} E_t\hat{Q}_{t+1} + \frac{\tau_k}{1-\delta + \tau_k} E_t\hat{r}_{K,t+1} + \eta_t^Q \]
\[ r_{K_t} = \text{rate of return to capital} \]
\[ \eta_t^Q = \text{tacked on} \]
output from producer of intermediate good of type \( j \)

\[ y^j_t = a^a_t K_{t-1}(j)^a L_t(j)^{1-a} - \Phi \]

\( \Phi \) = fixed cost

\( a^a_t \) = productivity shock

\( a^a_t = \rho \alpha a^a_{t-1} + \eta^a_t \)

\( L_t(j) \) = aggregate of labor hired from each household \( \tau \)

\[ L_t(j) = \left\{ \int_0^{1} \left[ \frac{\lambda_{w,t}}{1 + \lambda_{w,t}} \right] \frac{d\tau}{1 + \lambda_{w,t}} \right\}^{1 + \lambda_{w,t}} \]

\( \lambda_{w,t} = \lambda_w + \eta^w_t \)

\( \eta^w_t \) = shock to workers’ market power

wage stickiness:

a fraction \( \xi_w \) of workers are not allowed to change their wage but instead have their wage increase from the previous value by

\( (P_{t-1}/P_{t-2})^{\gamma_w} \)

\( \gamma_w \) = degree of indexing
\[
\hat{W}_t = \frac{\beta}{1+\beta} E \hat{W}_{t+1} + \frac{1}{1+\beta} \hat{W}_{t-1} \\
+ \frac{\beta}{1+\beta} E_t \hat{W}_{t+1} - \frac{1+\beta y_w}{1+\beta} \hat{W}_t + \frac{y_w}{1+\beta} \hat{W}_{t-1} \\
- h_w \left[ \hat{W}_t - \lambda^L \hat{L}_t - \frac{\lambda_c}{1-h} \left( \hat{C}_t - h \hat{C}_{t-1} \right) - \alpha^L_t - \gamma^w_t \right] \\
h_w = \frac{1}{1+\beta} \frac{(1-\beta\xi_w)(1-\zeta_w)}{[1+(1+\lambda_w)\lambda_t\lambda_w]\xi_w}
\]

\(\text{labor demand}\\n\frac{W_{L_{i}(j)}}{r_{K_{i}}K_{i-1}(j)} = \frac{1-a}{a}\\n\)

\(z_t = \text{capital utilization}\\n(6) \hat{L}_t = -\hat{W}_t + (1+\psi) \hat{K}_t + \hat{K}_{t-1}\\n\psi = \text{parameter based on cost}\\n\text{of utilizing capital}\)

\(\text{intermediate goods sold to}\\n\text{final goods producer with market}\\n\text{power of firm } j \text{ governed by}\\n\lambda_{p,t} = \lambda_p + \eta^p_t\\n\xi_p = \text{fraction allowed to adjust}\\n\text{prices}\\n\gamma_p = \text{indexing parameter}\\n\)
(7) \( \hat{\pi}_t = \frac{\beta}{1 + \beta_p} E_t \hat{\pi}_{t+1} + \frac{\gamma_p}{1 + \beta_p} \hat{\pi}_{t-1} 
\hfill + h_p \left[ a \hat{r}_{K,t} + (1 - a) \hat{w}_t - \hat{\alpha}_t^p + \eta_t^p \right] \)
\[ h_p = \frac{1}{1 + \beta_p} \frac{(1 - \beta_p)(1 - \xi_p)}{\xi_p} \]

goods market equilibrium

(8) \( \hat{Y}_t = [1 - \delta(\bar{K}/\bar{Y}) - (\bar{G}/\bar{Y})] \hat{C}_t \n\hfill + \delta(\bar{K}/\bar{Y}) \hat{I}_t + (\bar{G}/\bar{Y}) \hat{\alpha}_t \]

production function then determines \( r_{K,t} \)

(9) \( \hat{Y}_t = \phi \hat{\alpha}_t^p + \phi a \hat{K}_{t-1} + \phi a \psi \hat{r}_{K,t} \n\hfill + \phi(1 - a) \hat{L}_t \)
\[ \phi = 1 + \frac{\phi}{\text{s.s. costs}} \]

monetary policy (Taylor Rule)

