Practice Problems for 210A

1. Consider a household that maximizes lifetime utility:

\[ V = \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} U(C_t, L_t) \]

subject to the sequence of budget constraints:

\[ A_{t+1} = (1+r) A_t + W_t (1 - L_t) - C_t, \quad A_0 \text{ given, and} \]

where \( C_t \) is consumption, \( L_t \) is leisure, the time endowment is normalized to 1, \( A_t \) is assets at the beginning of period \( t \), and \( W_t \) is the real wage.

A. What are the states and the controls?

B. Find the first order conditions using the direct substitution method discussed in class. Do it two ways: first by substituting out for \( C_t \), second by substituting out for \( L_t \) rather than \( C_t \).

Manipulate the first-order conditions from both cases to show that they are equivalent.

2. (From Ramey and Vine (2006) AER Special Case 2) Suppose a factory has two margins with which to adjust production, the number of workers \( N \) and the hours per worker \( h \). Output is given by \( Y = h \cdot N \). The factory takes output as given and must decide whether to vary the number of workers or the hours per worker. The cost function is given by:

\[
C = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2} \gamma_2 ((h_t - 1)N_t)^2 + \frac{1}{2} \gamma_4 (N_t - N_{t-1})^2 \right\}
\]

The first term captures the convex cost of allowing hours to deviate from a standard schedule, where the standard schedule is normalized to unity. This captures costs of using overtime or using short-weeks. The second term is a cost of changing the number of workers. I have left out terms that will appear as constants in the Euler equations. The factory seeks to choose a sequence of \( h \)'s and \( N \)'s to minimize costs, given the \( Y \)'s.

A. Substitute \( h \) out using the relation between \( Y \), \( h \), and \( N \). Find the stochastic Euler equation in the \( N \)'s and \( Y \)'s.

B. Assume that \( Y_t = \rho \cdot Y_{t-1} + \varepsilon_t \), where \( 0 < \rho < 1 \) and \( \varepsilon \) is white noise. Find the optimal path of \( N_t \) as a function of \( N_{t-1} \) and \( Y_t \) that satisfies the transversality condition. Use the quadratic formula to find expressions for the coefficients as a function of the underlying parameter values.

C. How does the factory’s use of \( N_t \) and \( h_t \) to respond to a given shock to \( Y_t \) depend on the value of \( \rho \)? Give economic intuition.
3. Consider a representative agent who derives utility from nondurable consumption and the service flow from the stock of durable consumer goods. The consumer maximizes utility:

\[
V = E_0 \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t [U(c_t) + \psi(K_t)],
\]

subject to

\[
A_{t+1} = (1 + r) A_t + y_t - c_t - p_d d_t,
\]

\[
K_{t+1} = (1 - \delta) K_t + d_t
\]

where

- \(c_t\) = consumption in period \(t\).
- \(K_t\) = stock of durable goods, beginning of period \(t\).
- \(A_t\) = financial assets at the beginning of period \(t\).
- \(r\) = interest rate, in terms of nondurable goods - equal to the discount rate.
- \(y_t\) = income, assumed exogenous.
- \(d_t\) = purchases of durables.
- \(p_d\) = price of durable goods.
- \(\delta\) = depreciation rate of durable goods.

A. Derive the first-order conditions for the consumer maximization problem.

B. Assume that the consumer has perfect foresight about the path of \(p_d\). Let \(1 + \pi_t = \frac{p_{d,t+1}}{p_{d,t}}\). Use the first-order conditions to derive an expression that relates the ratio of the expected marginal utility of durable goods consumption to nondurable consumption to a rental cost of capital. (Note: use the approximation that \(\pi \cdot \delta \approx 0\).)

C. Suppose that the durable good in question is a house. How does the anticipated fall in future real estate prices affect the consumer’s optimal stock of housing relative to nondurable consumption?

D. Now suppose that the instantaneous utility function is given by:

\[
U = -\frac{1}{2} (\theta (c_t - \bar{c})^2 - \frac{1}{2} (K_t - \bar{K})^2)
\]

Assume that the values of the bliss points are such that the second order conditions hold. Also, assume that the relative price of durable goods \(p_d\) is constant at \(p_{\bar{d}}\).

Hall (1978) showed that with this type of utility function, nondurable consumption follows a random walk. Will the stock of durable goods \((K_t)\) also follow a random walk? Demonstrate.
4. Consider a firm that minimizes expected costs:

$$C = E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{1}{2} \gamma_2 Y_t^2 + \frac{1}{2} \alpha_1 (H_{t-1} - \alpha_2 S_t)^2 \right]$$

$$H_t = H_{t-1} + Y_t - S_t$$

where $Y_t$ is production during period $t$, $H_t$ is the stock of inventories at the end of period $t$, and $S_t$ is sales during period $t$. Assume that sales follow a white noise process, i.e., $S_t = \varepsilon_t$ where $\varepsilon_t$ is white noise and has mean 0. Note that we have left out terms that will appear as constants in the solution for simplicity, so we are normalizing the mean of sales to 0.

A. What is the optimal rule for inventories as a function of lagged inventories and the sales process? Assume both roots are real, and that $\lambda_1$ is the stable root. Interpret.

B. How does the response of inventories to a shock to sales depend on the ratio of $\alpha_1$ to $\gamma_2$? Give economic intuition.
5. Consider the following model of an economy with a constant population of 1 and no technological progress. **Representative households** choose $c_t$, $h_t$, and $a_{t+1}$ to maximize the present discounted value of utility:

$$U = \sum_{t=0}^{\infty} \beta^t [\ln(c_t) + \phi \ln(1 - h_t)] \quad \text{subject to} \quad a_{t+1} = (1 + r_t)a_t + (1 - \tau)w_t h_t - c_t - T_t$$

where $c$ is consumption, $h$ is hours, $a$ is financial assets of the household, $r$ is the return on financial assets, $\tau$ is the tax on labor income, $w$ is the wage, and $T$ is lump-sum taxes.

The **competitive representative firm** chooses $y$, $k$, and $h$ to maximize profits, given by:

$$\text{Profits}_t = y_t - (r_t + \delta)k_t - w_t h_t \quad \text{subject to the production function} \quad y_t = h_t^\alpha k_t^{1-\alpha}$$

The **government** purchases goods ($g_t$) and finances it with taxes on labor income and lump-sum taxes. Its budget constraint is: $g_t = \tau_t w_t h_t + T_t$

The additional constraints are:

$$k_{t+1} = (1 - \delta)k_t + i_t \quad \text{Capital Accumulation, } 0 < \delta < 1$$

$$c_t + i_t + g_t = y_t \quad \text{Resource Constraint}$$

$$a_t = k_t \quad \text{Household assets equal the capital stock}$$

A. In many problems, the social planner solution is identical to the decentralized competitive equilibrium. Is this one of them? Explain.

B. Find the first-order conditions for the representative firm.

C. Find the first-order conditions for the representative household.

**The rest of the questions refer to the steady-state competitive equilibrium.**

D. Show how the distortionary labor income tax $\tau$ affects the capital-labor ratio ($k/h$), for a given $g_t$.

E. Show how the distortionary labor income tax $\tau$ affects steady-state hours worked, for a given government spending-output ratio (i.e. constant $g/y$).

(Hints: Write $h/(1-h)$ as a function of $c/y$ by judicious substitution of the production function in key places. Then figure out $i/y$ as a function of $k/h$ (again using judicious substitution of the production function) to get $c/y$ from the resource constraint.)