II. The Arrow-Debreu Model of Competitive Equilibrium - Definition and Existence
B. Mathematics Refresher: Correspondences – Point to set mappings

Overview: There are many settings where the economic response to prices may not be unique, but rather set-valued. When two perfect substitute consumptions are priced equally, the household demand is for any mix of the two goods totalling its desired quantity. When a linear production technology is confronted by prices generating a zero profit at a range of outputs, the firm's response is a range of output levels including zero. In this setting, economic behavior is modeled as a point-to-set mapping, traditionally known to economists as a ***correspondence***. The related continuity concepts are ***upper*** and ***lower hemicontinuity***. The analog to the Brouwer Fixed Point Theorem, for a convex-valued upper hemicontinuous correspondence is the Kakutani Fixed Point Theorem. The Theorem of the Maximum provides sufficient conditions to describe optimizing behavior as an upper hemicontinuous correspondence.

23.1 Correspondences

We will call a point-to-set mapping a ***correspondence***. Let A and B be nonempty sets. For each $x \in A$ we associate a ***nonempty*** set $\beta \subset B$ by a rule φ. Then we say $\beta = \varphi(x)$ and φ is a correspondence; $\varphi : A \to B$. Note that if $x \in A$

and $y \in B$ it is meaningless or false to say $y = \varphi(x)$, rather we say $y \in \varphi(x)$. The ***graph*** of the correspondence is a subset of $A \times B : \{(x, y) \mid x \in A, y \in B \text{ and } y \in \varphi(x)\}$.

23.2 Upper hemicontinuity (also known as upper semicontinuity)

Definition : Let $\varphi : S \to T$, φ be a correspondence, and S and T be closed subsets of $\mathbf{R}^N$ and $\mathbf{R}^K$, respectively. Let $x^\nu, x^\circ \in S, \nu = 1, 2, 3, \ldots$; let $x^\nu \to x^\circ$, $y^\nu \in \varphi(x^\nu)$, for all $\nu = 1, 2, 3, \ldots$, and $y^\nu \to y^\circ$. Then φ is said to be ***upper hemicontinuous*** (also known as upper semicontinuous) at x° if and only if $y^\circ \in \varphi(x^\circ)$.

Example 23.1 <u>An upper hemicontinuous correspondence</u>. Let $\varphi(x)$ be defined as follows. $\varphi : \mathbf{R} \to \mathbf{R}$. For

$$x < 0, \varphi(x) = \{y \mid x - 4 \le y \le x - 2\}$$
$$x = 0, \varphi(x) = \{y \mid -4 \le y \le +4\}$$
$$x > 0, \varphi(x) = \{y \mid x + 2 \le y \le x + 4\}.$$

Note that $\varphi(\cdot)$ is convex valued. For each $x \in \mathbf{R}, \varphi(x)$ is a convex set. (See Figure 23.2)

Example 23.2 <u>A correspondence not upper hemicontinuous at 0</u>. Let $\varphi(x)$ be defined much as in Example 23.1 but with a discontinuity at 0. $\varphi : \mathbf{R} \to \mathbf{R}$. For

$$x < 0, \varphi(x) = \{y \mid x - 4 \leq y \leq x - 2\}$$
$$x = 0, \varphi(0) = \{0\}$$
$$x > 0, \varphi(x) = \{y \mid x + 2 \leq y \leq x + 4\}.$$

Note that $\varphi(\cdot)$ is convex valued. For each $x \in \mathbf{R}, \varphi(x)$ is a convex set. (See Figure 23.3)

Theorem 23.1 : φ is upper hemicontinuous if and only if its graph is closed in $S \times T$.

23.3 Lower hemicontinuity (also known as lower semicontinuity)

Definition : Let $\varphi : S \to T$, where S and T are closed subsets of $\mathbf{R}^N$ and $\mathbf{R}^K$, respectively. Let $x^\nu \in S$, $x^\nu \to x^\circ, y^\circ \in \varphi(x^\circ), \nu = 1, 2, 3, \ldots$. Then φ is said to be lower hemicontinuous (also known as lower semicontinuous) at x° if and only if there is $y^\nu \in \varphi(x^\nu)$, $y^\nu \to y^\circ$. Lower hemicontinuity asserts the presence of a sequence of points in the correspondence evaluated at a convergent sequence of points in the domain.

Intuitively, φ is lower hemicontinuous if, when it has caught a value, φ must be able to sneak up on it.

Example 23.3 : A lower hemicontinuous correspondence. Let $\varphi(x)$ be defined as follows. $\varphi : \mathbf{R} \to \mathbf{R}$. For

$$x \neq 0,\ \varphi(x) = \{y \mid x - 4 \leq y \leq x\}$$
$$x = 0,\ \varphi(x) = \{y \mid -3 \leq y \leq -1\}.$$

The graph of $\varphi(\cdot)$ is shown in Figure 23.4. Note that $\varphi(\cdot)$ is convex valued. For each $x \in \mathbf{R}$, $\varphi(x)$ is a convex set. For all $x^\circ \in \mathbf{R}$, $\varphi(\cdot)$ is lower hemicontinuous at x°. The only point where this requires some care is at $x^\circ = 0$. Let $x^\nu \to 0$, $y^\circ \in \varphi(0)$. To demonstrate lower hemicontinuity, we must show that there is $y^\nu \in \varphi(x^\nu)$ so that $y^\nu \to y^\circ$. Note that $-3 \leq y^\circ \leq -1$. But for ν large, there is $y^\nu \in \varphi(x^\nu)$, so that $y^\nu = y^\circ$. Hence, trivially, $y^\nu \to y^\circ$. Note that $\varphi(\cdot)$ is not upper hemicontinuous at $x^\circ = 0$. This follows simply because $y = -4$ is the limit of a sequence of values in $\varphi(x^\nu)$ but $-4 \notin \varphi(0)$.

