Economics 200B | UCSD
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Notes on Syllabus Section V: Social Welfare Functions and Social Choice Theory

Overview: The *Arrow Possibility Theorem* is the foundation of social choice theory. Bottom line: group decision-making mechanisms either are dictatorial or may lead to inconsistent decisions. There is no non-dictatorial universally consistent democratic decision-making mechanism. The theorem posits four plausible requirements of a ***transitive*** group decision-making mechanism, qualities fulfilled by majority voting on pairwise alternatives: Pareto principle [a unanimous preference is a group preference], Unrestricted domain [any complex of individual preferences can be accommodated by the decision mechanism], Independence [choices between two alternatives are independent of preferences on other possibilities], Non-dictatorship [there is no one whose preferences are always followed independent of others']. The logic of Sen's proof of the theorem is (1) there is always a decisive group (a subset of the population that gets its way, independent of others), (2) a group decisive on one proposition is decisive on all, (3) for each decisive group there is a <u>proper</u> subset that is also decisive, (4) apply observation (3) repeatedly so that there is a one-person decisive group. (4) gives you the dictator.

Black's theorem on majority rule with *single-peaked* preferences salvages the consistency of majority rule by restricting the domain. In the case where voter preferences have a linear form, the median voter's preferences are decisive.

Social Choice Theory, Arrow Possibility Theorem

Read about Bergson-Samuelson social welfare function in Varian, MasColell et al.

Paradox of Voting (Condorcet)

Cyclic majority:

Voter preferences:	1	2	3
	A	B	C
	B	C	A
	C	A	B

Majority votes A > B, B > C. Transitivity requires A > C but majority votes C > A.

Conclusion: Majority voting on pairwise alternatives by rational (transitive) agents can give rise to intransitive group preferences.

Is this an anomaly? Or systemic. Arrow Possibility Theorem says systemic.

Arrow (Im) Possibility Theorem:

We'll follow Sen's treatment (Handbook of Mathematical Economics). For simplicity we'll deal in strong orderings (strict preference) only

X = Space of alternative choices

Π = Space of transitive strict orderings on X

H = Set of voters, numbered #H

$\Pi^{\#H}$ = #H - fold Cartesian product of Π , space of preference profiles

f: $\Pi^{\#H} \rightarrow \Pi$, f is an Arrow Social Welfare Function.

P_i represents the preference ordering of typical household i. $\{P_i\}$ represents a preference profile, $\{P_i\} \in \Pi^{\#H}$. P represents the resulting group (social) ordering.
" $x P_i y$ " is read "x is preferred to y by i" for $i \in H$

P (without subscript) denotes the social ordering, $f(P_1, P_2, ..., P_{\#H})$.

Unrestricted Domain: Π = all logically possible strict orderings on X.
$\Pi^{\#H}$ = all logically possible combinations of #H elements of Π.

Non-Dictatorship: There is no $j \in H$, so that $x \, P \, y \Leftrightarrow x \, P_j \, y$, for all $x, y \in X$, for all $\{P_i\} \in \Pi^{\#H}$.

(Weak) Pareto Principle: Let $x \, P_i \, y$ for all $i \in H$. Then x P y.

For $S \subseteq X$, Define $C(S) = \{ x \mid x \in S, \; x \, P \, y, \text{ for all } y \in S, y \neq x \}$

Independence of Irrelevant Alternatives: Let $\{P_i\} \in \Pi^{\#H}$ and $\{P'_i\} \in \Pi^{\#H}$, so that for all $x, y \in S \subseteq X$, $x \, P_i \, y$ if and only if ($\Leftrightarrow$) $x \, P'_i \, y$. Then $C(S) = C'(S)$.

General Possibility Theorem (Arrow): Let f satisfy (Weak) Pareto Principle, Independence of Irrelevant Alternatives, Unrestricted Domain, and let #H be finite, $\#X \geq 3$. Then there is a dictator; there is no f satisfying non-dictatorship and the three other conditions.

Definition (Decisive Set): Let $x,y \in X$, $G \subseteq H$. G is decisive on (x, y) denoted $\overline{D}_G(x,y)$ if [$x \, P_i \, y$ for all $i \in G$] implies [x P y] independent of P_j, $j \in H$, $j \notin G$.

Definition (Almost Decisive Set): Let $x,y \in X$, $G \subseteq H$. G is almost decisive on (x, y) denoted $D_G(x, y)$ if [$x \, P_i \, y$ for all $i \in G$; $y \, P_j \, x$ for all $j \notin G$] implies [x P y] .

