Technical Notes

Notation and definitions.

$\tilde{X} =$ random variable.

$\tilde{X} = \tilde{Y} \Rightarrow \tilde{X}$ and $\tilde{Y}$ are equivalent.

$\tilde{X} = 0 \Rightarrow \tilde{X}$ is identically equal to zero.

$E_t[\tilde{X}] =$ expectation conditional on information available at year $t$, $t = 0, 1, 2, \ldots$.

Law of Iterated Expectations: $E_u[E_t[\tilde{X}]] = E_t[\tilde{X}]$ for any $u < t$.

Cash flow stream = sequence of cash flows $C_{A_0}, \tilde{C}_{A_1}, \tilde{C}_{A_2}, \ldots$.

$C_{A_0} =$ cash flow at year 0.

$\tilde{C}_{A_t} =$ random cash flow at year $t$, $t = 1, 2, \ldots$.

$C_{A_0}, \tilde{C}_{A_t} > 0 \Rightarrow$ cash is received.

$C_{A_0}, \tilde{C}_{A_t} < 0 \Rightarrow$ cash is paid.

$\tilde{C}_{A_t} = 0 \Rightarrow$ no cash flow occurs at year $t$.

Asset = ownership of a particular cash flow stream.

$V_{A_0} =$ current market value of $\tilde{C}_{A_1}, \tilde{C}_{A_2}, \ldots$ ($C_{A_0}$ not included).

$\tilde{V}_{A_1} =$ future market value of $\tilde{C}_{A_1, t+1}, \tilde{C}_{A_1, t+2}, \ldots$ ($\tilde{C}_{A_t}$ not included).

$\tilde{r}_{A_1} =$ return on asset at year 1:

$$\tilde{r}_{A_1} = \frac{\tilde{C}_{A_1} + \tilde{V}_{A_1} - V_{A_0}}{V_{A_0}}.$$

$\tilde{r}_{A_t} =$ return on asset at year $t > 1$:

$$\tilde{r}_{A_t} = \frac{\tilde{C}_{A_t} + \tilde{V}_{A_t} - \tilde{V}_{A_{t-1}}}{\tilde{V}_{A_{t-1}}}.$$

Capital market assumptions.

Assumption 1. Perfectly competitive.

a. All agents are price takers;

b. Traded assets may be arbitrarily scaled, and combined into portfolios;

c. Agents have symmetric information.
Assumption 2. Arbitrage-free. Agent cannot make trades that yield strictly positive cash flows with certainty.

Assumption 3. Complete. All risks can be traded

Assumption 4. Constant forecasts. For all $t > 0$ and all $u < t$, the year $u$ forecast of $\tilde{r}_{At}$ does not vary with $u$:

$$E_u[\tilde{r}_{At}] = E_0[\tilde{r}_{At}] = r_{At}.$$ 

Assumption 5. Flat yield curve. For any given asset, expected returns are constant over time:

$$r_{At} = r_A, \quad t = 1, 2, 3, ...$$

Valuation Theorem. a. Suppose the capital market is perfectly competitive and arbitrage-free. Then there exist random variables $\tilde{m}_t^S$, $t = 1, ..., T - S$, $S = 0, 1, ..., T - 1$, called stochastic discount factors (SDFs), such that the current market value of an asset is determined by

$$V_{A0} = \sum_{t=1}^{T} E_0[\tilde{m}_t^0 \tilde{C}_{At}],$$

and the future market values are determined by

$$\tilde{V}_{AS} = \sum_{t=1}^{T-S} E_0[\tilde{m}_t^S \tilde{C}_{A,S+t}], \quad S > 0.$$

b. If the market is also complete, then the random variables $\tilde{m}_t^S$ are unique.¹

Proposition 2.1. The current market value of $\tilde{C}_{AS}$ is given by

$$V_{A0} = \frac{E_0[\tilde{C}_{AS}]}{(1 + r_A)^S}.$$ 

Proof. Since $\tilde{C}_{A,S+t} = 0$ for all $t > 0$, the Valuation Theorem implies

$$\tilde{V}_{AS} = \sum_{t=1}^{T-S} E_S[\tilde{m}_{S+t}^S \cdot 0] = 0.$$ 

This means we can write the return $\tilde{r}_{AS}$ as

$$\tilde{r}_{AS} = \frac{\tilde{C}_{AS} + \tilde{V}_{AS} - \tilde{V}_{A,S-1}}{\tilde{V}_{A,S-1}} = \frac{\tilde{C}_{AS} - \tilde{V}_{A,S-1}}{\tilde{V}_{A,S-1}}.$$ 

¹For a proof, see Asset Pricing by John H. Cochrane.
Rearrange: 
\[ \hat{V}_{A,S-1}(1 + \hat{r}_{AS}) = \hat{C}_{AS}. \]

Now take conditional expectation at year \( S - 1 \) of both sides of the equation:
\[
E_{S-1} \left[ \hat{V}_{A,S-1}(1 + \hat{r}_{AS}) \right] = E_{S-1} \left[ \hat{C}_{AS} \right].
\]

