Consider an individual who chooses a control variable \( x \) to solve
\[
\max_x \int u(x, x) \, dE(x, \lambda) = \int u(x, x) \, f(x, \lambda) \, dx,
\]
where \( f(\cdot) \) has a density shifted along the parameter \( \lambda \).

For example, \( x \) might be the return on a risky asset (investment in training for a job) and \( \lambda \) the amount invested in it (with the rest invested in a safe asset), and \( u(\cdot) \) a kind of indirect utility (wealth under the riskiness of the assets' return).

Assume the support of \( x \in [0, m] \) as before, so that \( F(0; \lambda) \equiv 0 \) and \( F(m; \lambda) \equiv 1 \). The first-order condition is
\[
\int u_x(x, x) \, dE(x, \lambda) = \int u_x(x, x) \, f(x, \lambda) \, dx = 0.
\]
Assume \( u_{xx}(x, x) < 0 \), so the second-order condition
\[
\int u_{xx}(x, x) \, f(x, \lambda) \, dx < 0
\]
is always satisfied strictly.

By the Implicit Function Theorem, letting \( x(\lambda) \) be the optimal \( x \),

\[
\frac{dx(\lambda)}{d\lambda} = -\frac{\int u_{xx}(x, x) \, f_x(x, \lambda) \, dx}{\int u_{xx}(x, x) \, f(x, \lambda) \, dx},
\]
so that

\[
\text{Sign} \left( \frac{d\lambda}{d\lambda} \right) = \text{Sign} \left( \int u_{xx}(x, x) \, f_x(x, \lambda) \, dx \right).
\]

Now a small increase in \( \lambda \) induces an PO SD increase in \( F(\cdot) \) iff \( F_{\lambda}(x, \lambda) \leq 0 \) for all \( x \), \( \lambda \) (from the "finite" change condition \( \Delta 2 \) and the Mean Value Theorem), thus \( \frac{dx(\lambda)}{d\lambda} < 0 \) for all such changes.

\[
\text{Sign} \left( \frac{d\lambda}{d\lambda} \right) = \text{Sign} \left( \int u_{xx}(x, x) \, f_x(x, \lambda) \, dx \right) \text{ since } F_{\lambda}(m, \lambda) = F_{\lambda}(0, \lambda) = 0.
\]

Thus the sign of \( \frac{dx(\lambda)}{d\lambda} \) is the same as that of \( \frac{d\lambda}{d\lambda} \).
Integrating by parts once again we obtain

\[- \int_0^m u_{\bar{x}x}(x, \bar{x}) \bar{x}_x(x, \bar{x}) \, dx = \int_0^m \left[ u_{\bar{x}x}(x, \bar{x}) \right]_{\bar{x}=0}^{\bar{x}=m} \, d\bar{x} \]

\[+ \int_0^m \left[ \int_0^m \bar{x}_x(\bar{x}, \bar{x}) \, d\bar{x} \right] u_{\bar{x}xx}(x, \bar{x}) \, dx \]

= \int_0^m \left[ \int_0^m \bar{x}_x(\bar{x}, \bar{x}) \, d\bar{x} \right] u_{\bar{x}xx}(x, \bar{x}) \, dx \quad \text{because} \quad \int_0^m \bar{x}_x(\bar{x}, \bar{x}) \, d\bar{x} = 0 \]

by the mean-preserving part of (C'). Now \( \int_0^m \bar{x}_x(\bar{x}, \bar{x}) \, d\bar{x} \geq 0 \), \( \forall \bar{x} \),

introducing mean-preserving spreads in \( F(\cdot) \), for all \( u(\cdot) \) iff \( u_{\bar{x}xx}(\cdot) \geq 0 \), \( \forall \bar{x}, \bar{x} \). (Same for \( \bar{x}_x(\cdot) \) by "duality".)

(Notice the use of the Mean Value theorem again to translate finite-change condition (C') into a calculus condition. Note that we used same expression to sign effects of changes in \( \bar{x} \) that do different things: FOSD and MPS. This is weird but OK, because the formula is an identity and we never use both interpretations at the same time.)

Intuition: If \( u_{\bar{x}xx}(\cdot) \geq 0 \), \( \forall \bar{x}, \bar{x}, \) raising \( \bar{x} \) raises \( u_{\bar{x}x}(\cdot) \), making the person less risk-averse (yes, less!).

Therefore, the optimal response to a change in \( \bar{x} \) that increases the richness of \( F(\cdot) \), the distribution of \( \bar{x} \),

is to move the control variable \( \bar{x} \), in the direction that makes you less risk-averse. (Note that most of our intuition about the effects of changes in \( F(\cdot) \) are first-order, as captured in the first result (in FOSD) above; the mean-preserving spread eliminates these effects by holding the mean of \( F(\cdot) \) constant, and is entirely second-order, hence more subtle.)

The two effects can be combined using the formula above.