Note: $\overline{D}_G(x,y)$ implies D(x, y) but D(x,y) does not imply $\overline{D}_G(x,y)$ (though it does not contradict either).

Field Expansion Lemma: Assume (Weak) Pareto Principle, Independence of Irrelevant Alternatives, Unrestricted Domain, Non-Dictatorship.
Let $x, y \in X$, $G \subseteq H$, $D_G(x, y)$. Then for arbitrary $a, b \in X$, $a \neq b$, $\overline{D}_G(a,b)$.

<u>Field Expansion Lemma</u>: Assume (Weak) Pareto Principle, Independence of Irrelevant Alternatives, Unrestricted Domain, Non-Dictatorship.
Let $x, y \in X$, $G \subseteq H$, $D_G(x, y)$. Then for arbitrary $a, b \in X$, $a \neq b$, $\overline{D}_G(a,b)$.

<u>Proof:</u> Introduce $a, b \in X$, $a \neq b$. We'll consider three cases
1. $x \neq a \neq y$, $x \neq b \neq y$
2. $a = x$. This is typical of the three other cases (which we'll skip, assuming their treatments are symmetric) $b = x$, $a = y$, $b = y$.
3. $a = x$ and $b = y$.

Case 1 (a, b, x, y are all distinct) : Let G have preferences : $a > x > y > b$.
Unrestricted Domain allows us to make this choice. Let H \ G have preferences: $a > x$, $y > b$,
$y > x$, a ? b (unspecified).
Pareto implies a P x, y P b.
$D_G(x, y)$ implies x P y.
P transitive implies a P b, independent of H \ G's preferences.
Independence implies $\overline{D}_G(a,b)$.

Case 2 ($a = x$): Let G have preferences: $a > y > b$. Let H\G have preferences: $y > a$, $y > b$, a ? b (unspecified). $D_G(x, y)$ implies that xPy or equivalently aPy. Pareto principle implies yPb. Transitivity implies aPb. By Independence, then $\overline{D}_G(a,b)$.

Case 3 ($a = x$, $b = y$): Introduce a third state z, distinct from a and b, x and y. Since $\#X \geq 3$, this is possible. We now consider a succession of examples.

Let G have preferences: $(x=)a > (y=)b > z$. Let H\G have preferences: $b > a$, $b > z$, a?z (unspecified). $D_G(x, y)$ implies that xPy or equivalently aPb. Pareto principle implies bPz. Transitivity implies (x=)aPz. By Independence, then $\overline{D}_G(x, z)$.

Now consider G: $b > x > z$; H\G: b ?z, z?x (unspecified), $b > x$. We have xPz by $\overline{D}_G(x,z)$. By Pareto we have bPx. By transitivity we have (y=)bPz. By Independence, then $\overline{D}_G(y, z)$. [Is this step necessary?]

Now consider G: y(=b) > z > x(=a); H\G z>x, x?y, z?y. $\overline{D}_G$ (y, z) implies yPz. Pareto implies zPx. Transitivity implies yPx. Independence implies $\overline{D}_G$ (y, x) = $\overline{D}_G$ (b, a). [Is this step necessary?]

Repeating the argument in Case 2, consider G: a(=x) > z > b(=y). Let H\G have preferences: z > a, z > b, a ? b (unspecified). $\overline{D}_G$ (x, z) implies xPz. Pareto implies zPb. Transitivity implies x(=a)Pb. Independence implies $\overline{D}_G$ (a, b) = $\overline{D}_G$ (x, y).

QED

The Field Expansion Lemma tells us that a set that is almost decisive on any (x, y), $x \neq y$, is decisive on arbitrary (a, b).

Note that under the Pareto Principle, there is always at least one decisive set, H.

Group Contraction Lemma: Let $G \subseteq H$, #G > 1, G decisive. Then there are G_1, G_2 , disjoint, nonempty, so that $G_1 \cup G_2 = G$, so that one of G_1, G_2 is decisive.

Proof: By Unrestricted Domain, we get to choose our example. Let

G_1 : x > y > z

G_2 : y > z > x

H \ G : z > x > y

G is decisive so $\overline{D}_G(y,z)$ so y P z .

Case 1: x P z

Then G_1 is decisive by the Field Expansion Lemma and Independence of Irrelevant Alternatives.

