Technical Notes

Random variables

\( \tilde{X}_t \) = random variable realized at year \( t \).

Outcome of \( \tilde{X}_t \) is \textit{uncertain} at years \( s < t \).

Outcome of \( \tilde{X}_t \) is \textit{known} at years \( u \geq t \).

Note that \( \tilde{X}_0 \) is nonrandom, since it is known at year 0.

\( \tilde{X}_t = \tilde{Y}_t \Rightarrow \tilde{X}_t \) and \( \tilde{Y}_t \) have identical distributions.

\( \tilde{X}_t = \alpha \Rightarrow \) the constant \( \alpha \) is the only possible outcome of \( \tilde{X}_t \).

Expected values

\( E_0[\tilde{X}_t] = \) expected value or mean of \( \tilde{X}_t \).

Linearity: \( E_0[\alpha \tilde{X}_t + \beta \tilde{Y}_t] = \alpha E_0[\tilde{X}_t] + \beta E_0[\tilde{Y}_t] \).

Alternative notation: \( X_t = E_0[\tilde{X}_t] \).

Suppose \( 0 < t < u \).

\( E_t[\tilde{X}_u] = \) conditional expectation of \( \tilde{X}_u \) at year \( t \).

Assets and capital market

\textit{Asset} = designated sequence of cash flows \( \tilde{C}_{A0}, \tilde{C}_{A1}, \ldots \tilde{C}_{AT} \).

\( \tilde{C}_{At} > 0 \Rightarrow \) asset owner \textit{receives} cash at year \( t \).

\( \tilde{C}_{At} < 0 \Rightarrow \) asset owner \textit{pays} cash at year \( t \).

\textit{Market values} (prices) = \( \tilde{V}_{A0}, \tilde{V}_{A1}, \ldots \tilde{V}_{AT} \).

Note that \( \tilde{C}_{A0} \) and \( \tilde{V}_{A0} \) are nonrandom.

Asset sale at year \( t \):

Seller nets \( \tilde{C}_{At} + \tilde{V}_{At} \) at year \( t \).

Buyer pays \( \tilde{V}_{At} \) at year \( t \), receives or pays \( \tilde{C}_{At+1} \) at year \( t + 1 \).

\( \tilde{V}_{At} < 0 \Rightarrow \) seller pays cash to buyer. This defines a \textit{liability}. 

Total cash value or liquidation value at year $t = \tilde{C}_{At} + \tilde{V}_{At}$.

Present Value (PV) = total cash value at year 0 = $C_{A0} + V_{A0}$.

Capital market assumptions

Assumption 1. Perfectly competitive.
   - All agents are *price takers*;
   - Traded assets may be *arbitrarily scaled*, and *combined* into portfolios;
   - Agents have *symmetric information*.

Assumption 2. Arbitrage-free. Agent cannot make trades that yield strictly positive cash flows with certainty.

Assumption 3. Complete. All risks can be traded.

Assumption 4. Liquid. Assets can be fully liquidated in any future year.

Valuation Theorem. Assume there are no cash flows after year $T$, and Ass. 1-4 hold. Then there exists a unique collection of random variables $\tilde{M}_1, ..., \tilde{M}_T$, called *pricing factors*, such that the market values of any asset are given by

$$V_{A0} = \sum_{t=1}^{T} E_0[\tilde{M}_t \tilde{C}_{At}],$$

$$\tilde{V}_{At} = \sum_{u=t+1}^{T} E_t \left[ \left( \frac{\tilde{M}_u}{\tilde{M}_t} \right) \tilde{C}_{Au} \right], \quad t = 1, ..., T - 1.$$  

and $\tilde{V}_T = 0$.


Ass. 1-2 imply the existence of pricing factors and linearity of market values in cash flows. Adding Ass. 3 gives uniqueness of the pricing factors and $\tilde{M}_t \neq 0$ with probability 1. Adding Ass. 4 implies that only one set of pricing factors is applied for all years.

The pricing factors are also called *stochastic discount factors*.