Since \( \hat{V}_{A,S-1} \) is known at year \( S - 1 \), \( \hat{V}_{A,S-1} \) may be treated as a constant, and
the left-hand side of the preceding equation may be written as
\[
E_{S-1} \left[ \hat{V}_{A,S-1}(1 + \hat{r}_{AS}) \right] = \hat{V}_{A,S-1} E_{S-1} [(1 + \hat{r}_{AS})]
\]
\[
= \hat{V}_{A,S-1} (1 + E_{S-1} [\hat{r}_{AS}]) = \hat{V}_{A,S-1} (1 + r_{AS}),
\]
where the last equality uses Assumption 4. Thus,
\[
\hat{V}_{A,S-1} (1 + r_{AS}) = E_{S-1} \left[ \hat{C}_{AS} \right]. \tag{1}
\]

Next, since \( \hat{C}_{A,S-1} = 0 \), the return \( \hat{r}_{A,S-1} \) becomes
\[
\hat{r}_{A,S-1} = \frac{\hat{C}_{A,S-1} + \hat{V}_{A,S-1} - \hat{V}_{A,S-2}}{\hat{V}_{A,S-2}} = \frac{\hat{V}_{A,S-1} - \hat{V}_{A,S-2}}{\hat{V}_{A,S-2}}.
\]
Rearrange:
\[ \hat{V}_{A,S-2}(1 + \hat{r}_{A,S-1}) = \hat{V}_{A,S-1}. \tag{2} \]

Combining equations (1) and (2) gives
\[
\hat{V}_{A,S-2}(1 + \hat{r}_{A,S-1})(1 + r_{AS}) = E_{S-1} \left[ \hat{C}_{AS} \right].
\]

Now take conditional expectation at year \( S - 2 \) of both sides of the equation, and apply the Law of Iterated Expectations:
\[
E_{S-2} \left[ \hat{V}_{A,S-2}(1 + \hat{r}_{A,S-1})(1 + r_{AS}) \right] = E_{S-2} \left[ \hat{C}_{AS} \right].
\]

Since \( \hat{V}_{A,S-2} \) and \( r_{AS} \) are known at year \( S - 2 \), they may be treated as constants, and
the left-hand side of the preceding equation may be written as
\[
E_{S-2} \left[ \hat{V}_{A,S-2}(1 + \hat{r}_{A,S-1})(1 + r_{AS}) \right]
\]
\[
= \hat{V}_{A,S-2} (1 + r_{AS}) E_{S-2} [(1 + \hat{r}_{A,S-1})]
\]
\[
= \hat{V}_{A,S-2} (1 + r_{AS})(1 + E_{S-2} [\hat{r}_{A,S-1}])
\]
\[
= \hat{V}_{A,S-2} (1 + r_{AS})(1 + r_{A,S-1}),
\]

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where the last equality uses Assumption 4. Thus,

\[ \tilde{V}_{A,S-2} (1 + r_{AS}) (1 + r_{A,S-1}) = E_{S-2} \left[ \tilde{C}_{AS} \right]. \]

Continuing in this way for years \( S-3, S-4, \ldots \), we eventually reach year 0, and we arrive at the equation

\[ V_{A0} (1 + r_{AS}) (1 + r_{A,S-1}) \cdots (1 + r_{A1}) = E_0 \left[ \tilde{C}_{AS} \right]. \]

Moreover, Assumption 5 implies

\[ (1 + r_{AS}) (1 + r_{A,S-1}) \cdots (1 + r_{A1}) = (1 + r_A)^S, \]

so we may write

\[ V_{A0}(1 + r_A)^S = E_0 \left[ \tilde{C}_{At} \right], \]

which gives the result.

\[ \square \]

**Proposition 2.2.** The current market value of the portfolio consisting of Assets 1, \ldots, \( J \) is given by

\[ V_{A0} = \sum_{j=1}^{J} V_{A0}^j. \]

**Proof.** Since \( \tilde{C}_{At} = \sum_{j=1}^{J} \tilde{C}_{At}^j \), we have, using the Valuation Theorem:

\[
V_{A0} = \sum_{t=1}^{T} E_0 \left[ \tilde{m}_t^0 \tilde{C}_{At} \right] = \sum_{t=1}^{T} E_0 \left[ \tilde{m}_t^0 \left( \sum_{j=1}^{J} \tilde{C}_{At}^j \right) \right] \\
= \sum_{t=1}^{T} E_0 \left[ \sum_{j=1}^{J} \tilde{m}_t^0 \tilde{C}_{At}^j \right] = \sum_{t=1}^{T} \sum_{j=1}^{J} E_0 \left[ \tilde{m}_t^0 \tilde{C}_{At}^j \right] \\
= \sum_{j=1}^{J} \sum_{t=1}^{T} E_0 \left[ \tilde{m}_t^0 \tilde{C}_{At}^j \right] = \sum_{j=1}^{J} V_{A0}^j.
\]

\[ \square \]