Case 2: z P x

transitivity implies y P x

Field Expansion Lemma & Independence of Irrelevant Alternatives implies G_2 is decisive. QED

Proof of the Arrow Possibility Theorem: Pareto Principle implies that H is decisive. Group contraction lemma implies that we can successively eliminate elements of H so that remaining subsets are still decisive. Repeat. Then there is $j \in H$ so that {j} is decisive. Then j is a dictator. QED

Salvaging Majority Rule: Single Peaked Preferences and the Median Voter Theorem

Arrow Possibility Theorem implies that majority rule or any similar decision-making mechanism on pairwise alternatives cannot generally lead to transitive group preferences.

Restriction on space of possible preferences --- purposely violate 'Unrestricted Domain'; limit the space of possible profiles. Single peaked preferences: Suppose all propositions to be decided can be linearly ordered, left to right. All voters agree on the left to right ordering. They disagree on their choices.

Everyone has his favorite point; but *chacun à son gout (to each his own)* --- the favorite point differs among voters. For each voter, as we move to the left of his favorite his utility goes down; as we move to the right of his favorite his utility goes down.

Let L be the "is to the left of" ordering. All voters agree on the L ordering. Arrange the propositions a_1, a_2, ... so that a_1 L a_2 L a_3 L a_4 .... , and so forth. For each voter $i \in H$, there is a favorite proposition a^{*i}. All propositions to the left of a^{*i} are inferior --- according to i's preferences --- and the farther to the left the worse they get. All propositions to the right of a^{*i} are inferior to a^{*i} , and the farther to the right they get, the worse they are. Thus, for propositions u, v, w, x,

$$u \; L \; v \; L \; a^{*i} \; L \; w \; L \; x$$

implies $a^{*i} \, P_i \, v \, P_i \, u$, and $a^{*i} \, P_i \, w \, P_i \, x$. This situation describes "single-peaked preferences."

Arrange the favorite points of all agents $i \in H$, a^{*i} , in the left to right ordering. Assume (for convenience) an odd number of voters to avoid ties. Find the proposition $\mathcal{A}$ in the middle of this left to right array (so that half but one of others' favorites are to the left, half but one to the right). Then $\mathcal{A}$ is said to be the median preference point. It will command a majority vote against any alternative.

Theorem 1 (Duncan Black): If preferences are single-peaked, then majority voting on pairwise alternatives yields transitive group decisions.

Theorem 2 (Median voter theorem, Duncan Black): Let $\hat{A}$ be a median preference point. Then there is a majority (non-minority) of voters favoring $\hat{A}$ over any alternative, a'. (The favorite of the median voter is undominated in majority rule).

Proof of theorem 2: By inspection.

Proof of theorem 1: This requires some work. What do we want to show? Let P be the majority rule preference relation. Without loss of generality, let A P B, B P C, and let preferences be single peaked. Then we must show that A P C.

Consider (an exhaustive list of) six special cases:

1. A L B L C
2. B L C L A
3. C L A L B
4. C L B L A (equivalent argument to case 1)
5. A L C L B (equivalent argument to case 2)
6. B L A L C (equivalent argument to case 3)

Describe each household's preferences by a utility function $u^i()$. A household votes in favor of x over y when $u^i(x) > u^i(y)$. We will ignore ties.

Case 1: Consider those households $i \in H$, so that $u^i(A) > u^i(B)$. These households constitute a majority since A P B. But with the ordering of case 1, they must all have $u^i(B) > u^i(C)$ (otherwise they would fail single peakedness; they'd have two peaks). Hence we have A P C, as claimed.

Case 2: We claim case 2 is an empty set under A P B, B P C and single peakedness.

We have that a majority of voters has $u^i(B) > u^i(C)$. With the Case 2 ordering and single peakedness that means that a majority has $u^i(B) > u^i(A)$. Then we cannot have A P B, so case 2 cannot occur under the hypothesis.

Case 3: Really requires some work. We break H into four subgroups:

Households $i \in H$, so that:

Group I : $u^i(A) > u^i(B)$; $u^i(B) > u^i(C)$. Transitivity of $u^i(\ )$ implies $u^i(A) > u^i(C)$.

Group II : $u^i(A) > u^i(B)$; $u^i(B) < u^i(C)$

Group III : $u^i(A) < u^i(B)$; $u^i(B) > u^i(C)$. Single peakedness and the case 3 ordering implies that $u^i(A) > u^i(C)$ for Group III

Group IV : $u^i(A) < u^i(B)$; $u^i(B) < u^i(C)$. Single peakedness and the case 3 ordering implies that group IV is the empty set.

A P B implies $I \cup II$ constitutes a majority.

B P C implies $I \cup III$ constitutes a majority. Note preferences on A versus C in I and III. Then $I \cup III$ constitutes a majority for $u^i(A) > u^i(C)$, so A P C as required.

QED