Notes on Syllabus Section III: Pareto efficiency and competitive equilibrium

Overview: The notion of Pareto efficiency is that all opportunities for welfare improvement are fully utilized, without making a judgment as to whether one household is to be favored over another. More precisely, an attainable allocation is Pareto efficient if there is no alternative attainable allocation that improves some household's utility without reducing another's.

The First Fundamental Theorem of Welfare Economics (1FTWE) says that any competitive equilibrium allocation is Pareto efficient. Most of the structure of the assumptions of the theorem are embodied in the definition of competitive equilibrium including that households fully spend their income. 1FTWE does not require convexity and assumes absence of external effects.

The Second Fundamental Theorem of Welfare Economics (2FTWE) is a partial converse. 2FTWE says that any Pareto efficient allocation can be supported as a competitive equilibrium with suitably chosen prices and subject to a redistribution of income (or endowment). 2FTWE requires convexity of tastes and technology in order to establish supporting prices using the Supporting Hyperplane Theorem.

The two fundamental theorems provide sufficient conditions for market mechanisms to provide an efficient allocation of resources while avoiding judgment on interpersonal welfare comparison.

19.1 Pareto efficiency

> "In 1954, referring to the first and second theorems of classical welfare economics, Gerard wrote 'The contents of both Theorems ... are old beliefs in economics. Arrow and Debreu have recently treated these questions with techniques permitting proofs.' This statement is precisely correct; once there were beliefs, now there was knowledge."
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> ——— Hugo Sonnenschein (2005)

Definition : An allocation x^i, $i{\in}H$, is attainable if $x^i \in X^i$, $i{\in}H$ and there is $y^j{\in}Y^j$, $j{\in}F$, so that $0 \leq \sum_{i\in H} x^i \leq \sum_{j\in F} y^j + \sum_{i\in H} r^i$. (the inequality holds co-ordinatewise)

Note the inequality, $\leq$, $(\sum_{i\in H} x^i \leq \sum_{j\in F} y^j + \sum_{i\in H} r^i)$, in the definition of "attainable." This amounts to assuming that commodities can be discarded costlessly (free disposal).

Definition : Consider two allocations of bundles to consumers, $v^i, w^i \in X^i$, $i \in H$. v^i is said to be Pareto superior (or Pareto preferable) to w^i if for each $i \in H$, $v^i \succeq_i w^i$, and for some $h \in H$, $v^h \succ_h w^h$.

Note that Pareto superiority (='Pareto preferability') is an incomplete ordering. There are many allocation pairs that are Pareto incomparable.

Definition : An attainable allocation of bundles to consumers, $w^i \in X^i$, $i \in H$, is said to be Pareto efficient (or Pareto optimal) if there is no other attainable allocation $v^i \in X^i$ so that v^i is Pareto superior to w^i.

Definition : $\{p^\circ, x^{\circ i}, y^{\circ j}\}$, $p^\circ \in \mathbf{R}^N_+$, $i \in H$, $j \in F$, $x^{\circ i} \in \mathbf{R}^N$, $y^{\circ j} \in \mathbf{R}^N$, is said to be a competitive equilibrium if

(i) $y^{\circ j} \in Y^j$ and $p^\circ \cdot y^{\circ j} \geq p^\circ \cdot y$ for all $y \in Y^j$, for all $j \in F$,
(ii) $x^{\circ i} \in X^i$, $p^\circ \cdot x^{\circ i} \leq M^i(p^\circ) = p^\circ \cdot r^i + \sum_{j \in F} \alpha^{ij} p^\circ \cdot y^{\circ j}$ and $x^{\circ i} \succeq_i x$ for all $x \in X^i$ with $p^\circ \cdot x \leq M^i(p^\circ)$ for all $i \in H$, and
(iii) $0 \geq \sum_{i \in H} x^{\circ i} - \sum_{j \in F} y^{\circ j} - \sum_{i \in H} r^i$ with $p^\circ_k = 0$ for coordinates k so that the strict inequality holds.

This definition is sufficiently general to include the equilibria developed in Theorems 14.1, 18.1, and 24.7.