**Proposition 2.3.** The OCC of the portfolio consisting of Assets 1, \ldots, \( J \) is given by

\[ r_A = \sum_{j=1}^{J} \frac{V_{A0}^j}{V_{A0}} r_{A}^j. \]
Proof. The Valuation Theorem implies

\[ \tilde{V}_{A1} = \sum_{t=1}^{T-1} E_t \left[ \tilde{m}_{1+t} \tilde{C}_{A,t+1} \right], \]

where the random variables \( \tilde{m}_{t+1}, t = 1, 2, \ldots \), are the SDFs for valuing assets at year 1. Thus, for the portfolio consisting of Assets 1, \ldots, J, we may apply the argument of Prop. 2.2 to obtain

\[ \tilde{V}_{A1} = \sum_{j=1}^{J} \tilde{V}_j. \]

It follows that \( \tilde{r}_{A1} \) may be expressed as

\[
\tilde{r}_{A1} = \frac{\tilde{C}_{A1} + \tilde{V}_{A1} - V_{A0}}{V_{A0}} = \sum_{j=1}^{J} \frac{\tilde{C}_j^{A1} + \sum_{j=1}^{J} \tilde{V}_j - \sum_{j=1}^{J} V_j}{V_{A0}} \\
= \sum_{j=1}^{J} \left( \frac{V_j^{A0}}{V_{A0}} \times \frac{\tilde{C}_j^{A1} + \tilde{V}_j^{A1} - V_j^{A0}}{V_j^{A0}} \right) = \sum_{j=1}^{J} \frac{V_j^{A0}}{V_{A0}} \tilde{r}_j^{A1}.
\]

Take unconditional expectation:

\[
E_0 [\tilde{r}_{A1}] = E_0 \left[ \sum_{j=1}^{J} \frac{V_j^{A0}}{V_{A0}} \tilde{r}_j^{A1} \right] = \sum_{j=1}^{J} \frac{V_j^{A0}}{V_{A0}} E_0 [\tilde{r}_j^{A1}].
\]

Finally, Assumption 5 implies

\[
E_0 [\tilde{r}_{A1}] = r_A, \quad E_0 [\tilde{r}_j^{A1}] = r_j^{A}.
\]

Thus,

\[
r_A = \sum_{j=1}^{J} \frac{V_j^{A0}}{V_{A0}} r_j^{A}.
\]

Proposition 2.4. The current market value of the cash flow stream \( \tilde{C}_{A1}, \tilde{C}_{A2}, \ldots \) is given by

\[
V_{A0} = \sum_{t=1}^{\infty} \frac{C_{At}}{(1 + r_A)^t}.
\]
Proof. First consider the finite cash flow stream $\tilde{C}_{A1}, \tilde{C}_{A2}, ..., \tilde{C}_{AT}$. The future cash flows may be viewed as a portfolio consisting of $T$ separate assets, with each asset generating the single cash flow $\tilde{C}_{At}$. Prop. 2.1 shows that the value of $\tilde{C}_{At}$ considered as a separate asset is given by

$$\frac{C_{At}}{(1 + r_A)^t}.$$ 

Thus, Prop. 2.2 implies

$$V_{A0} = \sum_{t=1}^{T} \frac{C_{At}}{(1 + r_A)^t}.$$ 

For an infinite cash flow stream, we have

$$V_{A0} = \lim_{T \to \infty} \sum_{t=1}^{T} \frac{C_{At}}{(1 + r_A)^t} = \sum_{t=1}^{\infty} \frac{C_{At}}{(1 + r_A)^t},$$

provided the limit exists.

Corollary 2.1. The market value of $\tilde{C}_{A,S+1}, \tilde{C}_{A,S+2}, ...$ at year $S > 0$ is given by

$$\tilde{V}_{AS} = \sum_{t=1}^{\infty} E_S \left[ \frac{\tilde{C}_{A,S+t}}{(1 + r_A)^t} \right].$$

Proof. First consider a single cash flow $\tilde{C}_{A,S+t}$ received at year $S + t > S$. The arguments from the proof of Prop. 2.1 may be used to obtain, for $u < S + t$,

$$\tilde{V}_{A,S+t-u}(1 + E_{S+t-u}[r_{A,S+t}]) (1 + E_{S+t-u}[r_{A,S+t-1}])$$

$$\cdots (1 + E_{S+t-u}[r_{A,S+t-u+1}]) = E_{S+t-u} \left[ \tilde{C}_{A,S+t} \right].$$