Returns and OCC

$\tilde{r}_{At} = \text{net rate of return at year } t \text{ on a one-year investment in the asset, defined by}$

$$\tilde{V}_{A,t-1}(1 + \tilde{r}_{At}) = \tilde{C}_{At} + \tilde{V}_{At}.$$
Valuation Theorem \Rightarrow \text{ returns } \hat{r}_{A1}, \hat{r}_{A2}, ..., \hat{r}_{AT} \text{ exist uniquely whenever } \hat{V}_{A,t-1} \neq 0. \text{ In this case:}

\[
\hat{r}_{At} = \frac{\hat{C}_{At} + \hat{V}_{At} - \hat{V}_{A,t-1}}{\hat{V}_{A,t-1}}.
\]

\( E_{t-1}[\hat{r}_{At}] \) = expected return at year \( t - 1 \)

\( = \text{ Opportunity Cost of Capital (OCC) of the asset at year } t - 1. \)

**Portfolios and Value Additivity**

Assume Ass. 1-4. Consider a portfolio, called Asset \( P \), composed of Assets \( j \in J \):

\[
\hat{C}_{Pt} = \sum_{j \in J} \hat{C}_{jt}, \quad t = 0, 1, ..., T.
\]

**Proposition 2.1.** The market values of the portfolio consisting of Assets \( j \in J \) are given by

\[
\hat{V}_{Pt} = \sum_{j \in J} \hat{V}_{jt}, \quad t = 0, 1, ..., T.
\]

**Proof.** The Valuation Theorem gives

\[
V_{P0} = \sum_{t=1}^{T} E_0 [\hat{M}_t \hat{C}_{Pt}]
\]

\[
= \sum_{t=1}^{T} E_0 \left[ \hat{M}_t \sum_{j \in J} \hat{C}_{jt} \right] = \sum_{t=1}^{T} \sum_{j \in J} E_0 \left[ \hat{M}_t \hat{C}_{jt} \right]
\]

\[
= \sum_{j \in J} \sum_{t=1}^{T} E_0 \left[ \hat{M}_t \hat{C}_{jt} \right] = \sum_{j \in J} V_{j0}.
\]

Similarly for future market values.

Thus, the value of a portfolio of assets is the *sum* of the values of the assets making up the portfolio. This property is called **Value Additivity**.

**Corollary 2.1.** The return at year \( t > 0 \) on the portfolio consisting of Assets \( j \in J \) is given by

\[
\hat{V}_{P,t-1} \hat{r}_{Pt} = \sum_{j \in J} \hat{V}_{j,t-1} \hat{r}_{jt}.
\]

**Proof.** The definition of a return gives

\[
\hat{V}_{P,t-1} \hat{r}_{Pt} = \hat{C}_{Pt} + \hat{V}_{Pt} - \hat{V}_{P,t-1}
\]

\[
= \sum_{j \in J} \left( \hat{C}_{jt} + \hat{V}_{jt} - \hat{V}_{j,t-1} \right) = \sum_{j \in J} \hat{V}_{j,t-1} \hat{r}_{jt}.
\]
Discounted Cash Flow (DCF) formula

**Proposition 2.2.** The current market value of $\tilde{C}_{At}$ is given by

$$V_{A0} = DF_{At} \times C_{At},$$

where

$$DF_{At} = \frac{1}{E_0 [(1 + \tilde{r}_{A1}) (1 + \tilde{r}_{A2}) \cdots (1 + \tilde{r}_{At})]}.$$  

**Proof.** The Valuation Theorem gives $\tilde{V}_{At} = 0$, since $\tilde{C}_{Au} = 0$ for $u > t$. Hence

$$\tilde{V}_{At-1}(1 + \tilde{r}_{At}) = \tilde{C}_{At} + \tilde{V}_{At} = \tilde{C}_{At},$$

which implies

$$\tilde{V}_{At-1} = \frac{\tilde{C}_{At}}{1 + \tilde{r}_{At}}.$$  

Moreover, since $\tilde{C}_{A,t-1} = 0$,

$$\tilde{V}_{At-2}(1 + \tilde{r}_{A,t-1}) = \tilde{C}_{A,t-1} + \tilde{V}_{A,t-1} = \tilde{V}_{A,t-1} = \frac{\tilde{C}_{At}}{1 + \tilde{r}_{At}},$$

which implies

$$\tilde{V}_{At-2} = \frac{\tilde{C}_{At}}{(1 + \tilde{r}_{A,t-1})(1 + \tilde{r}_{At})}.$$ 

Continue for years $t - 2, t - 3, ..., $ until we reach year 1:

$$V_{A0} = \frac{\tilde{C}_{At}}{(1 + \tilde{r}_{A1})(1 + \tilde{r}_{A2}) \cdots (1 + \tilde{r}_{At})}.$$  

Now rearrange and take expectation at year 0:

$$V_{A0} E_0 [(1 + \tilde{r}_{A1})(1 + \tilde{r}_{A2}) \cdots (1 + \tilde{r}_{At})] = E_0 [\tilde{C}_{At}].$$

Defining $DF_{At}$ as in (2), and applying the notation $C_{At} = E_0 [\tilde{C}_{At}]$, gives (1). \hfill \square

