Notes on Syllabus Section IV: Core of a Market Economy

Overview: It is a commonplace in the microeconomics principles course to say that in a large 'competitive' market no single actor has any significant bargaining power. They necessarily accept prevailing market prices and allocation. The core convergence model proves that statement. The notion of the ***core*** of an economy is a generalization of the contract curve, an outcome of bargaining. We concentrate on a pure exchange economy and consider the allocations that can be sustained as the number of agents in the economy becomes large. The bargaining concept is that each individual or each freely forming group (***coalition***) considers how well they can do on their own, without relying on others. For any proposed allocation, each coalition is thought to consider whether it can improve its own outcome compared to the proposal using only its own resources. If so, the coalition is said to ***block*** the proposed allocation. The core of the economy represents the allocations so that no individual or group can block. The core is necessarily Pareto efficient. It will be shown that the core includes the competitive equilibrium. Following the simplification of Debreu and Scarf, we consider the economy becoming large through ***replication***, doubling, tripling, ..., Q-fold multiplication of the population by creating identical copies of the original population. The core becomes smaller as the economy becomes large as the number and variety of blocking coalitions proliferates. The only allocation that remains in the core in the limit as $Q \to \infty$ is the competitive equilibrium.

Bargaining and equilibrium: The core of a market economy

Set $X^i = \mathbf{R}^N_+$, all i.

Each $i \in H$ has an endowment $r^i \in \mathbf{R}^N_+$ and a preference quasi-ordering $\succeq_i$ defined on $\mathbf{R}^N_+$.

An allocation is an assignment of $x^i \in \mathbf{R}^N_+$ for each $i \in H$. A typical allocation, $x^i \in \mathbf{R}^N_+$ for each $i \in H$, will be denoted $\{x^i, i \in H\}$. An allocation, $\{x^i, i \in H\}$, is ***feasible*** if $\sum_{i \in H} x^i \leq \sum_{i \in H} r^i$, where the inequality holds coordinatewise.

We assume preferences fulfill weak monotonicity (C.IV*), continuity (C.V), and strict convexity (C.VI(SC)).

The core of a pure exchange economy

Definition : A ***coalition*** is any subset $S \subseteq H$. Note that every individual comprises a (singleton) coalition.

Definition : An allocation $\{x^i, h \in H\}$ is ***blocked*** by $S \subseteq H$ if there is a coalition $S \subseteq H$ and an assignment $\{y^i, i \in S\}$ so that:

(i) $\sum_{i \in S} y^i \leq \sum_{i \in S} r^i$ (where the inequality holds coordinatewise),
(ii) $y^i \succeq_i x^i$, for all $i \in S$, and
(iii) $y^h \succ_h x^h$, for some $h \in S$

Definition : The ***core*** of the economy is the set of feasible allocations that are not blocked by any coalition $S \subseteq H$.

- Any allocation in the core must be individually rational. That is, if $\{x^i, i \in H\}$ is a core allocation then we must have $x^i \succeq_i r^i$, for all $i \in H$.
- Any allocation in the core must be Pareto efficient.

(i) The competitive equilibrium is always in the core (Theorem 21.1).

Theorems 22.2 and 22.3 say that

(ii) For a large economy, the set of competitive equilibria and the core are virtually identical. All core allocations are (nearly) competitive equilibria.

The competitive equilibrium allocation is in the core

Definition : $p \in \mathbf{R}^N_+$, $p \neq 0$, $x^i \in \mathbf{R}^N_+$, for each $i \in H$, constitutes a competitive equilibrium if

(i) $p \cdot x^i \leq p \cdot r^i$, for each $i \in H$,
(ii) $x^i \succeq_i y$, for all $y \in R^N_+$, such that $p \cdot y \leq p \cdot r^i$, and
(iii) $\sum_{i \in H} x^i \leq \sum_{i \in H} r^i$ (the inequality holds coordinatewise) with $p_k = 0$ for any $k = 1, 2, \ldots, N$ so that the strict inequality holds.

Theorem 21.1 : Let the economy fulfill C.II, C.IV*, C.VI(SC) and let $X^i = \mathbf{R}^N_+$. Let p, x^i, $i \in H$, be a competitive equilibrium. Then $\{x^i, i \in H\}$ is in the core of the economy.

Proof : We will present a proof by contradiction. Suppose the theorem were false. Then there would be a blocking coalition $S \subseteq H$ and a blocking assignment $y^i, i \in S$. We have

$\sum_{i \in S} y^i \leq \sum_{i \in S} r^i$(feasibility, the inequality holds coordinatewise)
$y^i \succeq_i x^i$, for all $i \in S, and$
$y^h \succ_h x^h$, some $h \in S$.