First Fundamental Theorem of Welfare Economics

1FTWE: It says that every competitive equilibrium is Pareto efficient.

Does NOT require convexity of tastes or technology. Does assume no external effects.

To prove the First Fundamental Theorem of Welfare Economics, it is useful to have the budget constraint fulfilled as an equality in equilibrium, as noted in Lemmas 14.1, 17.1, or 24.4. For full generality, it is useful at this point to have alternative sufficient conditions for that equality that do not depend on convexity of preferences, C.VI(C).

(C.IV*) (Weak Monotonicity) Let $x, y \in X^i$ and $x >> y$. Then $x \succ_i y$.

Lemma 19.1 : Assume C.IV*, $X^i = \mathbf{R}^N_+$, and let $M^i(p)$ and $D^i(p)$ be well defined. Let $x \in D^i(p)$. Then $p \cdot x = M^i(p)$.

Theorem 19.1 (First Fundamental Theorem of Welfare Economics) : For each $i \in H$, assume C.II, C.IV, and either assume C.VI(C) or assume C.IV*, $X^i = \mathbf{R}^N_+$. Let $p^\circ \in \mathbf{R}^N_+$ be a competitive equilibrium price vector of the economy. Let $w^{\circ i} \in X^i$, $i \in H$, be the associated individual consumption bundles, and let $y^{\circ j}$, $j \in F$, be the associated firm supply vectors. Then $w^{\circ i}$ is Pareto efficient.

Intuition for the proof of 1FTWE: Proof by contradiction. If there's a better attainable consumption plan it must be more expensive than CE consumption plan evaluated at equilibrium prices. Then it must be more profitable (and attainable) to the firm sector as well. Then it must be available and more profitable to some firm. But that contradicts the definition of CE.

Proof : $w^{\circ i} \succeq_i x$, for all $x \in X^i$ so that $p^\circ \cdot x \le M^i(p^\circ)$, for all $i \in H$. This is a property of the equilibrium allocation. Consider an allocation x^i that household $i \in H$ regards as more desirable than $w^{\circ i}$. If the allocation x^i is preferable, it must also be more expensive. That is,

$$x^i \succ_i w^{\circ i} \text{ implies } p^\circ \cdot x^i > p^\circ \cdot w^{\circ i}.$$

Similarly, profit maximization in equilibrium implies that production plans more profitable than $y^{\circ j}$ at prices p are not available in Y^j. $p^\circ \cdot y > p^\circ \cdot y^{\circ j}$ implies $y \notin Y^j$. Noting that markets clear at the equilibrium allocation, we have

$$\sum_{i \in H} w^{\circ i} \le \sum_{j \in F} y^{\circ j} + r.$$

Note that, for each household $i \in H$,

$$p^\circ \cdot w^{\circ i} = M^i(p^\circ) = p^\circ \cdot r^i + \sum_j \alpha^{ij}(p^\circ \cdot y^{\circ j}),$$

by Lemmas 14.1, 17.1, and 24.4 or by Lemma 19.1.

Summing over households,

$$\begin{aligned}\sum_{i\in H} p^\circ \cdot w^{\circ i} = \sum_i M^i(p^\circ) &= \sum_i \left[p^\circ \cdot r^i + \sum_j \alpha^{ij}(p^\circ \cdot y^{\circ j})\right]\\ &= p^\circ \cdot \sum_i r^i + p^\circ \cdot \sum_i \sum_j \alpha^{ij} y^{\circ j}\\ &= p^\circ \cdot \sum_i r^i + p^\circ \cdot \sum_j \sum_i \alpha^{ij} y^{\circ j}\\ &= p^\circ \cdot r + p^\circ \cdot \sum_j y^{\circ j} \ \left(\text{since for each j, } \sum_i \alpha^{ij} = 1\right).\end{aligned}$$

