Notes on Trend and Cycle Components

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1 Definitions

Suppose the series \( \{Y_t\} \) has a trend component \( \{Y_t^{tr}\} \). Write

\[
Y_t = Y_t^{tr} \frac{Y_t}{Y_t^{tr}} = Y_t^{tr} \cdot \hat{Y}_t,
\]

which defines the cycle component as

\[
\hat{Y}_t = \frac{Y_t}{Y_t^{tr}}.
\]

Take logs:

\[
\ln Y_t = \ln Y_t^{tr} + \ln \hat{Y}_t.
\]

Now define

\[
y_t = \ln Y_t, \quad y_t^{tr} = \ln Y_t^{tr},
\]

\[
\hat{y}_t = \ln \hat{Y}_t = \ln Y - \ln Y_t^{tr}.
\]

Then we may write:

\[
y_t = y_t^{tr} + \hat{y}_t.
\]

Note that \( \hat{y}_t \) approximates the percentage deviation of output from trend. To see this, consider the Taylor expansion around the trend component:

\[
\ln Y_t \simeq \ln Y_t^{tr} + \left. \frac{d \ln Y_t}{d Y_t} \right|_{Y_t = Y_t^{tr}} \cdot (Y_t - Y_t^{tr}) = \ln Y_t^{tr} + \frac{1}{Y_t^{tr}} (Y_t - Y_t^{tr}),
\]

which implies

\[
\ln Y_t - \ln Y_t^{tr} = \hat{y}_t \simeq \frac{Y_t - Y_t^{tr}}{Y_t^{tr}}.
\]
This is a good approximation given that values of \( \hat{y}_t \) are small empirically.

2 Modeling the trend component

a Polynomial trend

The trend component is assumed to be a polynomial in \( t \):

\[
y_{t}^{tr} = \sum_{k=0}^{K} \mu_k t^k.
\]

Example: Log-linear trend (\( K = 1 \)):

\[
y_{t}^{tr} = \mu_0 + \mu_1 t \quad \Rightarrow \quad Y_t^{tr} = Y_0^\gamma t,
\]

where

\[Y_0 = e^{\mu_0}, \quad \gamma = e^{\mu_1}.
\]

Note that a polynomial can be used to approximate any deterministic trend. Moreover, estimation and inference are straightforward.

b Hodrick-Prescott (HP) filter

For a given series \( \{y_t\}_{t=1}^T \), the trend component \( \{y_{t}^{tr}\}_{t=1}^T \) is chosen to solve

\[
\min_{\{y_{t}^{tr}\}_{t=1}^T} \sum_{t=1}^T [(y_t - y_{t}^{tr})^2 + \lambda((y_{t+1}^{tr} - y_{t+1}^{tr}) - (y_{t}^{tr} - y_{t-1}^{tr}))^2].
\]

The first term captures how closely trend tracks the data, while the second term captures smoothness of the trend in terms of second differences. The smoothing parameter \( \lambda \) governs the weight assigned to tracking vs. smoothness in the minimization problem. As \( \lambda \) rises, the trend becomes smoother, and correspondingly more variations are assigned to the cycle component. As \( \lambda \to \infty \), positive second differences are not allowed, and \( y_{t}^{tr} \) becomes linear in \( t \).

The choice \( \lambda = 1600 \) is reasonable for business cycle analysis using quarterly data, as it works to remove cyclical movements with periods in excess of 4-6 years.
The HP filter is very flexible, and takes no stand on the form of the trend. It can induce spurious volatility into the cycle component, however.

\section{Stochastic trend}

The trend component is assumed to be a random process.

\textbf{Example:} Suppose \( \{y_t\} \) is a random walk with drift:

\[ y_t = y_{t-1} + \gamma + \varepsilon_t, \]

where \( \varepsilon_t \) is white noise. This is a "unit root" specification. Define:

\[ y^r_t = y_t - 1 + \gamma, \quad \hat{y}_t = \varepsilon_t. \]

In this case the trend is the linear projection, and the cycle is the innovation. Moreover, innovations to output have a permanent effect on the trend.

To obtain a stationary representation of \( \{y_t\} \), take the first difference:

\[ \Delta y_t = y_t - y_{t-1} = \gamma + \varepsilon_t. \]

Note that \( \Delta y_t \) approximates the growth rate of output:

\[ \Delta y_t = \ln Y_t - \ln Y_{t-1} \approx \frac{Y_t - Y_{t-1}}{Y_{t-1}}. \]

Thus, for a unit root specification, we consider output growth rates rather than log levels.

There are numerous other approaches to modeling trend and cycle components, e.g., state-space and frequency domain methods.