Limiting LQRE as a Model of Limiting Outcomes in Van Huyck, Battalio, and Beil's Coordination Games (web appendix to "Comparing Models of Strategic Thinking in Van Huyck, Battalio, and Beil's Coordination Games")

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Revised 5 September 2008

Recall that McKelvey and Palfrey (1995), GH, and others have suggested using LQRE with precisions declining over time as a reduced-form model of learning; and by extension the limiting LQRE, as precision approaches infinity, as a model of limiting learning outcomes. Because VHBB's games were played repeatedly with feedback, and subjects converged quickly to equilibrium, with selection varying sharply across treatments, their data also provide a good opportunity to test the latter claim.

Anderson et al. (2001; "AGH") take this approach in studying an approximation of VHBB's (1990) minimum games with continuous strategy spaces but all else unchanged. In a rare example of analytical rather than computational results for LQRE, they show that the limiting LQRE is the analog (extrapolating back to VHBB's seven-effort strategy spaces) of all-1 in the game used in treatment A, of all-7 in the game used in treatment B, and of all-4 in the game used in treatment C_d. (In each case the limiting LQRE happens to coincide with the risk-dominant equilibrium by Harsanyi and Selten's 1988 definition; see Crawford 1991, p. 56, fn. 27.) AGH then note that in each case the limiting LQRE is virtually the same as subjects' median limiting decision in VHBB's minimum treatments, 1 in treatment A, 7 in B, and 5 in C_d. (Almost all subjects made the median choice by the fifth period of play in A and B; but considerable dispersion remained after 3 or 5 periods in the two C_d sessions, and the *modal* choice was 7 in each of the C_d sessions.) AGH argue that limiting LQRE's excellent fit for these three treatments is strong evidence in favor of using it as a model of limiting learning outcomes.

In this appendix we analyze three caveats to this conclusion. First, AGH's analysis is limited to VHBB's 1990 minimum games, and they mention VHBB's equally relevant 1991 median games only in passing (p. 191, fn. 15). Second, AGH's results for the minimum games are limited to the continuously approximated strategy spaces, and Yi (1999, 2003) shows that this approximation can matter for some such games, making the limiting LQRE different for the discrete versions that subjects actually played. Third, AGH's results are for the standard case where players think of their partners' decisions as independent. Yet we have seen in our paper's analysis of initial responses that a correlated version of LQRE may be more descriptive of initial responses (for example, fitting better in two of the three minimum treatments and no worse in the third); and learning from repeated observation of the order statistic (subjects observed the entire effort profile only in some runs of treatment A) cannot be expected to "decorrelate" subjects' beliefs. Thus a priori the limiting *correlated* LQRE may be a more natural notion to compare with VHBB's subjects' limiting decisions that the limiting *independent* LQRE. We now take up each of these caveats in turn.

Maintaining independence and AGH's continuous approximations to the discrete strategy spaces, Yi (2003) shows that the limiting LQRE is all-7 in VHBB's (1991) treatment Γ, but Yi (1999) shows computationally that the limiting LQRE is all-4 in VHBB's treatment Ω in the discrete case, and it is unlikely to be different in the continuous case. For, Ω's payoff structure makes small deviations very costly even in the continuous case, and as Yi's (2003) analysis shows, the fact that deviations are not costly in the continuous version of treatment Γ is what drives the LQRE up to all-7 there. Thus, the independent, continuous limiting LQRE is either two or three full effort levels (out of a possible six) away from the median and modal limiting decision of 4 or 5 in treatment Γ (where three of six subject groups converged to all-4 and the other three converged almost perfectly to all-5). The independent, continuous limiting LQRE is also likely to be one or three effort levels away from the median and modal limiting decision of 7 in treatment Ω (though one of three subject groups converged to all-5 instead of all-7).

Remarkably, Yi (2003) also showed that with continuously approximated strategy spaces, if the linear deviation costs of VHBB's minimum experiments and AGH's analysis are replaced by the quadratic deviation costs of VHBB's median treatment Γ, then the limiting LQRE is the analog of the efficient all-7 equilibrium for *any* order statistic, even including the minimum. The linearly increasing payoff advantage of higher-effort equilibria outweighs the locally negligible quadratic deviation cost, pushing the limiting LQRE all the way to all-7, at the upper limit of the game's strategy spaces. Yet it seems highly unlikely that quadratic deviation costs would make even moderately large subject groups in VHBB's minimum games converge to all-7. Van Huyck et al. (2007) report the results of some experiments with nearly continuous strategy spaces (100 efforts), quadratic deviation costs, and order statistics as low as 2 (where 1 is the minimum) in five- and seven-person groups. Although these subject groups, particularly the five-person groups, move significantly upward in 21 periods of play (Van Huyck et al.'s Table 2), they stop well short of the analog of all-7.

Maintaining independence but going to the discrete games that subjects actually played makes no difference in the minimum treatments, and is again unlikely to make a difference in median treatment Ω; but it changes the limiting LQRE from all-7 to all-4 in median treatment Γ (Yi 1999). Thus it improves the fit in treatment Γ and leaves it unchanged in the other treatments.

Going to correlated versions in the discrete games subjects actually played changes the limiting LQRE in treatment A from all-1 to all-4 (because it makes treatment A equivalent to C_d), significantly degrading the fit; and in treatment Ω from all-4 to all-7, significantly improving the fit; and leaves the limiting LQRE unchanged in the other treatments. Going to correlated versions but maintaining AGH's continuous approximations has the same effect in treatment A; is likely to have the same effect in treatment Ω; and again leaves the limiting LQRE unchanged in the other three treatments.

Finally, it is evident from VHBB's data that there is a great deal of history-dependence in the learning dynamics. Crawford (1995) confirmed via explicit, detailed analysis of the dynamics that this history-dependence extends even to treatment A, where it was not immediately evident from the data only because of the very strong, robust convergence to all-1, at the lower limit of the game's strategy spaces, in that treatment. Given this and the persistence of initial differences in beliefs and decisions, it is implausible that a static concept such as limiting LQRE could reliably describe limiting learning outcomes in all settings. And as we have seen, when VHBB's median treatments are considered as well as their minimum treatments, the limiting LQRE does not fit the data well.

References for Appendix

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