Suppose, contrary to the theorem, there is an attainable Pareto preferable allocation $v^i \in X^i$, $i \in H$, so that $v^i \succeq_i w^{\circ i}$, for all i with $v^h \succ_h w^{\circ h}$ for some $h \in H$. The allocation v^i must be more expensive than $w^{\circ i}$ for those households made better off and no less expensive for the others. Then we have

$$\sum_{i\in H} p^\circ \cdot v^i > \sum_{i\in H} p^\circ \cdot w^{\circ i} = \sum_{i\in H} M^i(p^\circ) = p^\circ \cdot r + p^\circ \cdot \sum_{j\in F} y^{\circ j}.$$

But if v^i is attainable, then there is $y'^j \in Y^j$ for each $j \in F$, so that

$$\sum_{i\in H} v^i = \sum_{j\in F} y'^j + r.$$

But then, evaluating this production plan at the equilibrium prices, p°, we have

$$p^\circ \cdot r + p^\circ \cdot \sum_{j\in F} y^{\circ j} < p^\circ \cdot \sum_{i\in H} v^i = p^\circ \cdot \sum_{j\in F} y'^j + p^\circ \cdot r.$$

So $p^\circ \cdot \sum_{j\in F} y^{\circ j} < p^\circ \cdot \sum_{j\in F} y'^j$. Therefore, for some $j \in F$, $p^\circ \cdot y^{\circ j} < p^\circ \cdot y'^j$.

But $y^{\circ j}$ maximizes $p^\circ \cdot y$ for all $y \in Y^j$; there cannot be $y'^j \in Y^j$ so that $p \cdot y'^j > p \cdot y^{\circ j}$. Hence, $y'^j \notin Y^j$. The contradiction shows that v^i is not attainable. QED

Second Fundamental Theorem of Welfare Economics

2FTWE: Every Pareto efficient allocation (in a convex economy) can be supported as a competitive equilibrium with efficiency prices subject to a reallocation of endowment (redistribution of income).

Reflects the Separating Hyperplane Theorem for disjoint convex sets.

Theorem 8.2 (Separating Hyperplane Theorem) : Let $A, B \subset \mathbf{R}^N$; let A and B be non-empty, convex, and disjoint, that is, $A \cap B = \phi$. Then there is $p \in \mathbf{R}^N$, $p \neq 0$, so that $p \cdot x \geq p \cdot y$ for all $x \in A, y \in B$.

In addition, a minor lemma helps with the technical structure of the proof.

Lemma 19.2 : Assume C.II, C.III, C.IV, C.VI(C). Let $x^\circ \in X^i$. Then there is $x^\nu \in X^i, \nu = 1, 2, 3, \ldots, x^\nu \succ_i x^\circ$, so that $x^\nu \to x^\circ$.

Recall $A^i(x^i) \equiv \{x \mid x \in X^i, x \succeq_i x^i\}$. Under the assumptions of convexity and continuity of preferences, $A^i(x^i)$ is a closed convex set. Starting from the allocation $x^i, i \in H$, we can take the sum of sets $\sum_{i \in H} A^i(x^i)$; this sum, called A, is also a convex set and represents the set of aggregate consumptions preferred or indifferent to x^i. Consider a subset of A that includes aggregate consumptions strictly preferred to x^i (approximately the interior of A). Let us denote this set by $\mathcal{A}$, which is also a convex set. A point in $\mathcal{A}$ represents an aggregate consumption mix that can provide an allocation Pareto preferable to x^i, $i \in H$. The set of aggregate attainable allocations is the (coordinate-wise) nonnegative elements of $Y + \{r\}$. We will denote this set as $B = (Y + \{r\}) \cap \mathbf{R}^N_+$, a convex set. Starting from a Pareto efficient allocation x^i, $i \in H$, under monotonicity, the sets $\mathcal{A}$ and B must be disjoint. If not, there would be an attainable Pareto preferable allocation. But this is precisely the setting where we can employ the Separating Hyperplane Theorem. The normal to the separating hyperplane is the price system that decentralizes the efficient allocation. The existence of such a price system is the import of Theorem 19.2.