Example 23.4 An upper hemicontinuous correspondence that is not lower hemicontinuous. This example is merely Examples 23.1 and 23.2 revisited. $\varphi(\cdot)$ in both Examples 23.1 and 23.2 is not lower hemicontinuous at $x^\circ = 0$. In both cases $0 \in \varphi(0)$ but for a typical sequence $x^\nu \to 0$, there is no $y^\nu \in \varphi(x^\nu)$ so that $y^\nu \to 0$.

23.4 Continuous correspondence

Definition : Let $\varphi : A \to B$, with φ a correspondence. $\varphi(\cdot)$ is said to be continuous at x° if $\varphi(\cdot)$ is both upper and lower hemicontinuous at x°.

Example 23.5 A continuous correspondence. The following correspondence, $\varphi(\cdot)$, is both upper and lower hemicontinuous throughout its range and hence is a continuous correspondence. For

$$x < 0, \varphi(x) = \{y \mid 2x \le y \le -x\}$$
$$x = 0, \varphi(x) = \{0\}$$
$$x > 0, \varphi(x) = \{y \mid -2x \le y \le -x\} \cup \{y \mid 3x \le y \le 4x\}.$$

Note that if φ is point valued (i.e., a function) with a compact range then upper hemicontinuity, continuity (in the sense of a function), and lower hemicontinuity are equivalent.

23.6 Optimization subject to constraint: Composition of correspondences; the Maximum Theorem

Maximization subject to constraint Let $f(\cdot)$ be a real-valued function, and let $\varphi(\cdot)$ be a correspondence intended to represent an opportunity set. Then we let $\mu(\cdot)$ represent the correspondence consisting of the maximizers of $f(\cdot)$ subject to choosing the maximizer in the opportunity set $\varphi(\cdot)$. Formally, we state

The Maximum Problem : Let $T \subseteq \mathbf{R}^N, S \subseteq \mathbf{R}^M, f : T \to \mathbf{R}$, and $\varphi : S \to T$, where φ is a correspondence, and let $\mu : S \to T$, where $\mu(x) \equiv \{y^\circ \mid y^\circ \text{ maximizes } f(y) \text{ for } y \in \varphi(x)\}$. (See Figure 23.6)

Theorem 23.3 (The Maximum Theorem) Let $f(\cdot), \varphi(\cdot)$, and $\mu(\cdot)$ be as defined in the Maximum Problem. Let f be continuous on T and let φ be continuous (both upper and lower hemicontinuous) at x° and compact-valued in a neighborhood of x°. Then μ is upper hemicontinuous at x°.

Example 23.6 Applying the Maximum Theorem. Let $S = T = \mathbf{R}$. Let $f(y) = y^2$. Let

$$\varphi(x) = \{y \mid -x \leq y \leq x\} \text{ for x} \geq 0$$
$$\varphi(x) = \{y \mid x \leq y \leq -x\} \text{ for x} < 0.$$

Then $\mu(x) = \{x, -x\}$, since $\mu(x)$ is the set of maximizers of y^2 for $y \in \varphi(x)$. Note that $\varphi(x)$ is both upper and lower hemicontinuous throughout $\mathbf{R}$ and is convex valued. $\mu(x)$ is upper hemicontinuous by the Maximum Theorem. It is not, however, convex valued.

23.7 Kakutani Fixed-Point Theorem

Theorem 23.4 (Kakutani Fixed-Point Theorem) Let S be an N-simplex. Let $\varphi : S \to S$ be a correspondence that is upper hemicontinuous everywhere on S. Further, let $\varphi(x)$ be a convex set for all $x \in S$. Then there is $x^* \in S$ so that $x^* \in \varphi(x^*)$.

Example 23.7 Applying the Kakutani Fixed-Point Theorem. Let $\varphi : [0,1] \rightarrow [0,1]$. Let

$$\begin{aligned}
\varphi(x) &= \{1 - x/2\} \text{ for } 0 \leq x < .5 \\
\varphi(0.5) &= [.25, .75] \\
\varphi(x) &= \{x/2\} \text{ for } 1 \geq x > .5,
\end{aligned}$$

where φ is upper hemicontinuous and convex valued. The fixed point is $x^\circ = 0.5$. (See Figure 23.10.)

Corollary 23.1 Let $K \subseteq \mathbf{R}^M, K \neq \emptyset$, be compact and convex. Let $\Psi : K \rightarrow K$, with $\Psi(x)$ upper hemicontinuous and convex valued for all $x \in K$. Then there is $x^* \in K$ so that $x^* \in \Psi(x^*)$.