But x^i is a competitive equilibrium allocation. That is, for all $i \in H$, $p \cdot x^i = p \cdot r^i$ (recalling Lemma 17.1), and $x^i \succeq_i y$, for all $y \in R^N_+$ such that $p \cdot y \leq p \cdot r^i$.

Note that $\sum_{i\in S} p\cdot x^i = \sum_{i\in S} p\cdot r^i$. Then for all $i \in S$, $p\cdot y^i \geq p\cdot r^i$. That is, x^i represents i's most desirable consumption subject to budget constraint. y^i is at least as good under preferences $\succeq_i$ fulfilling C.II, C.IV*, C.VI(SC), (local non-satiation). Therefore, y^i must be at least as expensive. Furthermore, for h, we must have $p \cdot y^h > p \cdot r^h$. Therefore, we have

$$\sum_{i\in S} p \cdot y^i > \sum_{i\in S} p \cdot r^i.$$

Note that this is a strict inequality. However, for coalitional feasibility we must have

$$\sum_{i\in S} y^i \leq \sum_{i\in S} r^i.$$

But since $p \geq 0$, $p \neq 0$, we have $\sum_{i\in S} p \cdot y^i \leq \sum_{i\in S} p \cdot r^i$. This is a contradiction. The allocation $\{y^i, i \in S\}$ cannot simultaneously be smaller or equal to the sum of endowments r^i coordinatewise and be more expensive at prices $p, p \geq 0$. The contradiction proves the theorem. QED

Convergence of the core of a large economy. Replication

In replication, the economy keeps cloning itself: duplicate to triplicate, ..., to Q-tuplicate, and so on. The set of core allocations keeps getting smaller, although it always includes the set of competitive equilibria (per Theorem 21.1).

We will treat a Q-fold replica economy, denoted Q-H. Q will be a positive integer; $Q = 1, 2, \ldots$. In a Q-fold replica economy we take an economy consisting of households $i \in H$, with endowments r^i and preferences $\succeq_i$, and create a similar larger economy with Q times as many agents in it, totaling $\#H \times Q$ agents. There will be Q agents with preferences $\succeq_1$ and endowment r^1, Q agents with preferences $\succeq_2$ and endowment $r^2, \ldots,$

and Q agents with preferences $\succeq_{\#H}$ and endowment $r^{\#H}$. Each household $i \in H$ now corresponds to a household type. There are Q individual households of type i in the replica economy Q-H. Note that the competitive equilibrium prices in the original H economy will be equilibrium prices of the Q-H economy. Household i's competitive equilibrium allocation x^i in the original H economy will be a competitive equilibrium allocation to all type i households in the Q-H replica economy. Agents in the Q-H replica economy will be denoted by their type and a serial number. Thus, the agent denoted i, q will be the qth agent of type i, for each $i \in H, q = 1, 2, \ldots, Q$.

Q-fold replica economy, denoted Q-H. $Q = 1, 2, \ldots$.

$\#H \times Q$ agents.

Q agents with preferences $\succeq_1$ and endowment r^1,

Q agents with preferences $\succeq_2$ and endowment $r^2, \ldots,$ and Q agents with preferences $\succeq_{\#H}$ and endowment $r^{\#H}$. Each household $i \in H$ now corresponds to a household type. There are Q individual households of type i in the replica economy Q-H.

Equal treatment

Theorem 22.1 (Equal treatment in the core) : Assume C.IV, C.V, and C.VI(SC). Let $\{x^{i,q}, i \in H, q = 1, \ldots, Q\}$ be in the core of Q-H, the Q-fold replica of economy H. Then for each $i, x^{i,q}$ is the same for all q. That is, $x^{i,q} = x^{i,q'}$ for each $i \in H, q \neq q'$.

Proof of Theorem 22.1 : Recall that the core allocation must be feasible. That is,

$$\sum_{i \in H} \sum_{q=1}^{Q} x^{i,q} \leq \sum_{i \in H} \sum_{q=1}^{Q} r^i.$$

Equivalently,

$$\frac{1}{Q}\sum_{i\in H}\sum_{q=1}^{Q} x^{i,q} \le \sum_{i\in H} r^i.$$

Suppose the theorem to be false. Consider a type i so that $x^{i,q} \neq x^{i,q'}$. For each type i, we can rank the consumptions attributed to type i according to $\succeq_i$.

For each i, let x^{i^*} denote the least preferred of the core allocations to type $i, x^{i,q}, q = 1, \ldots, Q$. For some types i, all individuals of the type will have the same consumption and x^{i^*} will be this expression. For those in which the consumption differs, x^{i^*} will be the least desirable of the consumptions of the type. We now form a coalition consisting of one member of each type: the individual from each type carrying the worst core allocation, x^{i^*}.