Combining the Valuation Theorem and Prop. 2.2, it follows that the current market value of the cash flow stream $\tilde{C}_{A1}, ..., \tilde{C}_{AT}$ is given by

$$V_{A0} = \sum_{t=1}^{T} DF_{At} \times E_0 [\tilde{C}_{At}].$$
Assumption 5. Constant OCC. For any asset, the returns $\tilde{r}_{A1}, \tilde{r}_{A2}, \ldots, \tilde{r}_{AT}$ satisfy

\[ \tilde{r}_{At} = r_A + \tilde{\varepsilon}_{At}, \]

where $E_s[\tilde{\varepsilon}_{At}] = 0$ for all $s < t$.

Henceforth assume Ass. 1-5.

The proof of the following proposition makes use of two general properties of conditional expectation:

1. If $0 \leq s \leq t$, then $E_t[\tilde{Y}_s \tilde{X}_u] = \tilde{Y}_s E_t[\tilde{X}_u]$, since $\tilde{Y}_s$ is a known constant at year $t$.
2. Law of Iterated Expectations: $E_s[E_t[\tilde{X}_u]] = E_s[\tilde{X}_u]$.

Proposition 2.3. The current market value of $\tilde{C}_{At}$ is given by

\[ V_{A0} = \frac{C_{At}}{(1 + r_A)^t}, \]  

where $r_A$ is the OCC of $\tilde{C}_{At}$.

Proof. The proof is by induction. Rearrange (3) and take expectation at year $t - 1$:

\[ E_{t-1}[\tilde{V}_{A,t-1}(1 + \tilde{r}_{At})] = E_{t-1}[\tilde{C}_{At}]. \]  

$\tilde{V}_{A,t-1}$ is a known constant at year $t - 1$. Thus:

\[ E_{t-1}[\tilde{V}_{A,t-1}(1 + \tilde{r}_{At})] = \tilde{V}_{A,t-1}E_{t-1}[(1 + \tilde{r}_{At})]. \]

Now apply Ass. 5 and use the linearity of expected value:

\[ \tilde{V}_{A,t-1}E_{t-1}[(1 + \tilde{r}_{At})] = \tilde{V}_{A,t-1}E_{t-1}[(1 + r_A + \tilde{\varepsilon}_{At})] \]

\[ = \tilde{V}_{A,t-1}(1 + r_A + E_{t-1}[\tilde{\varepsilon}_{At}]) = \tilde{V}_{A,t-1}(1 + r_A). \]

Substitute into (6) and rearrange to obtain

\[ \tilde{V}_{A,t-1} = \frac{E_{t-1}[\tilde{C}_{At}]}{1 + r_A}. \]

Now suppose the market value at year $t - k$ satisfies

\[ \tilde{V}_{A,t-k} = \frac{E_{t-k}[\tilde{C}_{At}]}{(1 + r_A)^k}. \]
Since $\check{C}_{t-k-1} = 0$, the year $t - k$ return is determined by
$$
\check{V}_{A,t-k-1}(1 + \check{r}_{A,t-k}) = \check{V}_{A,t-k} = \frac{E_{t-k}[\check{C}_{At}]}{(1 + r_A)^k}.
$$
Take expectation of the preceding equation at year $t - k - 1$:
$$
E_{t-k-1}[\check{V}_{A,t-k-1}(1 + \check{r}_{A,t-k})] = \frac{E_{t-k-1}[E_{t-k}[\check{C}_{At}]]}{(1 + r_A)^k}.
$$

Apply the steps used above:
$$
E_{t-k-1}[\check{V}_{A,t-k-1}(1 + \check{r}_{A,t-k})] = \check{V}_{A,t-k-1}(1 + r_A + E_{t-k-1}[\check{r}_{A,t-k}]) = \check{V}_{A,t-k-1}(1 + r_A).
$$
Moreover, the Law of Iterated Expectations gives
$$
E_{t-k-1}[E_{t-k}[\check{C}_{At}]] = E_{t-k-1}[\check{C}_{At}].
$$
Substitute into (7) and rearrange to obtain
$$
\check{V}_{A,t-k-1} = \frac{E_{t-k-1}[\check{C}_{At}]}{(1 + r_A)^{k+1}}.
$$

Working backward to year $t - k - 1 = 0$, and applying the notation $C_{At} = E_0[\check{C}_{At}]$, gives (5).