Theorem 19.2 : Assume P.I and C.I– C.V,C.VI(C). Let x^{*i}, y^{*j}, $i \in H$, $j \in F$, be an attainable Pareto efficient allocation. Then there is $p \in \mathbf{R}^N, p \neq 0$ so that

(i) x^{*i} minimizes $p \cdot x$ on $A^i(x^{*i})$, $i \in H$, and
(ii) y^{*j} maximizes $p \cdot y$ on Y^j, $j \in F$.

Proof : Let $x^* = \sum_{i \in H} x^{*i}$, and let $y^* = \sum_{j \in F} y^{*j}$. Note that $x^* \leq y^* + r$ (the inequality applies coordinatewise). Let $A = \sum_{i \in H} A^i(x^{*i})$. Let $B = Y + \{r\}$. A and B are convex sets. Let $\mathcal{A} = \sum_{i \in H}\{x \mid x \in X^i, x \succ_i x^{*i}\} = \sum_{i \in H}\{X^i \backslash G^i(x^{*i})\}$, a convex set whose closure is A (by Lemma 19.2). Set $\mathcal{A}$ represents aggregate consumption bundles that can provide an allocation that is a Pareto improvement over x^{*i}, $i \in H$. $\mathcal{A}$ and B are disjoint. x^* is an element of A but x^* is not interior to A or B. By the Separating Hyperplane Theorem, there is a normal $p \in \mathbf{R}^N, p \neq 0$, so that

$$p \cdot x \geq p \cdot v \text{ for all } x \in \mathcal{A} \text{ and all } v \in B.$$

By continuity of preferences and continuity of the dot product we have also $p \cdot x \geq p \cdot v$ for all $x \in A$ and all $v \in B$. But $x^* \leq y^* + r$, $p \geq 0$. So $p \cdot x^* \leq p \cdot (y^* + r)$. Then x^* minimizes $p \cdot x$ on A and

$(y^* + r)$ maximizes $p \cdot v$ on B. . However, x^* is the sum of many elements, one for each of $A^i(x^{*i})$, $i{\in}H$, and y^* is the sum of many elements, one for each Y^j, $j \in F$. Then the additive structure of A and B implies that x^{*i} minimizes $p \cdot x$ on $A^i(x^{*i})$ and y^{*j} maximizes $p \cdot y$ on Y^j. That is,

$$p \cdot x^* = \min_{x \in A} p \cdot x = \min_{x^i \in A^i(x^{*i})} p \cdot \sum_{i \in H} x^i = \sum_{i \in H} \Big(\min_{x \in A^i(x^{*i})} p \cdot x \Big),$$

and

$$p \cdot (r + y^*) = \max_{v \in B} p \cdot v = p \cdot r + \sum_{j \in F} \Big(\max_{y^j \in Y^j} p \cdot y^j \Big).$$

So x^{*i} minimizes $p \cdot x$ for all $x \in A_i(x^{*i})$ and y^{*j} maximizes $p \cdot y$ for all $y \in Y^j$. QED

Corollary 19.1 = 2FTWE, says that the supporting prices introduced in Theorem 19.2 can be used, along with a suitably chosen redistribution of endowment, to support any chosen efficient allocation as an equilibrium.

Corollary 19.1 (Second Fundamental Theorem of Welfare Economics) : Assume P.I and C.I–C.V, C.VI(C). Let x^{*i}, y^{*j} be an attainable Pareto efficient allocation. Then there is $p \in \mathbf{R}^N, p \neq 0$ and $\hat{r}^i \in \mathbf{R}^N$, $\hat{r}^i \geq 0$, $\hat{\alpha}^{ij} \geq 0$, so that

$$\sum_{i \in H} \hat{r}^i = r,$$

$$\sum_{i \in H} \hat{\alpha}^{ij} = 1 \text{ for each j},$$

$$p \cdot y^{*j} \quad \text{maximizes} \quad p \cdot y \text{ for } y \in Y^j,$$

and

$$p \cdot x^{*i} = p \cdot \hat{r}^i + \sum_{j \in F} \hat{\alpha}^{ij} (p \cdot y^{*j}).$$

Further, for each $i \in H$, one of the following properties holds:

CASE 1 $(p \cdot x^{*i} > \min_{x \in X^i} p \cdot x) : x^{*i} \succeq_i x$ for all $x \in X^i$ so that

$$p \cdot x \leq p \cdot \hat{r}^i + \sum_{j \in F} \hat{\alpha}^{ij}(p \cdot y^{*j}), or$$

CASE 2 $(p \cdot x^{*i} = \min_{x \in X^i} p \cdot x)$: x^{*i} minimizes $p \cdot x$ for all x so that $x \succeq_i x^{*i}$.