Consider the average core allocation to type i, to be denoted $\bar{x}^i$.

$\bar{x}^i = \frac{1}{Q}\sum_{q=1}^{Q} x^{i,q}$.

We have, by strict convexity of preferences (C.VI(SC)),

$$\bar{x}^i = \frac{1}{Q}\sum_{q=1}^{Q} x^{i,q} \succ_i x^{i^*} \text{ for those types } i \text{ so that } x^{i,q} \text{ are not identical,}$$

and

$$x^{i,q} = \bar{x}^i = \frac{1}{Q}\sum_{q=1}^{Q} x^{i,q} \sim_i x^{i^*} \text{ for those types } i \text{ so that } x^{i,q} \text{ are identical.}$$

From feasibility, above, we have that

$$\sum_{i\in H}\bar{x}^i = \sum_{i\in H}\frac{1}{Q}\sum_{q=1}^{Q} x^{i,q} = \frac{1}{Q}\sum_{i\in H}\sum_{q=1}^{Q} x^{i,q} \le \sum_{i\in H} r^i.$$

In other words, a coalition composed of one of each type (the worst off of each) can achieve the allocation $\bar{x}^i$. However, for each agent in the coalition, $\bar{x}^i \succeq_i x^{i^*}$ for all i and $\bar{x}^i \succ_i x^{i^*}$ for some i. Therefore, the coalition of the worst off individual of each type blocks the allocation $x^{i,q}$. The contradiction proves the theorem. QED

Core$(Q) = \{x^i, i \in H\}$ where $x^{i,q} = x^i, q = 1, 2, \ldots, Q$, and the allocation $x^{i,q}$ is unblocked.

Core convergence in a large economy

As Q grows there are more blocking coalitions, and they are more varied. Any coalition that blocks an allocation in Q-H still blocks the allocation in $(Q + 1)$-H, but there are new blocking coalitions and allocations newly blocked in $(Q + 1)$-H.

Recall the Bounding Hyperplane Theorem:

Theorem 8.1, Bounding Hyperplane Theorem (Minkowski) Let K be convex, $K \subseteq \mathbf{R}^N$. There is a hyperplane H through z and bounding for K if z is not interior to K. That is, there is $p \in \mathbf{R}^N, p \neq 0$, so that for each $x \in K, p \cdot x \geq p \cdot z$.

Theorem 22.2 (Debreu-Scarf) : Assume C.IV*, C.V, C.VI(SC), and let $X^i = \mathbf{R}^N_+$. Let $\{x^{\circ i}, i \in H\} \in$ core(Q) for all $Q = 1, 2, 3, 4, \ldots$. Then $\{x^{\circ i}, i \in H\}$ is a competitive equilibrium allocation for $Q - H$, for all Q.

Proof : We must show that there is a price vector p so that for each household type $i, p \cdot x^{\circ i} \leq p \cdot r^i$ and that $x^{\circ i}$ optimizes preferences $\succeq_i$ subject to this budget. The strategy of proof is to create a set of net trades preferred to those that achieve $\{x^{\circ i}, i \in H\}$. We will show that it is a convex set with a supporting hyperplane through the origin. The normal to the supporting hyperplane will be designated p. We will then argue that p is a competitive equilibrium price vector supporting $\{x^{\circ i}, i \in H\}$.

For each $i \in H$, let $\Gamma^i = \{z \mid z \in \mathbf{R}^N, z + r^i \succ_i x^{oi}\}$. What is this set of vectors Γ^i? Γ^i is defined as the set of net trades from endowment r^i so that an agent of type i strictly prefers these net trades to the trade $x^{oi} - r^i$, the trade that gives him the core allocation. We now define the convex hull (set of convex combinations) of the family of sets $\Gamma^i, i \in H$. Let $\Gamma = \{\sum_{i \in H} a_i z^i \mid z^i \in \Gamma^i, a_i \geq 0, \sum a_i = 1\}$, the set of convex combinations of preferred net trades. The set Γ is the convex hull of the union of the sets Γ^i. (See Figure 22.1.) Note that $(x^{\circ i} - r^i) \in \text{boundary}(\Gamma^i), (x^{\circ i} - r^i) \in \overline{\Gamma}^i$, and $(x^{\circ i} - r^i) \in \overline{\Gamma}$ for all i.

The strategy of proof now is to show that Γ and the constituent sets Γ^i are arrayed strictly above a hyperplane through the origin. The normal to the hyperplane will be the proposed equilibrium price vector.