Combining the Valuation Theorem and Prop. 2.3, and letting $T \to \infty$, it follows that the current market value of the infinite-horizon cash flow stream $\check{C}_{A1}, \check{C}_{A2}, \ldots$ is given by
$$
V_{A0} = \sum_{t=1}^{\infty} \frac{C_{At}}{(1 + r_A)^t},
$$
provided the limits exist. Call this the Valuation Formula.

Growing Annuities

A *Growing Annuity* is an asset whose expected cash flow stream satisfies
$$
C_{At} = \begin{cases} 
B(1 + g)^{t-1}, & t = 1, 2, \ldots, T \\
0, & t = T + 1, T + 2, \ldots.
\end{cases}
$$

Proposition 2.4. The current market value of a Growing Annuity is given by
$$
V_{A0} = \begin{cases} 
\frac{B}{r_A-g} \left[ 1 - \left( \frac{1+g}{1+r_A} \right)^T \right], & g \neq r_A \\
\frac{TB}{1+r_A}, & g = r_A.
\end{cases}
$$
Proof. Suppose \( g \neq r_A \). For \( T = 1 \) we have, using the Valuation Formula,

\[
V_{A0} = \frac{B}{1 + r_A} + \sum_{t=2}^{\infty} \frac{0}{(1 + r_A)^t}
\]

\[
= \frac{B}{1 + r_A} \times \frac{r_A - g}{r_A - g} = \frac{B}{r_A - g} \left[ \frac{r_A - g}{1 + r_A} \pm \frac{1}{1 + r_A} \right]
\]

\[
= \frac{B}{r_A - g} \left[ \frac{1 + r_A - 1 - g}{1 + r_A} \right] = \frac{B}{r_A - g} \left[ \frac{1 - 1 + g}{1 + r_A} \right].
\]

Now fix \( T > 1 \) and suppose the result holds for \( 1, 2, \ldots, T - 1 \). Again using the Valuation Formula, we have

\[
V_{A0} = \sum_{t=1}^{T} \frac{B(1 + g)^{t-1}}{(1 + r_A)^t} + \sum_{t=T+1}^{\infty} \frac{0}{(1 + r_A)^t}
\]

\[
= \sum_{t=1}^{T-1} \frac{B(1 + g)^{t-1}}{(1 + r_A)^t} + \frac{B(1 + g)^{T-1}}{(1 + r_A)^T}
\]

\[
= \frac{B}{r_A - g} \left[ 1 - \left( \frac{1 + g}{1 + r_A} \right)^{T-1} \right] + \frac{B(1 + g)^{T-1}}{(1 + r_A)^T} \times \frac{r_A - g}{r_A - g}
\]

\[
= \frac{B}{r_A - g} \left[ 1 - \frac{(1 + g)^{T-1}(1 + r_A) - (1 + g)^{T-1}(r_A - g)}{(1 + r_A)^T} \right]
\]

\[
= \frac{B}{r_A - g} \left[ 1 - \frac{(1 + g)^T}{(1 + r_A)^T} \right].
\]

Thus by induction the result is valid for all \( T \).

For \( g = r_A \):

\[
V_{A0} = \sum_{t=1}^{T} \frac{B(1 + r_A)^{t-1}}{(1 + r_A)^t} + \sum_{t=T+1}^{\infty} \frac{0}{(1 + r_A)^t}
\]

\[
= \sum_{t=1}^{T} \frac{B}{1 + r_A} = \frac{TB}{1 + r_A}.
\]

\[
\square
\]

Proposition 2.5. If \( g < r_A \), then the current market value of a Growing Annuity satisfies

\[
\lim_{T \to \infty} V_{A0} = \frac{B}{r_A - g}.
\]
Proof. Define the constant \( \alpha \) by

\[
\alpha = \frac{1 + g}{1 + r_A}.
\]

Then \(-1 < g < r_A \) implies \( 0 < \alpha < 1 \), so that \( \lim_{T \to \infty} \alpha^T = 0 \). Moreover, the Growing Annuity Formula may be written as

\[
V_{A0} = \frac{B}{r_A - g} \left[ 1 - \alpha^T \right].
\]

Since this is a continuous function of \( \alpha^T \), we have

\[
\lim_{T \to \infty} V_{A0} = \frac{B}{r_A - g} \left[ 1 - \lim_{T \to \infty} \alpha^T \right] = \frac{B}{r_A - g}.
\]