Letting $u = t$ and invoking Assumptions 4 and 5 gives

$$\tilde{V}_{AS}(1 + r_A)^t = E_S \left[ \tilde{C}_{A,S+t} \right],$$

or

$$\tilde{V}_{AS} = \frac{E_S \left[ \tilde{C}_{A,S+t} \right]}{(1 + r_A)^t}.$$ 

Next, for $T > S$, the market value at year $S$ of the cash flow stream $\tilde{C}_{A,S+1}, \tilde{C}_{A,S+2}, ... \tilde{C}_{A,T}$ is given by

$$\tilde{V}_{AS} = \sum_{t=1}^{T-S} E_S \left[ \tilde{m}_{S+t}^S \tilde{C}_{A,S+t} \right],$$

or
where the random variables \( \tilde{n}^{S+t} \), \( t = 1, 2, \ldots \), are the SDFs for valuing assets at year \( S \). Applying the arguments from the proof of Prop. 2.4 gives

\[
\tilde{V}_{AS} = \sum_{t=1}^{T-S} E_S \left[ \tilde{C}_{A,S+t} \right] \frac{1}{(1 + r_A)^t},
\]

and the result follows by letting \( T \to \infty \), provided the limit exists.

\[ \square \]

**Corollary 2.2.** The current market value of \( \tilde{V}_{AS} \) is given by

\[
\frac{V_{AS}}{(1 + r_A)^S}.
\]

Moreover, the Valuation Formula converges if and only if

\[
\lim_{S \to \infty} \frac{V_{AS}}{(1 + r_A)^S} = 0.
\]

**Proof.** Consider the cash flow stream in which \( \tilde{C}_{Au} = 0 \) at years \( u \leq S \), and the flows \( \tilde{C}_{A,S+1}, \tilde{C}_{A,S+2}, \ldots \) are received at years \( S + 1, S + 2, \ldots \). Using the Valuation Formula (with the index of summation changed to \( u \)), we can write

\[
V_{A0} = \sum_{u=1}^{\infty} \frac{C_{Au}}{(1 + r_A)^u}
= \sum_{u=1}^{S} \frac{0}{(1 + r_A)^u} + \sum_{u=S+1}^{\infty} \frac{C_{Au}}{(1 + r_A)^u}.
\]

Let \( t = S - u \), so that \( u = S + t \) and the indices \( u = S + 1, S + 2, \ldots \) correspond to \( t = 1, 2, \ldots \). Thus,

\[
V_{A0} = \sum_{u=S+1}^{\infty} \frac{C_{Au}}{(1 + r_A)^u} = \sum_{t=1}^{\infty} \frac{C_{A,S+t}}{(1 + r_A)^{S+t}} = \frac{1}{(1 + r_A)^S} \sum_{t=1}^{\infty} \frac{C_{A,S+t}}{(1 + r_A)^t}.
\]

Moreover, from Cor. 2.1 we have

\[
\tilde{V}_{AS} = \sum_{t=1}^{\infty} E_S \left[ \tilde{C}_{A,S+t} \right] \frac{1}{(1 + r_A)^t}.
\]

Take unconditional expectation of both sides and apply the Law of Iterated Expectations:

\[
V_{AS} = E_0 \left[ \tilde{V}_{AS} \right] = E_0 \left[ \sum_{t=1}^{\infty} E_S \left[ \tilde{C}_{A,S+t} \right] \frac{1}{(1 + r_A)^t} \right].
\]
Combining equations (3) and (4) gives the result.

Next, suppose the Valuation Formula converges. This means that, for any \( \varepsilon > 0 \), there exists \( \tilde{S} \) such that \( S > \tilde{S} \) implies

\[
\left| \sum_{t=S+1}^{\infty} \frac{C_A}{(1 + r_A)^t} \right| < \varepsilon.
\]

Let \( u = t - S \), so that \( t = S + u \), and also the indices \( t = S + 1, S + 2, \ldots \) correspond to \( u = 1, 2, \ldots \). Thus,

\[
\sum_{t=S+1}^{\infty} \frac{C_A}{(1 + r_A)^t} = \sum_{u=1}^{\infty} \frac{C_{A,S+u}}{(1 + r_A)^{S+u}} = \frac{1}{(1 + r_A)^S} \sum_{u=1}^{\infty} \frac{C_{A,S+u}}{(1 + r_A)^u}.
\]

Using Cor. 2.1 (with the index of summation changed to \( u \)), we have

\[
V_{AS} = E_0 \left[ \hat{V}_{AS} \right] = E_0 \left[ \sum_{u=1}^{\infty} \frac{E_S \left[ \hat{C}_{A,S+u} \right]}{(1 + r_A)^u} \right] = \sum_{u=1}^{\infty} \frac{C_{A,S+u}}{(1 + r_A)^u}.
\]

Substitute to obtain

\[
\sum_{t=S+1}^{\infty} \frac{C_A}{(1 + r_A)^t} = \frac{V_{AS}}{(1 + r_A)^S}.
\]

It follows that for every \( \varepsilon > 0 \), there exists \( \tilde{S} \) such that \( S > \tilde{S} \) implies

\[
\left| \sum_{t=S+1}^{\infty} \frac{C_A}{(1 + r_A)^t} \right| = \left| \frac{V_{AS}}{(1 + r_A)^S} \right| < \varepsilon,
\]

which means

\[
\lim_{S \to \infty} \frac{V_{AS}}{(1 + r_A)^S} = 0.
\]

The reverse implication follows immediately from these arguments.

\[\square\]

