Notes on Syllabus Section VI: TIME AND UNCERTAINTY, FUTURES MARKETS

Overview: The mathematical abstraction of the Arrow-Debreu general equilibrium model is subject to interpretation. To treat allocation over time we conceive of the commodity space as encompassing futures markets. A commodity is specified by what it is, location, and availability date. The market meets at a single date and arranges all allocation from the present to a finite time horizon. Futures markets perform the savings, investment, borrowing, and lending functions. This specification eliminates the distinction between income and wealth. Each household enters the market with a dated portfolio of endowment and firm ownership and leaves with a dated portfolio of consumption plans. Pareto efficiency is efficient allocation over time.

The next step is to treat uncertainty and allow the markets to fulfill an insurance function. There is assumed to be a complete list of possible future events, common to all firms and households and reflected in the market. Each commodity is defined by what it is, location, availability date, and state of the world deliverable. Commodity prices reflect the market's judgment on likelihood of realization of the uncertain event. There is no default. A firm's technology includes specification of inputs and outputs subject to realization of uncertain events. Profit maximization is well defined: all firm plans, contingent on the outcomes of uncertainty, are arranged in the market prior to the realization of uncertain events. The firm chooses a production plan evaluated on the futures market to maximize firm value on the current market, with no probability judgment or attitude toward risk. Each household enters the market with a dated portfolio of event-contingent endowment and firm ownership and leaves with a dated portfolio of event-contingent consumption plans. Household choices reflect household tastes and risk aversion. Pareto efficiency is in terms of the desirability of attainable state-contingent consumption portfolios. There is no assurance that realized consumption will be efficient subject to hindsight, merely that no attainable reallocation of contingent portfolios will generate *ex ante* a Pareto improvement in portfolio desirability.

Gerard says: theory's in the math. The rest is interpretation (See Debreu quote in textbook, p. 204). So how can we interpret the Arrow-Debreu model as a model of markets and resource allocation over time? Think: Futures Markets,Chicago Board of Trade. A commodity is characterized

by *what* it is (its description), and

by *where* it is available (its location), and

by *when* it is available (its date).

The market takes place separate from and prior to all consumption and production. At the market date, there are actively traded dated goods for all dates. If a good can be available at a date in the future, futures contracts for the good deliverable at that date are traded at the market date.

With that interpretation of a commodity, interpret the model as follows:

• make the markets for goods over time look just like those in the Arrow-Debreu general equilibrium model and the same formal results will follow. You'll be able to establish an intertemporal equilibrium, and intertemporally efficient allocation. All that remains is to interpret what economic institutions it requires for intertemporal goods allocation to look like the general equilibrium model.

For the market:

• all economically significant scarce resources are traded in the market; goods distinct from one another in production or consumption are distinct co-ordinates in N-dimensional commodity space.

• There is a single market date where all futures contracts are traded. At the market date all supplies and demands are expressed and equated. Budget constraints and firm profits are expressed effective with this date. Thus the complete future is collapsed for market purposes into plans expressed at the single market date.

For the firm:

• there is a single scalar maximand, profit, discounted value of dated outputs minus dated inputs. $\pi^j(p)$ is the value of the firm.

• all of the firm's dated production possibilities are fully expressed in the firm technology set from the present to the finite horizon.

For the household:

• there is a single maximand, $\succeq_i$ or equivalently the scalar u^i defined on consumption plans of dated commodities from the present to the finite horizon.

• there is a single scalar budget constraint, defined as the discounted value of the portfolio of planned purchases constrained by household wealth. Wealth is

defined as the discounted value of endowment plus the value of the household's share of firms.

For the economy:

• firm profits are distributed to households. Walras' Law holds.

N finite. N includes as a separate count every good, at every location where it is deliverable, and at every date at which it is deliverable.

There is only a single meeting of the market. Equilibrium consists of prices that clear markets for all dated goods deliverable at each date. Each household has only a single budget constraint representing receipts and expenditures undertaken at the market date for goods deliverable at all dates from the present to the finite horizon. Firms have only a single calculation of profit representing the net return on receipts for outputs and expenditures for inputs over all dates from the present to the finite horizon.

All receipts and expenditures for spot (current) goods and future deliveries are evaluated at the single market date. The distinctions between income and wealth, firm profits and firm value, are eliminated.