Alternate Proof. Using the Valuation Formula, we have

\[
V_{A0} = \sum_{t=1}^{T} \frac{B(1 + g)^{t-1}}{(1 + r_A)^t} = \frac{B}{(1 + r_A)} \sum_{t=1}^{T} \frac{(1 + g)^{t-1}}{(1 + r_A)^{t-1}}
\]

\[
= \frac{B}{(1 + r_A)} \sum_{t=1}^{T} \alpha^{t-1} = \frac{B}{(1 + r_A)} \sum_{u=0}^{T-1} \alpha^u,
\]

where the substitution \( u = t - 1 \) has been made. Since \( 0 < \alpha < 1 \), the Geometric Series Theorem yields

\[
\lim_{T \to \infty} \sum_{u=0}^{T-1} \alpha^u = \frac{1}{1 - \alpha}.
\]

Thus,

\[
\lim_{T \to \infty} V_{A0} = \frac{B}{(1 + r_A)} \times \frac{1}{1 - \alpha} = \frac{B}{(1 + r_A)} \times \frac{1}{1 - \frac{1+g}{1+r_A}} = \frac{B}{r_A - g}.
\]

Future market values

Corollary 2.2. The market value at year \( t \) of the stream \( \bar{C}_{A,t+1}, \bar{C}_{A,t+2}, \ldots \) satisfies

\[
V_{At} = \sum_{u=t+1}^{\infty} \frac{C_{Au}}{(1 + r_A)^{u-t}},
\]

provided the limits exist.
Proof. The proof of Prop. 2.3 showed that the value of \( C_{Au} \) at year \( t < u \) is

\[
\tilde{V}_{At} = \frac{E_t[C_{Au}]}{(1 + r_A)^{u-t}}.
\]

Thus, the value of the stream \( C_{A,t+1}, C_{A,t+2}, \ldots, C_{AT} \) at year \( t \) is

\[
\tilde{V}_{At} = \sum_{u=t+1}^{T} \frac{E_t[C_{Au}]}{(1 + r_A)^{u-t}}. \tag{9}
\]

Next, take expectation of (9) at year 0,

\[
E_0[\tilde{V}_{At}] = E_0 \left[ \sum_{u=t+1}^{T} \frac{E_t[C_{Au}]}{(1 + r_A)^{u-t}} \right].
\]

Linearity of expected value and the Law of Iterated Expectations give

\[
E_0 \left[ \sum_{u=t+1}^{T} \frac{E_t[C_{Au}]}{(1 + r_A)^{u-t}} \right] = \sum_{u=t+1}^{T} \frac{E_0[E_t[C_{Au}]]}{(1 + r_A)^{u-t}}
\]

\[
= \sum_{u=t+1}^{T} \frac{E_0[C_{Au}]}{(1 + r_A)^{u-t}}.
\]

Finally, (8) is obtained by applying the notation \( V_{At} = E_0[\tilde{V}_{At}], C_{Au} = E_0[C_{Au}] \) and taking the limit as \( T \to \infty \).

\[\Box\]

Corollary 2.3. The current market value of \( \tilde{V}_{At} \) is given by

\[
\frac{V_{At}}{(1 + r_A)^t}.
\]

Moreover, the Valuation Formula converges if and only if

\[
\lim_{t \to \infty} \frac{V_{At}}{(1 + r_A)^t} = 0. \tag{10}
\]

Proof. Consider an asset such that \( C_{Au} = 0 \) at each year \( u \leq t \). The Valuation Formula and Cor. 2.2 give

\[
V_{A0} = \sum_{u=t+1}^{\infty} \frac{C_{Au}}{(1 + r_A)^u}.
\]

\[
= \frac{1}{(1 + r_A)^t} \sum_{u=t+1}^{\infty} \frac{C_{Au}}{(1 + r_A)^{u-t}} = \frac{V_{At}}{(1 + r_A)^t}.
\]
Next, the Valuation Formula converges if and only if
\[
\lim_{t \to \infty} \left( \sum_{u=t+1}^{\infty} \frac{C_{Au}}{(1 + r_A)^u} \right) = 0.
\]
In view of (11), this is equivalent to (10). \qed

**Internal Rate of Return (IRR)**

**Proposition 2.6.** Suppose \( C_{B1}, C_{B2}, \ldots \) and \( I_0 \) satisfy IRR Regularity Conditions 1 and 2. Then the IRR exists uniquely, and the IRR Rule is equivalent to the NPV Rule.

**Proof.** Define the function \( V_B(r) \) by
\[
V_B(r) = \sum_{t=1}^{\infty} \frac{C_{Bt}}{(1 + r)^t} - I_0, \quad r > -1.
\]
It follows that \( r_I \) satisfies the IRR Equation if and only if \( V_B(r_I) = 0 \), i.e., \( V_B \) intersects the \( r \)-axis at \( r_I \).