(10) \( \hat{R}_t = \rho \hat{R}_{t-1} + (1 - \rho) \langle \pi_t \rangle + 
\hfill r_s(\hat{\pi}_t - \bar{\pi}_{t-1}) + r_Y(\hat{Y}_t - \bar{Y}_t^p) \}
\hfill + r_{\Delta s}(\hat{\pi}_t - \hat{\pi}_{t-1}) 
\hfill + r_{\Delta Y}[\hat{Y}_t - \hat{Y}_t^p - (\hat{Y}_{t-1} - \hat{Y}_{t-1}^p)] + \eta_t^p \)

\( \pi_t = \) inflation target 
\( \bar{\pi}_t = \rho_s \bar{\pi}_{t-1} + \eta_t^s \)
\( \bar{Y}_t^p = \) output level if prices perfectly flexible
\[
\begin{align*}
y_t &= (\hat{C}_t, \hat{C}_{t-1}, \hat{R}_t, \hat{R}_{t-1}, \hat{K}_t, \hat{K}_{t-1}, \hat{I}_t, \hat{I}_{t-1}, \\
\hat{Q}_t, \hat{w}_t, \hat{w}_{t-1}, \hat{L}_t, \hat{\pi}_t, \hat{\pi}_{t-1}, \hat{Y}_t, \hat{\gamma}_{K,t})' \\
x_t &= (\hat{a}_t^0, \hat{a}_t^1, \eta_t^0, \hat{a}_t^2, \eta_t^1, \hat{a}_t^3, \eta_t^2, \hat{a}_t^4, \eta_t^3, \pi_t, \eta_t^4)' \\
\text{equations (1)-(10) (along with lag definitions) can be written as} \\
\mathbf{A}\mathbf{E}_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t \\
\text{while shocks satisfy} \\
\mathbf{x}_{t+1} = \mathbf{\Phi}\mathbf{x}_t + \mathbf{e}_{t+1} \\
\text{(note also } \mathbf{E}_t\mathbf{x}_{t+1} = \mathbf{\Phi}\mathbf{x}_t) \\
\end{align*}
\]

Observed data:
\[
\text{OLS regression of log real consumption on constant and time trend} \\
\text{residual } = z_{1t} = \hat{C}_t \\
\text{Same for log of investment yields } \hat{I}_t
\]

Other data: GDP, real wages, GDP deflator, nominal interest rate
\[
\text{Treat } \hat{Q}_t, \hat{r}_{K,t}, \hat{a}_t^0 \text{ as unobservable}
\]
state equation:
\[ \xi_{t+1} = F\xi_t + v_{t+1} \]
(came from solution to DSGE model)
observation equation:
\[ z_t = H'\xi_t \]

Let \( \theta \) = parameters of structural model
Use Kalman filter to evaluate likelihood function
\[ p(z_1, \ldots, z_T|\theta) \]
prior \( p(\theta) \)
posterior \( p(\theta|Z) \propto p(\theta)p(Z|\theta) \)

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we can sample from posterior using method such as Metropolis-Hastings
\[ \lambda(\theta) = \log p(\theta) + \log p(Z|\theta) \]
\[
\lambda(\theta) = \log p(\theta) + \log p(Z|\theta)
\]

(1) Find mode of posterior distribution using numerical optimization
\[\theta^* = \arg \max \lambda(\theta)\]

(2) Find Hessian of posterior distribution
\[H(\theta^*) = -\frac{\partial^2 \lambda(\theta)}{\partial \theta \partial \theta} \bigg|_{\theta=\theta^*}\]

(3) Set \(\theta^{(j)} = \) arbitrary starting value for \(j = 1\)

(4) Generate \(\tilde{\theta}^{(j+1)} \sim N(\theta^{(j)}, cH(\theta^*))\) for choice of \(c\) described below (could experiment to see if Student \(t\) works better)

(5) Generate \(u^{(j+1)} \sim U[0, 1]\)

(6) Set \(\theta^{(j+1)} = \tilde{\theta}^{(j+1)}\) if \(u^{(j+1)} < \alpha^{(j+1)}\)
\[
= \theta^{(j)}\quad \text{if}\quad u^{(j+1)} \geq \alpha^{(j+1)}
\]
\[\alpha^{(j+1)} = \min\{1, \exp[\lambda(\theta^{(j+1)}) - \lambda(\theta^{(j)})]\}\]
(7) Repeat steps (4)-(6) for 
\( j = 2, 3, \ldots, 10,000 \)

Values of \( \theta^{(j)} \) for \( j \in \{5,001, \ldots, 10,000\} \) represent sample from \( p(\theta|Z) \)

choose \( c \) so that have 20-30% acceptance at step (6)