**Proposition 2.5.** If the cash flow stream \( \hat{C}_{A1}, \hat{C}_{A2}, \ldots \) is delayed by \( S \) years, then its current market value is given by

\[
\hat{V}_{A0} = \frac{1}{(1 + r_A)^S} V_{A0}.
\]
Proof. Given the delay, the asset generates zero cash at years \( t = 1, 2, ..., S - 1 \), and the cash flows \( \tilde{C}_{A1}, \tilde{C}_{A2}, ... \) are received at years \( t = S + 1, S + 2, ... \). Thus, the Valuation Formula gives

\[
\tilde{V}_{A0} = \sum_{t=1}^{S} \frac{0}{(1 + r_A)^t} + \sum_{t=S+1}^{\infty} \frac{C_{A,t-S}}{(1 + r_A)^t}
\]

\[
= \sum_{t=S+1}^{\infty} \frac{C_{A,t-S}}{(1 + r_A)^t}.
\]

Define \( u = t - S \), so that \( t = S + u \), and the years \( t = S + 1, S + 2, ... \) correspond to years \( u = 1, 2, ... \). Substitute for \( t \) and rearrange to obtain:

\[
\tilde{V}_{A0} = \sum_{u=1}^{\infty} \frac{C_{Au}}{(1 + r_A)^{S+u}}
\]

\[
= \frac{1}{(1 + r_A)^S} \sum_{u=1}^{\infty} \frac{C_{Au}}{(1 + r_A)^u} = \frac{1}{(1 + r_A)^S} V_{A0}.
\]

\( \square \)

Proposition 2.6. The current market value of a Growing Annuity is given by

\[
V_{A0} = \left\{ \begin{array}{ll}
\frac{B}{r_A - g} \left[ 1 - \left( \frac{1+g}{1+r_A} \right)^T \right], & g \neq r_A \\
\frac{TB}{1+r_A}, & g = r_A
\end{array} \right.
\]

Proof. Suppose \( g \neq r_A \). For \( T = 1 \) we have, using Prop. 2.4:

\[
V_{A0} = \frac{B}{1+r_A} + \sum_{t=2}^{\infty} \frac{0}{(1 + r_A)^t}
\]

\[
= \frac{B}{1+r_A} \times \frac{r_A - g}{r_A - g} = \frac{B}{r_A - g} \left[ \frac{r_A - g}{1 + r_A} \pm \frac{1}{1 + r_A} \right]
\]

\[
= \frac{B}{r_A - g} \left[ \frac{1-r_A - 1-g}{1 + r_A} \right] = \frac{B}{r_A - g} \left[ \frac{1-g}{1 + r_A} \right].
\]

Now fix \( T > 1 \) and suppose the result holds for \( 1, 2, ..., T - 1 \). Again using Prop. 2.4, we have

\[
V_{A0} = \sum_{t=1}^{T} \frac{B(1+g)^{t-1}}{(1 + r_A)^t} + \sum_{t=T+1}^{\infty} \frac{0}{(1 + r_A)^t}
\]

\[
= \sum_{t=1}^{T-1} \frac{B(1+g)^{t-1}}{(1 + r_A)^t} + \frac{B(1+g)^{T-1}}{(1 + r_A)^T}
\]

\[
= \sum_{t=1}^{T-1} \frac{B(1+g)^{t-1}}{(1 + r_A)^t} + \frac{B(1+g)^{T-1}}{(1 + r_A)^T}
\]

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Thus by induction the result is valid for all $T$

For $g = r_A$:

$$V_{A0} = \sum_{t=1}^{T} \frac{B(1+r_A)^{t-1}}{(1+r_A)^t} + \sum_{t=T+1}^{\infty} \frac{0}{(1+r_A)^t} = \sum_{t=1}^{T} \frac{B}{1+r_A} = \frac{TB}{1+r_A}. $$

$\square$

**Proposition 2.7.** If $g < r_A$, then the current market value of a Growing Annuity satisfies

$$\lim_{T \to \infty} V_{A0} = \frac{B}{r_A - g}.$$ 

**Proof.** Define the constant $\alpha$ by

$$\alpha = \frac{1+g}{1+r_A}.$$ 

Then $-1 < g < r_A$ implies $0 < \alpha < 1$, so that $\lim_{T \to \infty} \alpha^T = 0$. Moreover, the Growing Annuity Formula may be written as

$$V_{A0} = \frac{B}{r_A - g} \left[ 1 - \alpha^T \right].$$

Since this is a continuous function of $\alpha^T$, we have

$$\lim_{T \to \infty} V_{A0} = \frac{B}{r_A - g} \left[ 1 - \lim_{T \to \infty} \alpha^T \right] = \frac{B}{r_A - g}. $$

$\square$

**Alternate Proof.** Using the Valuation Formula, we have

$$V_{A0} = \sum_{t=1}^{T} \frac{B(1+g)^{t-1}}{(1+r_A)^t} = \frac{B}{(1+r_A)} \sum_{t=1}^{T} (1+g)^{t-1}.$$ 