Using Condition 1, we have
\[
V_B(0) = \sum_{t=1}^{\infty} C_{Bt} - I_0 > 0,
\]
\[
\lim_{r \to -\infty} V_B(r) = -I_0 < 0.
\]
Thus \( V_B(r) \) must cross the \( r \)-axis at least once for \( r > 0 \). Since \( V_B(r) \) is a continuous function, it must intersect the \( r \)-axis, which means \( V_B(r) = 0 \) for some \( r > 0 \). Conclude that an IRR exists. Moreover, Condition 2 implies that \( V_B(r) \) is strictly decreasing in \( r \), so it cannot intersect the \( r \)-axis at more than one point. Conclude that the IRR is unique.

Next, using Condition 2, we have that \( r_I > r_B \) holds if and only if \( 0 = V_B(r_I) < V_B(r_B) = \text{NPV} \), whereas \( r_I < r_B \) holds if and only if \( 0 = V_B(r_I) > V_B(r_B) = \text{NPV} \). This proves that the IRR and NPV Rules are equivalent. \qed

**Debt valuation**

**Lemma 4.1.** For any \( l \) and \( t \), the market value at year \( t \) of type \( l \) bonds issued at years \( t+1, t+2, \ldots \) is zero.
Proof. Let \( \tilde{F}_l(t + v, T) \) be the face value of type \( l \) bonds of maturity \( T \) issued at year \( t + v > t \), and let \( \tilde{D}_l(t + v, T) \) be the market value of these bonds at year \( t + v \). Net payments to holders of these bonds at years \( u \geq t + v \) are as follows:

\[
\tilde{C}_{D_l}(t+v, T) = \begin{cases} 
-\tilde{D}_l(t + v, T), & u = t + v, \\
r^l_i \tilde{F}_l(t + v, T), & u = t + v + 1, \ldots, t + v + T - 1, \\
r^l_i \tilde{F}_l(t + v, T) + \tilde{F}_l(t + v), & u = t + v + T, \\
0, & u = t + v + T + 1, t + v + T + 2, \ldots .
\end{cases}
\]

From equation (9) in the proof of Corollary 2.2, it follows that the market value of the bonds at year \( t + v \) is

\[
\tilde{D}_l(t + v, T) = \sum_{u=t+v+1}^{\infty} \frac{E_{t+v} \left[ \tilde{C}_{D_l}(t+v, T) \right]}{(1 + r^l_i)^{u-(t+v)}}.
\]

Thus, the market value of the bonds at year \( t \) is

\[
\tilde{D}_l(t + v, T) = -\frac{E_{t+v} \left[ \tilde{D}_l(t + v, T) \right]}{(1 + r^l_i)^v} + \sum_{u=t+v+1}^{\infty} \frac{E_{t+v} \left[ \tilde{C}_{D_l}(t+v, T) \right]}{(1 + r^l_i)^{u-t}}
\]

\[
= \frac{1}{(1 + r^l_i)^v} \cdot E_t \left[ -\tilde{D}_l(t + v, T) + \sum_{u=t+v+1}^{\infty} \frac{E_{t+v} \left[ \tilde{C}_{D_l}(t+v, T) \right]}{(1 + r^l_i)^{u-(t+v)}} \right]
\]

\[
= \frac{1}{(1 + r^l_i)^v} \cdot E_t[0] = 0.
\]

It follows that the market value at year \( t \) of type \( l \) bonds issued in all future years at all maturities is given by

\[
\sum_{v=1}^{\infty} \sum_{T=1}^{\infty} \tilde{D}_l(t + v, T) = 0.
\]

\[\square\]

Proposition 4.1. If type \( l \) debt satisfies \( r^l_i = r^l_D \), then \( \tilde{D}_l = \tilde{F}_l^l \) for all \( t \).

Proof. Fix \( T > 0 \), and let \( \tilde{F}_l^l(T) \) denote the face value of type \( l \) bonds outstanding at year \( t \) that mature at year \( t + T \). Net payments to holders of these bonds at years \( u > t \) are as follows:

\[
\tilde{C}_{D_l}(T) = \begin{cases} 
{r^l_i} \tilde{F}_l^l(T), & u = t + 1, \ldots, t + T - 1, \\
r^l_i \tilde{F}_l^l(T) + \tilde{F}_l^l(T), & u = t + T, \\
0, & u = t + T + 1, t + T + 2, \ldots .
\end{cases}
\]
Equation (9) in the proof of Corollary 2.2 implies that the market value of the bonds at year \( t \) is

\[
\dot{D}_t(T) = \sum_{u=t+1}^{\infty} \frac{E_t \left[ \tilde{C}_{Du}^l(T) \right]}{(1 + r_D^l)^{u-t}} = \sum_{u=t+1}^{t+T} \frac{r_C^l \tilde{F}_t^l(T)}{(1 + r_D^l)^{u-t}} + \frac{\tilde{F}_t^l(T)}{(1 + r_D^l)^T}.
\]