$\mathrm{p_i}$ = discounted value of commodity i discounted from the delivery date to the market date.

Costs are incurred, revenues received, accounts debited and credited at the market date, long prior to delivery.

Household endowment $\mathrm{r^h} \equiv (r_1^h, r_2^h, \ldots, r_N^h)$: present and future goods.

Household consumption $\mathrm{x^h} \equiv (\mathrm{x}_1^h, x_2^h, \ \ldots, x_N^h)$: Each co-ordinate represents dated planned consumption of a particular good.

Firm j′s production $\mathrm{y^j}$: dated plan for inputs and outputs at a sequence of dates.

Input and output prices are discounted values, discounted to the market date. Firm j′s profit is $\pi^j(p) = \max_{y \in Y^j} p \cdot y = \mathrm{p} \cdot S^j(p)$. $\pi^{\mathrm{j}}(\mathrm{p})$ is the sum evaluated at the market date, over all dates from the present through the time horizon of the (discounted) value of outputs less the (discounted) value of inputs. Discounting is from the delivery date to the market date.

Maximizing firm (discounted) profit, maximizing shareholder wealth, and maximizing firm (stock market) value are identical.

$\succeq_{\mathrm{h}}$ [household h's preference quasi-order], and $\mathrm{u^h}$, represent preferences on time dated streams of consumption from the present through the future until the horizon.

$M^h(p) = p \cdot r^h + \sum_{j \in F} \alpha^{hj} \pi^j(p)$. Household h chooses a consumption plan for the present through the horizon to optimize a planned program of consumption

evaluated by h's preferences for consumption across goods and time, subject to the budget constraint that the discounted value of the consumption plan is bounded above by the discounted value of endowment plus the value of household share ownership.

The futures markets here perform the functions both of goods markets and of capital markets. Investment and saving are arranged through futures markets. All trade takes place prior to consumption or production. Markets do not reopen over time; if they did, there would be no need for trade on them.

Compared with the real economy, there are too many active markets at the market date, and too few in the future. The future is merely the unfolding of contracted plans.

A Sequence Economy

It is possible to restate the model of dated commodities above reducing the number of active markets at the single market date by modeling reopening of markets and debt instruments. That is a *sequence economy.* At each date there are spot markets for active trade in goods deliverable at that date. There are financial markets in debt instruments, borrowing and lending into the future. Firms and households have perfect foresight concerning the prices prevailing in the future. Budget constraint at each date: sales ofgoods and debt (borrowing) must finance purchases. Sequence economy model with complete debt markets corresponds to the concept of a perfect capital market. Requires foresight on spot prices.

UNCERTAINTY: CONTINGENT COMMODITIES

Gerard says: theory's in the math. The rest is interpretation.

A commodity is now characterized

by *what* it is (its description), and

by *where* it is available (its location), by *when* it is available (its date), and

by it's *state of the world* (the uncertain event in which it is deliverable).

Arrow (1954, 1962) . State of the world is exogenous, not a decision variable.

Described in the literature as "a full set of Arrow-Debreu futures contracts" or "a full set of Arrow-Debreu contingent commodities".

N finite: number of possible uncertain events is finite at every date (& time is finite).

Contingent commodity: specific good *deliverable if a specified event occurs.*

An event tree.

• make the markets for goods under uncertainty look just like those in the Arrow-Debreu general equilibrium model and the same formal results will follow. You'll be able to establish an equilibrium for goods across uncertain events, and an efficient allocation of risk bearing. All that remains is to interpret what economic institutions it requires.

y^j, firm j's production plan, in this setting, is chosen to maximize the value of $p \cdot y$ for y in Y^j. Firm output in any date/event is known with certainty and reliability. The firm does not need a forecast of the future nor an attitude toward risk. Households have attitudes toward risk and a notion of which events to plan on for consumption. Those feelings get priced into the market for contingent commodities. Firm j chooses y^j (a dated contingent commodity production plan) to maximize

$\pi^j(p) = \max_{y \in Y^j} p \cdot y = p \cdot S^l(p)$ =stock market value of the firm.

Note that the firm is not a risk-taker. It promises delivery only for goods it is physically possible to deliver, and only when it has acquired the inputs required to produce them (independent of the probability of the deliverable event occurring).