Proof : Applying Theorem 19.2, we have $p \in \mathbf{R}^N, p \neq 0$ so that y^{*j} maximizes $p \cdot y$ for all $y \in Y^j$ and so that x^{*i} minimizes $p \cdot x$ for all $x \in A^i(x^{*i})$. We must show two properties, (1) that $\hat{r}^i$, $\hat{\alpha}^{ij}$ can be found fulfilling the above equations and inequalities, and (2) that household behavior can be characterized as utility optimization subject to budget constraint in Case 1 and as cost minimization subject to utility level in Case 2.

By attainability of the allocation, we have

$$\sum_{i \in H} x^{*i} \leq \sum_{j \in F} y^{*j} + r.$$

Commodities k in which the strict inequality holds will have $p_k = 0$. Multiplying through by p, we have

$$\sum_{i \in H} p \cdot x^{*i} = \sum_{j \in F} p \cdot y^{*j} + p \cdot r.$$

But then it is merely simple arithmetic to find suitable $\hat{r}^i$, $\hat{\alpha}^{ij}$. A simple choice (one of many possible) is to let

$$\lambda^i = \frac{p \cdot x^{*i}}{\sum_{h \in H} p \cdot x^{*h}}$$

and set $\hat{r}^i = \lambda^i r$, $\hat{\alpha}^{ij} = \lambda^i$, for all $i \in H$, $j \in F$.

On the consumer side now, we wish to show that cost minimization subject to a utility constraint is equivalent to utility maximization subject to a budget constraint in Case 1. This follows from non-satiation and convexity of preferences, CVI(C). Suppose, on the contrary, there is x'^i so that $p \cdot x'^i \leq p \cdot x^{*i}$ and $x'^i \succ_i x^{*i}$. We will show that this leads to a contradiction. Since this is case 1, there is $\hat{x} \in X^i$, so that $\hat{x}$ is both less expensive and less desirable than x^{*i}. That is, $x^{*i} \succ_i \hat{x}$, $p \cdot x^{*i} > p \cdot \hat{x}$. By C.III, the points along the chord between x'^i and $\hat{x}$ are elements of X^i. All the points interior to the chord are less expensive than x^{*i} . That is, under C.VI(C) and C.V, there is α, $0 < \alpha < 1$, so that $[\alpha\hat{x} + (1-\alpha)x'^i] \sim_i x^{*i}$ and $p \cdot [\alpha\hat{x} + (1-\alpha)x'^i] < p \cdot x^{*i}$. But then, $[\alpha\hat{x} + (1-\alpha)x'^i] \in A^i(x^{*i})$ and $p \cdot [\alpha\hat{x} + (1-\alpha)x'^i] < p \cdot x^{*i}$, contradicting the result of Theorem 19.2, that x^{*i} is the minimizer of $p \cdot x$ in $A^i(x^{*i})$. The contradiction proves the result.

The assertion for Case 2 is merely a restatement of the property shown in Theorem 19.2.

QED

Case 1 in the proof (presumably the most common), occurs when the household expenditure at the efficient allocation exceeds the minimum level in the consumption set. Then the household is a utility maximizer subject to budget constraint. Case 2 occurs when the efficient allocation attributes expenditure to the household equal the minimum in its consumption set. In that case the household is an expenditure minimizer subject to utility constraint. Restricting attention to interior allocations would eliminate this complexity by confining attention to Case 1.

The Second Fundamental Theorem of Welfare Economics represents a significant defense of the market economy's resource allocation mechanism. It says (assuming convexity of tastes and technology) that any efficient allocation of resources can be decentralized using the price mechanism, subject to an initial redistribution of endowment.