We wish to show that $0 \notin \Gamma$. We will show that the possibility that $0 \in \Gamma$ corresponds to the possibility of forming a blocking coalition against the core allocation x^{oi}, a contradiction. The typical element of Γ can be represented as $\sum a_i z^i$, where $z^i \in \Gamma^i$. Suppose that $0 \in \Gamma$. Then there are $0 \leq a_i \leq 1, \sum_{i \in H} a_i = 1$ and $z^i \in \Gamma^i$ so that $\sum_{i \in H} a_i z^i = 0$. We'll focus on these values of a_i, z^i, and consider the k-fold replication of H, eventually letting k become arbitrarily large. Let the notation $[\cdot]$ represent the smallest integer greater than or equal to the argument $\cdot$. Consider the hypothetical net trade for a household of type i, $\frac{ka_i}{[ka_i]} z^i$. We have $\frac{ka_i}{[ka_i]} z^i \to z^i$ as $k \to \infty$. Therefore, by (C.V, continuity) for k sufficiently large,

$$[r^i + \frac{ka_i}{[ka_i]} z^i] \succ_i x^{oi} \qquad (\dagger)$$

Further,

$$\sum_{i \in H} [ka_i] \frac{ka_i}{[ka_i]} z^i = k \sum_{i \in H} a_i z^i = 0 \qquad (\ddagger).$$

It is now time to form a blocking coalition. We confine attention to those $i \in H$ so that $a_i > 0$. The blocking coalition is formed by $[\hat{k}a_i]$ households of type i where $\hat{k}$ is the smallest integer so that ($\dagger$) is fulfilled for all $i \in H$ for $a_i > 0$. That is, let $\hat{k} \equiv \inf\{k \in \mathcal{N} | (\dagger)$ is fulfilled for all $i \in H$ such that $a_i > 0\}$ where $\mathcal{N}$ is the set of positive integers. Consider Q larger than $\hat{k}$. Form the coalition S consisting of $[\hat{k}a_i]$ households of type i for all i so that $a_i > 0$. The blocking allocation to each household of type i is $r^i + \frac{\hat{k}a_i}{[\hat{k}a_i]} z^i$ (this expression has been amended from the text, by the addition of 'hat's on the k's). This allocation is attainable to the coalition by ($\ddagger$) and it is preferable to the coalition by ($\dagger$). This is how replication with large Q overcomes the indivisibility of the individual agents. Thus S blocks x^{oi}, which is a contradiction. Hence, as claimed, $0 \notin \Gamma$.

Having established that 0 is not an element of Γ, we should recognize that 0 is nevertheless very close to Γ. Indeed $0 \in$ boundary of Γ. This occurs inasmuch as $0 = (1/\#H) \sum_{i \in H} (x^{\circ i} - r^i)$, and the right-hand side of this expression is an element of $\overline{\Gamma}$, the closure of Γ. Thus 0 represents just the sort of boundary point through which a supporting hyperplane may go in the Bounding Hyperplane Theorem. The set Γ is trivially convex. Hence we can invoke the Bounding Hyperplane Theorem. There is $p \in \mathbf{R}^N, p \neq 0$, so that for all $v \in \Gamma$, $p \cdot v \geq p \cdot 0 = 0$. Noting $X^i = \mathbf{R}^N_+$, C.IV* and

C.VI(SC) , we know that $p \geq 0$. Now $(x^{\circ i} - r^i) \in \overline{\Gamma}$ for each i, so $p \cdot (x^{\circ i} - r^i) \geq 0$. But $\sum_{i \in H}(x^{\circ i} - r^i) = 0$, so $p \cdot \sum_{i \in H}(x^{\circ i} - r^i) = 0$. Hence $p \cdot (x^{\circ i} - r^i) = 0$ each i. Equivalently, $p \cdot x^{\circ i} = p \cdot r^i$. This gives us

$$0 = p \cdot \sum_{i \in H} \frac{1}{\#H}(x^{\circ i} - r^i) = \inf_{x \in \Gamma} p \cdot x = \sum_{i \in H} \frac{1}{\#H}\Big[\inf_{z^i \in \Gamma^i} p \cdot z^i\Big],$$

so

$$p \cdot (x^{\circ i} - r^i) = \inf_{z^i \in \Gamma^i} p \cdot z^i.$$

We have then for each i, that $p \cdot (x^{\circ i} - r^i) = \inf p \cdot y$ for $y \in \Gamma^i$. Equivalently, $x^{\circ i}$ minimizes $p \cdot (x - r^i)$ subject to $x \succeq_i x^{\circ i}$. In addition, $p \cdot x^{\circ i} = p \cdot r^i$. Further, by the specification of X^i and r^i, there is an ε-neighborhood of $x^{\circ i}$ contained in X^i. By C.IV*, C.V, and C.VI(SC) , and strict positivity of r^i, expenditure minimization subject to a utility constraint is equivalent to utility maximization subject to budget constraint. Hence $x^{\circ i}, i \in H$, is a competitive equilibrium allocation. QED