Since \( r_C^l = r_D^l \), we have

\[
\dot{D}_t(T) = \tilde{F}_t^l(T) \left[ 1 - \frac{1}{(1 + r_D^l)^T} \right] + \frac{\tilde{F}_t^l(T)}{(1 + r_D^l)^T} = \tilde{F}_t^l(T).
\]

Moreover, Lemma 4.1 shows that the market value at year \( t \) of type \( l \) bonds of all maturities issued at years \( t + 1, t + 2, \ldots \) is zero. Thus,

\[
\tilde{D}_t^l = \sum_{T=t}^{\infty} \dot{D}_t^l(T) = \sum_{T=t}^{\infty} \tilde{F}_t^l(T) = \tilde{F}_t^l.
\]

\[ \square \]

**Proposition 4.2.** If type \( l \) debt satisfies \( r_C^l = r_D^l \) and \( \tilde{F}_t^l = F_0^l(1 + g^l)^t \) for all \( t > 0 \), then the value of ITS is given by

\[
V_{0,IS}^l = \frac{r_D^l D_0^l \tau}{r_D^l - g^l}.
\]

**Proof.** In view of Prop. 4.1, \( r_C^l = r_D^l \) implies \( \tilde{F}_t^l = F_0^l \). Thus, we have \( \tilde{F}_t^l = D_0^l(1 + g^l)^t \), and the Growing Perpetuity Formula yields

\[
V_{0,IS}^l = \sum_{t=1}^{\infty} \frac{r_C^l \tilde{F}_t^l}{(1 + r_D^l)^t} = \sum_{t=1}^{\infty} \frac{r_D^l D_0^l (1 + g^l)^{t-1} \tau}{(1 + r_D^l)^t} = \frac{r_D^l D_0^l \tau}{r_D^l - g^l}.
\]

\[ \square \]

**Proposition 4.3.** Suppose for every \( l \), \( r_C^l = r_D^l \) and \( \tilde{F}_t^l = F_0^l \) for all \( t > 0 \). Then the value and OCC of ITS are given by

\[
V_{IS,0} = V_{D,0} \tau, \quad r_{IS} = r_D.
\]

**Proof.** Under the assumptions, Prop. 4.2 implies, for all \( l \),

\[
V_{0,IS}^l = \frac{r_D^l D_0^l \tau}{r_D^l - 0} = D_0^l \tau.
\]
Thus,
\[ V_{ITS,0} = \sum_{l=1}^{L} V_{ITS,0}^l = \sum_{l=1}^{L} D_0^l \tau = \left( \sum_{l=1}^{L} D_0^l \right) \tau = D_0 \tau, \]
\[ r_{ITS} = \sum_{l=1}^{L} \frac{V_{ITS,0}^l}{V_{ITS,0}} r_D^l = \sum_{l=1}^{L} \frac{D_0^l \tau}{D_0 \tau} r_D^l = \sum_{l=1}^{L} \frac{D_0^l}{D_0} r_D^l = r_D. \]

\[ \square \]

**Corollary 4.1.** Under the conditions of Proposition 4.3, expected net payouts to debtholders for all \( t > 0 \) are given by
\[ C_{Dt} = Int_t = r_D D_0. \]

**Proof.** Let \( \hat{F}_{Mt}^l \) denote the face value of type \( l \) debt that matures at year \( t \), and let \( \hat{F}_{Nt}^l \) and \( \hat{D}_{Nt}^l \) denote the face value and market value, respectively, of newly-issued type \( l \) debt at year \( t \). The face value of type \( l \) debt at year \( t + 1 \) is determined by
\[ \hat{F}_{t+1}^l = \hat{F}_{t}^l - \hat{F}_{Mt}^l + \hat{D}_{Nt}^l. \]

We have \( \hat{F}_{t+1}^l = \hat{F}_{t}^l = F_0^l \) by hypothesis, so the preceding equation implies \( \hat{F}_{Mt}^l = \hat{D}_{Nt}^l \).