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\[ \frac{B}{(1 + r_A)} \sum_{t=1}^{T} \alpha^{t-1} = \frac{B}{(1 + r_A)} \sum_{u=0}^{T-1} \alpha^u, \]

where the substitution \( u = t - 1 \) has been made. Since \( 0 < \alpha < 1 \), the Geometric Series Theorem yields

\[ \lim_{T \to \infty} \sum_{u=0}^{T-1} \alpha^u = \frac{1}{1 - \alpha}. \]

Thus,

\[ \lim_{T \to \infty} V_{A0} = \frac{B}{(1 + r_A)} \times \frac{1}{1 - \alpha} \]
\[ = \frac{B}{1 + \frac{r_A}{1 + \frac{r_A}{1 + g}}} = \frac{B}{r_A - g}. \]

\[ \square \]

**Proposition 2.8.** Suppose \( C_{B1}, C_{B2}, \ldots \) and \( I_0 \) satisfy IRR Regularity Conditions 1 and 2. Then the IRR exists uniquely, and the IRR Rule is equivalent to the NPV Rule.

**Proof.** Define the function \( N_B(r) \) by

\[ N_B(r) = \sum_{t=1}^{\infty} \frac{C_{Bt}}{(1 + r)^t} - I_0, \quad r > -1. \]

It follows that \( r_{\text{IRR}} \) satisfies the IRR Equation if and only if \( N_B(r_{\text{IRR}}) = 0 \), i.e., \( N_B \) intersects the \( r \)-axis at \( r_{\text{IRR}} \).

Using Condition 1, we have

\[ N_B(0) = \sum_{t=1}^{\infty} C_{Bt} - I_0 > 0, \]

\[ \lim_{r \to -\infty} N_B(r) = -I_0 < 0. \]

Thus \( N_B(r) \) must cross the \( r \)-axis at least once for \( r > 0 \). Since \( N_B(r) \) is a continuous function, it must intersect the \( r \)-axis, which means \( N_B(r) = 0 \) for some \( r > 0 \). Conclude that an IRR exists. Moreover, Condition 2 implies that \( N_B(r) \) is strictly decreasing in \( r \), so it cannot intersect the \( r \)-axis more than one point. Conclude that the IRR is unique.

Next, using Condition 2, we have that \( r_{\text{IRR}} > r_B \) holds if and only if \( 0 = N_B(r_{\text{IRR}}) < N_B(r_B) = \text{NPV} \), whereas \( r_{\text{IRR}} < r_B \) holds if and only if \( 0 = N_B(r_{\text{IRR}}) > N_B(r_B) = \text{NPV} \). This proves that the IRR and NPV Rules are equivalent.

\[ \square \]

**Lemma 4.1.** For any \( l \) and \( S \), the market value at year \( S \) of type \( l \) bonds issued at years \( S + 1, S + 2, \ldots \) is zero.
Proof. Let \( \bar{F}_l(S + u, T) \) be the face value of type \( l \) bonds of maturity \( T \) issued at year \( S + u \), and let \( \bar{D}_{S+u}^l(S + u, T) \) be the market value of these bonds at year \( S + u \). Net payments to holders of these bonds at years \( S + u, S + u + 1, S + u + 2 \ldots \) are as follows:

\[
\bar{C}_{D,S+u+t}^l(S+u,T) = \begin{cases} 
-\bar{D}_{S+u}^l(S + u, T), & t = 0, \\
 r_C^l \bar{F}_l(S + u, T), & t = 1, \ldots, T - 1, \\
r_C^l \bar{F}_l(S + u, T) + \bar{F}_l(S + u, T), & t = T, \\
0, & t = T + 1, T + 2, \ldots 
\end{cases}
\]

Corollary 2.1 implies that the market value of the bonds at year \( S + u \) is

\[
\bar{D}_{S+u}^l(S + u, T) = \sum_{t=1}^{\infty} \frac{E_{S+u} \left[ \bar{C}_{D,S+u+t}^l(S + u, T) \right]}{(1 + r_D^l)^t}.
\]

Thus, the market value of the bonds at year \( S \) is

\[
\bar{D}_S^l(S + u, T) = - \frac{E_S \left[ \bar{D}_{S+u}^l(S + u, T) \right]}{(1 + r_D^l)u} + \sum_{t=1}^{\infty} \frac{E_S \left[ \bar{C}_{D,S+u+t}^l(S + u, T) \right]}{(1 + r_D^l)^{u+t}}
\]

\[
= \frac{1}{(1 + r_D^l)^u} \left[ -E_S \left[ \bar{D}_{S+u}^l(S + u, T) \right] + \sum_{t=1}^{\infty} \frac{E_{S+u} \left[ \bar{C}_{D,S+u+t}^l(S + u, T) \right]}{(1 + r_D^l)^t} \right]
\]

\[
= \frac{1}{(1 + r_D^l)^u} \cdot [0] = 0.
\]

It follows that the market value at year \( S \) of type \( l \) bonds issued in all future years at all maturities is given by

\[
\sum_{u=1}^{\infty} \sum_{T=1}^{\infty} \bar{D}_S^l(S + u, T) = 0.
\]

\[\square\]

Proposition 4.1. If type \( l \) debt satisfies \( r_C^l = r_D^l \), then \( \bar{D}_S^l = \bar{F}_S^l \) for all \( S \).

