21.1. The core consists of the Pareto efficient individually ratational points. We denote the allocation to household 1 as (x^1, y^1) and to household 2 as (x^2, y^2). Efficiency requires $\frac{u_x^1}{u_y^1} = \frac{y^1+1}{x^1+1} = \frac{u_x^2}{u_y^2} = \frac{y^2+1}{x^2+1}$. Individual rationality requires that $u^1 \geq 10$, $u^2 \geq 10$. Hence the core is the segment of the diagonal of the Edgeworth box between $(x^1, y^1) = (9,9)$ and $(x^2, y^2) = (90, 90)$ and $(x^1, y^1) = (90,90)$ and $(x^2, y^2) = (9,9)$.

22.1. (a) For Q=1, the core allocations consist of all individually rational Pareto efficient allocations. To check for individual rationality note that

$u^1(10,10) = 10 > u^1(e^1) = \sqrt{99}$; $u^2(90, 90) = 90 > u^2(e^2) = \sqrt{99}$.

To check for Pareto efficiency, recall that the first order condition is equality of the household MRS's (see *General Equilibrium Theory*, section 1.3). Second order conditions are fulfilled by concavity of the utility functions.

$$\frac{\frac{\partial u^1}{\partial x}}{\frac{\partial u^1}{\partial y}} = \frac{y^1}{x^1} = 1 \; ; \; \frac{\frac{\partial u^2}{\partial x}}{\frac{\partial u^2}{\partial y}} = \frac{y^2}{x^2} = 1$$

. So equality of MRS's is satisfied.   Hence, the allocation is in the core for Q = 1.

(b) For Q = 2, we must construct a blocking coalition. Let the coalition S consist of two agents of type 1 and one of type 2. The total endowment of the coalition then is (199, 101). The only requirement of a blocking allocation is that it be feasible for the coalition and that it be utility improving in the coalition. Consider the following allocation:

To each agent of type 1, $a^{11} = a^{12} =$ (16, 8),

To the type 2 agent, a^{21} = (167, 85) .

We have then, $u^1(a^{11}) = u^1(a^{12}) = 8\sqrt{3} > 10$. We have $u^2(a^{21}) = 119 > 90$. Hence S is a blocking coalition and the allocation in part a is blocked for Q =2. This coalition illustrates the increasing variety of coalitions (the increasing variety of the proportionate composition of coalitions) available as the economy becomes large.

This example of a blocking coalition is typical of the process of eliminating the extreme elements of the core as the economy becomes large. The blocking coalition has unequal proportions of the members of each type, representing the flexibility in the composition of coalitions available as the economy becomes larger. Note that since the allocation is blocked for Q =2 it is blocked for all Q>2 as well.

c. A competitive equilibrium, equal treatment, allocation will be in the core for arbitrarily large Q. The following allocation is a CE.

To each agent of type 1: (50, 50)

To each agent of type 2:   (50, 50).

This allocation would be supported as a competitive equilibrium with prices ($^1/_2$, $^1/_2$).

22.2 (a) For each household of type i, let x^i denote the competitive equilibrium consumption in the original economy at equilibrium prices p^o. But in the Q-fold replica economy, at p^o, x^i is still optimizing for household's of type i, subject to budget. Let $x^{i,q}$ denote equilibrium consumption plans for the q^{th} household of type i, q = 1, …, Q, in the Q-fold replica economy. Then $x^{i,q} = x^i$. For the original economy, we have $\sum_{i \in H} x^i = \sum_{i \in H} r^i$. But then $\sum_{i \in H} \sum_{q=1}^{Q} x^{i,q} = \sum_{i \in H} Qx^i = Q\sum_{i \in H} r^i$ and we have market clearing for the Q-fold replica economy at the original equilibrium prices, p^o.

(b) The allocation is a competitive equilibrium with prices (.5, .5). Hence it is in the core.

(c) Utility levels at the initial endowment are 0. The allocation (9, 9), (1, 1) is attainable and individually preferable to the zero utility of endowment. Hence it is in the core for the original economy.

To demonstrate that the allocation is not in the core for a 2-fold replica economy we form a blocking coalition of two of type 2 and one of type 1. The coalition can attain

to type 1: (16, 8)

to the type 2's: (2, 1), (2, 1)

This allocation is attainable for the coalition and dominates a^1, a^2. Hence it blocks and a^1, a^2, is not in the core.

22.3 (a) $u^1(15,15) = 225 > 196 = u^1(98, 2)$, $u^2(85,85) = 7225 > u^2(2, 98)$. The allocation is Pareto efficient. Hence unblocked. So the allocation is in the core for Q = 1.

(b) Consider a coalition of two of type 1 and one of type 2. This coalition can achieve (30, 10) to each of the type 1's and (138, 82) to the type 2, resulting in respective utilities of 300, 11316. These utilities dominate the utilities in (a). Hence the allocation in (a) is blocked. This is a simple example of the increased scope for bargaining as the number of replications becomes large. Of course the blocking allocation need not be Pareto efficient, even within the attainable set of the blocking coalition.

(c) The competitive equilibrium is always in the core for all Q. All agents of each type get (50, 50).

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4.1 (90, 90); (10, 10) is Pareto efficient. $u^1(10,10) = 100 > 99 = u^1(r^1)$ so the allocation is unblocked for Q = 1.

For Q = 2, form the coalition of two of type 1 and one of type 2. Then the coalition can achieve the allocation (20, 10), (20, 10), (158, 81), leading to the utilities 200, 200, 12798 > 8100 = $u^2(90, 90)$. So the old allocation is blocked for Q = 2. Note that the blocking allocation need not fulfill equal treatment nor Pareto efficiency within the blocking allocation.

4.2 Let $\{x^h\}$ denote an equal-treatment allocation blocked in Q×H by coalition $S \subseteq Q \times H$. We know all core allocations are equal-treatment, so those are the only allocations we need to consider in the core. But $S \subseteq (Q+1) \times H$ also, so S blocks $\{x^h\}$ in (Q+1)×H. Thus, any allocation blocked in Q×H is blocked in (Q+1)×H. The set of equal treatment allocations unblocked is decreasing in Q, so the $\text{Core}((Q+1)\times H) \subseteq \text{Core}(Q\times H)$.