Net payouts to type \( l \) debtholders at year \( t \) are given by
\[ \tilde{C}_{Dt}^l = \tilde{I}nt_t^l + \hat{F}_{Mt}^l - \hat{D}_{Nt}^l = r_D^l \hat{F}_{t}^l - \hat{F}_{Mt}^l - \hat{D}_{Nt}^l. \]

Since \( r_C^l = r_D^l \), applying the argument of the proof of Prop. 4.1 to newly-issued debt gives \( F_{Nt}^l = \hat{D}_{Nt}^l \). Substituting for \( \hat{D}_{Nt}^l \) and taking current expectation gives
\[ C_{Dt}^l = r_C^l F_{t}^l + F_{Mt}^l - F_{Nt}^l = r_C^l F_{t}^l = r_C^l F_0^l = r_C^l D_0. \]

Expected net payouts to all debtholders are obtained as follows:
\[ C_{Dt} = \sum_{l=1}^{L} C_{Dt}^l = \sum_{l=1}^{L} r_D^l D_0 = \sum_{l=1}^{L} \frac{r_D^l}{D_0} D_0 \times D_0 = r_D D_0. \]

\[ \square \]

**Equity valuation**

**Proposition 4.4.** The Value of Equity equals the value of the net payout stream:
\[ E_0 = E_0^{NP}. \]
Proof. Since each share has a cash value of $\widetilde{Div}_1 + \widetilde{P}_1$ at year 1, the portfolio of all currently outstanding shares has a cash value of $(\widetilde{Div}_1 + \widetilde{P}_1)K_0$ at year 1. Moreover, the definition of $\check{C}_{E1}$ may be rearranged to obtain

$$(\widetilde{Div}_1 + \widetilde{P}_1)K_0 = \check{C}_{E1} + \check{P}_1\check{K}_1.$$  

Thus the cash value of the portfolio of shares at year 1 equals $\check{C}_{E1} + \check{P}_1\check{K}_1$. It follows that the Value of Equity must satisfy

$$E_0 = \frac{C_{E1}}{1 + r_{NP}^E} + \frac{P_1K_1}{1 + r_E^1},$$  

(12)

where $r_{NP}^E$ is the OCC of the stream $\check{C}_{E1}, \check{C}_{E2}, \ldots$, and $r_E^1$ is the OCC of $\check{P}_1\check{K}_1$.

Next, since each share has a cash value of $\widetilde{Div}_2 + \widetilde{P}_2$ at year 2, the portfolio of all shares outstanding at year 1 has a cash value of $(\widetilde{Div}_2 + \widetilde{P}_2)K_1$ at year 2. Moreover, the definition of $\check{C}_{E2}$ may be rearranged to obtain

$$(\widetilde{Div}_2 + \widetilde{P}_2)K_1 = \check{C}_{E2} + \check{P}_2\check{K}_2.$$  

Since $\check{P}_1\check{K}_1$ is the market value of the portfolio at year 1, it follows that the current market value of $\check{P}_1\check{K}_1$ must satisfy

$$\frac{P_1K_1}{1 + r_E^1} = \frac{C_{E2}}{(1 + r_{NP}^E)^2} + \frac{P_2K_2}{(1 + r_E^2)^2},$$  

where $r_E^2$ is the OCC of $\check{P}_2\check{K}_2$. Substituting into equation (12) gives

$$E_0 = \frac{C_{E1}}{1 + r_{NP}^E} + \frac{C_{E2}}{(1 + r_{NP}^E)^2} + \frac{P_2K_2}{(1 + r_E^2)^2}.$$  

Proceeding in this way for years $t = 2, 3, \ldots$, we obtain

$$E_0 = \sum_{t=1}^{\infty} \frac{C_{E1}}{(1 + r_{NP}^E)^t} = E_0^{NP}.$$  

\[\square\]

Proposition 4.5. The Return on Equity equals the OCC of the net payout stream:

$$r_E = r_{NP}^E.$$  

Proof. From the definition of $\check{C}_{E1}$ we have

$$(\widetilde{Div}_1 + \widetilde{P}_1)K_0 = \check{C}_{E1} + \check{E}_1.$$  

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Moreover, Prop. 4.4 gives $E_0 = E_0^{NP}$, and the argument of the proof may be used to obtain $E_1 = E_1^{NP}$. Thus,
\[
\tilde{r}_E = \frac{\ddot{D}iv_1 + \dot{P}_1 - P_0}{P_0} = \frac{(\ddot{D}iv_1 + \dot{P}_1)K_0 - P_0K_0}{P_0K_0}
\]
\[
= \frac{\ddot{C}_E + \ddot{E}_1 - E_0}{E_0} = \frac{\ddot{C}_E + \ddot{E}_1^{NP} - E_0^{NP}}{E_0^{NP}} = \tilde{r}_E^{NP}.
\]
Taking current expectation gives the result.
\[\square\]