Proof. Fix \( T > 0 \), and let \( \bar{F}_S^l(T) \) denote the face value of type \( l \) bonds outstanding at year \( S \) that mature at year \( S + T \). Net payments to holders of these bonds at years \( S + 1, S + 2, \ldots \) are as follows:

\[
\bar{C}_{D,S+t}^l(T) = \begin{cases} 
 r_C^l \bar{F}_S^l(T), & t = 1, \ldots, T - 1, \\
r_C^l \bar{F}_S^l(T) + \bar{F}_S^l(T), & t = T, \\
0, & t = T + 1, T + 2, \ldots 
\end{cases}
\]
Corollary 2.1 implies that the market value of the bonds at year $S$ is
\[
\tilde{D}_S(T) = \sum_{t=1}^{\infty} \frac{E_S [\tilde{C}_{D,S+t}(T)]}{(1 + r_D)^t} = \sum_{t=1}^{T} \frac{r_C^l \tilde{F}_S^l(T)}{(1 + r_D)^t} + \frac{\tilde{F}_S^l(T)}{(1 + r_D)^T}
\]
\[
= r_C^l \frac{\tilde{F}_S^l(T)}{r_D^l} \left[ 1 - \frac{1}{(1 + r_D)^T} \right] + \frac{\tilde{F}_S^l(T)}{(1 + r_D)^T}.
\]
Since $r_C^l = r_D^l$, we have
\[
\tilde{D}_S(T) = \tilde{F}_S(T) \left[ 1 - \frac{1}{(1 + r_D)^T} \right] + \frac{\tilde{F}_S(T)}{(1 + r_D)^T} = \tilde{F}_S(T).
\]
Moreover, Lemma 4.1 shows that the market value at year $S$ of type $l$ bonds of all maturities issued at years $S+1, S+2, \ldots$ is zero. Thus,
\[
\tilde{D}_S = \sum_{T=1}^{\infty} \tilde{D}_S^l(T) = \sum_{T=1}^{\infty} \tilde{F}_S^l(T) = \tilde{F}_S.
\]

\[\Box\]

**Proposition 4.2.** If type $l$ debt satisfies $r_C^l = r_D^l$ and $F_t^l = F_0^l(1 + g)^t$ for all $t > 0$, then the value of ITS is given by
\[
V_{0,ITS}^l = \frac{r_D^l D_0^l}{r_D^l - g^l}.
\]

**Proof.** In view of Prop. 4.1, $r_C^l = r_D^l$ implies $F_0^l = D_0^l$. Thus, we have $F_t^l = D_0^l(1 + g^l)^t$, and the Growing Perpetuity Formula yields
\[
V_{ITS,0}^l = \sum_{t=1}^{\infty} \frac{r_C^l F_{t-1}^l}{(1 + r_D^l)^t} = \sum_{t=1}^{\infty} \frac{r_D^l D_0^l(1 + g^l)^{t-1} \tau}{(1 + r_D^l)^t} = \frac{r_D^l D_0^l \tau}{r_D^l - g^l}.
\]

\[\Box\]

**Proposition 4.3.** Suppose for every $l$, $r_C^l = r_D^l$ and $F_t^l = F_0^l$ for all $t > 0$. Then the value and OCC of ITS are given by
\[
V_{ITS,0} = V_{D,0} \tau, \quad r_{ITS} = r_D.
\]

**Proof.** Under the assumptions, Prop. 4.2 implies, for all $l$,
\[
V_{0,ITS}^l = \frac{r_D^l D_0^l}{r_D^l - g^l} = D_0^l \tau.
\]
Thus,
\[
V_{ITS,0} = \sum_{l=1}^{L} V_{ITS,l} = \sum_{l=1}^{L} D_0^l \tau = \left( \sum_{l=1}^{L} D_0^l \right) \tau = D_0 \tau,
\]
\[
r_{ITS} = \sum_{l=1}^{L} V_{ITS,l} r_D^l = \sum_{l=1}^{L} \frac{D_0^l \tau}{D_0} r_D^l = \sum_{l=1}^{L} \frac{D_0^l}{D_0} r_D^l = r_D.
\]

\[\square\]

**Corollary 4.1.** Under the conditions of Proposition 4.3, expected net payouts to debtholders for all \( t > 0 \) are given by
\[
C_{Dt} = \text{Int}_t = r_D D_0.
\]

**Proof.** Let \( \tilde{F}^l_{Mt} \) denote the face value of type \( l \) debt that matures at year \( t \), and let \( \tilde{F}^l_{Nt} \) and \( \tilde{D}^l_{Nt} \) denote the face value and market value, respectively, of new type \( l \) debt issued at year \( t \). The face value of type \( l \) debt at year \( t + 1 \) is determined by
\[
\tilde{F}^l_{t+1} = \tilde{F}^l_t - \tilde{F}^l_{Mt} + \tilde{F}^l_{Nt}.
\]
We have \( F^l_{t+1} = F^l_t = F^l_0 \) by hypothesis, so the preceding equation implies \( F^l_{Mt} = F^l_{Nt} \).