Household income (wealth) is represented as $M^h(p) = p \cdot r^h + \sum_{j \in F} \alpha^{hj} \pi^j(p)$.

Household h's consumption vector x^h represents a state contingent dated list of projected consumptions. Household needs attitude toward risk and judgment of likelihood of realization of events. May be expected utility maximizer; optimizes utility of portfolio. Household preferences are on future uncertain consumption and on deciding on a desirable portfolio under uncertainty.

Most contingent commodity contracts expire without being executed by delivery. Again, in equilibrium there is no reopening of markets over time. Reopening is conceivable (but not modeled) if the event tree changes or subjective conditional probabilities change.

Equilibrium allocation of risky assets is Pareto efficient relative to $\succeq_h$. Given the endowments r^h and available technologies Y^j, there is no attainable reallocation of inputs to firms j or of contingent commodity outputs to households h that would move some household h higher in its ranking of portfolios, $\succeq_h$, without moving some other household h′ lower in its ranking of portfolios, $\succeq_{h'}$. This means that the allocation of risk bearing among households is Pareto efficient, an efficient allocation of risk bearing. There is no rearrangement of the risky assets, the contingent commodities, among households that would

be Pareto improving in terms of household portfolio preferences.

Ex ante efficiency distinct from ex post efficiency.

A Sequence Economy under uncertainty: Reducing the number of markets

Arrow insurance contract: Suppose there is a 'money' or numeraire in which we can describe a payment of generalized purchasing power. For each date event pair, t,s, the contract c(t,s) pays one unit of purchasing power if event s occurs at date t, and nil otherwise. Then instead of a full set of contingent commodity markets, we can use a mix of insurance contracts and spot markets (markets for actual goods deliverable in the current period) to achieve the same allocation as available in the contingent commodity equilibrium. Assumes correctly foreseen state-contingent goods prices.

There is then a formal equivalence between

• Contingent commodity markets, where the market meets once for all time and a very large number of contingent commodities are traded; most do not result in delivery of actual goods.

AND

• Securities markets for Arrow insurance contracts payable in abstract purchasing power: the securities market meets once; goods markets re-open at each date for spot trade. Most securities do not result in actual payment. Replace the full set of contingent commodity markets with a much smaller number of markets.

Ex ante and ex post efficiency under uncertainty

In the equilibrium of an Arrow-Debreu model with a full set of contingent commodity futures markets the *ex ante* allocation of risk bearing is Pareto efficient. However the realized allocation of resources *ex post* after the realization of uncertain events may or may not be efficient (with the benefit of 20-20 hindsight).

Consider households i and j in a 2-period model. Each has an expected utility function of the form

$$Eu^h = u^h(x_o^h) + \Theta_1^h u^h(x_1^h) + \Theta_2^h u^h(x_2^h)$$

where $u^h(x_o^h)$ is h's utility of period 0 consumption, Θ_1^h is h's subjective probability of event 1 in period 1, x_1^h is h's contingent consumption at date 1 state 1, Θ_2^h is h's subjective probability of state 2 date 1, x_2^h is h's contingent consumption at date 1 state 2 (this is crummy notation, but avoids more than one subscript at a time).

Prices are p_0 for date 0 consumption, p_1 for date 1 state 1 contingent consumption, and p_2 for date 1 state 2 contingent consumption. Let $'$ indicate first derivative. Then competitive equilibrium on the *ex ante* contingent commodity market (and an efficient allocation of risk bearing) requires

$$\frac{p_0}{p_1} = \frac{u^{i\prime}(x_0^i)}{\Theta_1^i u^{i\prime}(x_1^i)} = \frac{u^{j\prime}(x_0^j)}{\Theta_1^j u^{j\prime}(x_1^j)}$$

But *ex post* efficient allocation of resources requires $\dfrac{u^{i\prime}(x_0^i)}{u^{i\prime}(x_1^i)} = \dfrac{u^{j\prime}(x_0^j)}{u^{j\prime}(x_1^j)}$.

But these two conditions (assuming unique interior solutions) will only be consistent with one another if $\Theta_1^i = \Theta_1^j$, and there is no particular reason to expect this equality.