Net payouts to type \( l \) debtholders at year \( t \) are given by
\[
\tilde{C}^l_{Dt} = \text{Int}_t + \tilde{F}^l_{Mt} - \tilde{D}^l_{Nt} = r_C \tilde{F}^l_t + \tilde{F}^l_{Mt} - \tilde{D}^l_{Nt}.
\]
Since \( r_C = r_D \), we have \( \tilde{F}^l_{Nt} = \tilde{D}^l_{Nt} \). Substituting for \( \tilde{D}^l_{Nt} \) and taking expectation gives
\[
C^l_{Dt} = r_C F^l_t + F^l_{Mt} - F^l_{Nt} = r_C F^l_t + r_C F^l_0 = r_C D^l_0.
\]

Expected net payouts to all debtholders are obtained as follows:
\[
C_{Dt} = \sum_{l=1}^{L} C^l_{Dt} = \sum_{l=1}^{L} r_D^l D_0^l \times \frac{D_0}{D_0} = \sum_{l=1}^{L} r_D^l D_0^l \times D_0 = r_D D_0.
\]

\[\square\]

**Proposition 5.1.** The Value of Equity equals the value of the net payout stream:
\[
E_0 = E_0^{NP}.
\]
Proof. Since each share has a cash value of $\tilde{Div}_1 + \tilde{P}_1$ at year 1, the portfolio of all currently outstanding shares has a cash value of $(\tilde{Div}_1 + \tilde{P}_1)K_0$ at year 1. Moreover, the definition of $\tilde{C}_{E1}$ may be rearranged to obtain

$$(\tilde{Div}_1 + \tilde{P}_1)K_0 = \tilde{C}_{E1} + \tilde{P}_1K_1,$$

so that the cash value of the portfolio of shares at year 1 equals $\tilde{C}_{E1} + \tilde{P}_1K_1$. Thus, the Value of Equity must satisfy

$$E_0 = \frac{C_{E1}}{1 + r_{NP}^E} + \frac{P_1K_1}{1 + r_E^1},$$

(5)

where $r_{NP}^E$ is the OCC of the net payout stream, and $r_E^1$ is the OCC of $\tilde{P}_1K_1$.

Next, since each share has a cash value of $\tilde{Div}_2 + \tilde{P}_2$ at year 2, the portfolio of all shares outstanding at year 1 has a cash value of $(\tilde{Div}_2 + \tilde{P}_2)K_1$ at year 2. Moreover, the definition of $\tilde{C}_{E2}$ may be rearranged to obtain

$$(\tilde{Div}_2 + \tilde{P}_2)K_1 = \tilde{C}_{E2} + \tilde{P}_2K_2.$$

Since $\tilde{P}_1K_1$ is the market value of the portfolio at year 1, it follows that the current market value of $\tilde{P}_1K_1$ must satisfy

$$\frac{P_1K_1}{1 + r_E^1} = \frac{C_{E2}}{(1 + r_{NP}^E)^2} + \frac{P_2K_2}{(1 + r_E^2)^2},$$

where $r_E^2$ is the OCC of $\tilde{P}_2K_2$. Substituting into equation (5) gives

$$E_0 = \frac{C_{E1}}{1 + r_{NP}^E} + \frac{C_{E2}}{(1 + r_{NP}^E)^2} + \frac{P_2K_2}{(1 + r_E^2)^2}.$$ 

Proceding in this way for years $t = 2, 3, \ldots$, we obtain

$$E_0 = \sum_{t=1}^{\infty} \frac{C_{Et}}{(1 + r_{NP}^E)^t} = E_{NP}^0.$$

Proposition 5.2. The Return on Equity equals the OCC of the net payout stream:

$$r_E = r_{NP}^E.$$

Proof. From the definition of $\tilde{C}_{E1}$ we have

$$(\tilde{Div}_1 + \tilde{P}_1)K_0 = \tilde{C}_{E1} + \tilde{E}_1.$$
Thus, using Prop. 5.1, we may write
\[
\tilde{r}_E = \frac{\hat{\text{Div}}_1 + \hat{P}_1 - P_0}{P_0} = \frac{\hat{\text{Div}}_1 + \hat{P}_1)K_0 - P_0K_0}{P_0K_0}
\]
\[
= \frac{\hat{C}_{E1} + \hat{E}_1 - E_0}{E_0} = \frac{\hat{C}_{E1} + \hat{E}_1^{NP} - E_0^{NP}}{E_0^{AC}} = \tilde{r}_E^{NP}.
\]
Taking expectation at year 0 gives the result. \qed