2.9

$$V = u(x,y) + a(x-f(L^x,T^x)) + b(y-g(L^y,T^y)) + c(L^o-L^x-L^y) + d(T^o-T^x-T^y) \qquad (1)$$

(a) ***Suggested Answer*****:**

$$\frac{\partial V}{\partial x} = \frac{\partial u}{\partial x} + a = 0 \qquad (2)$$

$u_x = -a$

$$\frac{\partial V}{\partial y} = \frac{\partial u}{\partial y} + b = 0 \qquad (3)$$

$u_y = -b$

$$\frac{\partial V}{\partial T^x} = -a\frac{\partial f}{\partial T^x} - d = 0 \qquad (4)$$

$$\frac{\partial V}{\partial T^y} = -b\frac{\partial g}{\partial T^y} - d = 0 \qquad (5)$$

$$\frac{\partial V}{\partial L^x} = -a\frac{\partial f}{\partial L^x} - c = 0 \qquad (6)$$

$$\frac{\partial V}{\partial L^y} = -b\frac{\partial g}{\partial L^y} - c = 0 \qquad (7)$$

(b) $\frac{g_L}{g_T} = \frac{-c/b}{-d/b} = \frac{c}{d} = \frac{-c/a}{-d/a} = \frac{f_L}{f_T}$

(c) $\frac{f_L}{g_L}$ is the additional x available from a marginal reallocation of L to f divided by the additional y available from the same marginal reallocation to g. Hence the marginal rate of transformation.

(d)

$$\frac{u_x}{u_y} = \frac{a}{b} = \frac{-d/b}{-d/a} = \frac{g_T}{f_T}$$

$$\frac{u_x}{u_y} = \frac{a}{b} = \frac{-c/b}{-c/a} = \frac{g_L}{f_L}$$

(e) There is a typo in the problem. The correct statement is $w=p^x f_L=p^y g_L$; $r=p^x f_T=p^y g_T$.

From this we get
$$\frac{p^x}{p^y}=\frac{u_x}{u_y}=\frac{a}{b}=\frac{-d/b}{-d/a}=\frac{g_T}{f_T}$$
$$\frac{p^x}{p^y}=\frac{u_x}{u_y}=\frac{a}{b}=\frac{-c/b}{-c/a}=\frac{g_L}{f_L}$$

Problem 3.5 —- Suggested Answer

3.5 Consider an Edgeworth box (two households, A and B, two goods, x and y).

Household A is characterized as:

(a) endowment = (10, 0), ten units of x and zero of y;

(b) $U^A(x^A, y^A) = x^A + 4y^A$; A likes y four times as much as A likes x.

Household B is characterized as:

(a) endowment = (0, 10), ten units of y and zero of x;

(b) $U^B(x^B, y^B) = 5x^B + y^B$; B likes x five times as much as B likes y.

For both households, the two goods are perfect substitutes with MRS's respectively of (1/4) and 5.

(i) Draw an Edgeworth box for this economy. Show the endowment point, contract curve, competitive equilibrium (a) and the set of Pareto efficient points. Because of the linear preferences, the Pareto efficient set will not be a locus of smooth tangencies - - - don't bother differentiating anything. Show that $(x^A, y^A) = (0, 10)$, $(x^B, y^B) = (10, 0)$ is a competitive equilibrium.

Suggested Answer: The set $\{(x^A, y^A) = (10, C), (x^B, y^B) = (0, 10-C); (x^A, y^A) = (10-C, 0), (x^B, y^B) = (C, 10) | 0 \leq C \leq 10\}$ is the Pareto efficient set. The subset with $y^A \geq 2.5$, $x^B \geq 2$ is the contract curve. Set $(p_x, p_y) = (1/2, 1/2)$. Then $(x^A, y^A) = (0, 10)$, $(x^B, y^B) = (10, 0)$ fulfills budget constraint, market clearing, is maximal subject to budget constraint and nonnegativity for each household. So $(x^A, y^A) = (0, 10)$, $(x^B, y^B) = (10, 0)$ is a competitive equilibrium allocation.

(ii) Some writers would argue that:

the contract curve for this economy is equivalent to the set of competitive equilibria. That is, any individually rational Pareto efficient point in this Edgeworth box can be supported as a competitive equilibrium. These 'competitive equilibrium' allocations would include those of the form

$$\begin{aligned} (x^A, y^A), 2.5 &< y^A \leq 10, x^A = 0; \\ (x^B, y^B), x^B &= 10, y^B = 10 - y^A. \end{aligned}$$

Explain the reasoning for this argument (hint: think inside the box).

Suggested Answer: Supporting prices are $(p_x, p_y) = (\frac{\mathrm{y^A}}{10}, 1 - \frac{\mathrm{y^A}}{10})$. A budget line from the endowment point to the suggested allocation neatly separates the upper contour sets of the two households. For each household the suggested allocation is maximal subject to (1) budget constraint, (2) nonnegativity of own consumption, (3) nonnegativity of other household's consumption.

The assertion is false. Explain why it is mistaken (hint: think outside the box).

Suggested Answer: Point (3) in the argument above should not enter into the household's optimization. It should only optimize subject to points (1) and (2). At the posted prices, household B would like more x and less y, creating an excess demand for x, a disequilibrium.

Problems 4.7 and 4.8 are based on the following model. Consider the production of goods x and y in a competitive economy with two factors of production, land denoted T, and labor denoted L. Assume all functions are differentiable. Assume interior solutions (no boundary solutions). The available supply of labor is L^0. The available supply of land is T^0.

Good x is produced in a single firm, called firm x, by the production function $f(L^x, T^x) = x$, where L^x is L used to produce x, T^x is T used to produce x. $f(L^x, T^x) \geq 0$ for $L^x \geq 0$, $T^x \geq 0$; $f(0, 0) = 0$.

Good y is produced in a single firm by the production function $g(L^y, T^y) = y$ where L^y is L used to produce y, T^y is T used to produce y. $g(L^y, T^y) \geq 0$ for $L^y \geq 0$, $T^y \geq 0$; $g(0, 0) = 0$.

The resource constraints of the economy are

$$L^x + L^y = L^0$$
$$T^x + T^y = T^0.$$

The allocation of L and T is said to be technically efficient if there is no reallocation of L and T across firms that would increase the output of y without reducing the output of x. Technical efficiency is a necessary condition for Pareto efficiency. We'll characterize technical efficiency as maximizing the output of y for a given level of output of x. That is, choose L^y, T^y to maximize $g(L^y, T^y)$ subject to

$$f(L^x, T^x) = X^0$$
$$L^x + L^y = L^0$$
$$T^x + T^y = T^0.$$

Restating the problem as choosing L^y, T^y to maximize $g(L^y, T^y)$ subject to $f(L^0 - L^y, T^0 - T^y) = X^0$. The Lagrangian for this problem can be stated as $M = g(L^y, T^y) - \lambda[f(L^0 - L^y, T^0 - T^y) - X^0]$. Differentiating M with respect to L^y and T^y (letting subscripts denote partial derivatives) and setting the result equal to 0, we have

$$\frac{\partial M}{\partial L^y} = g_L - \lambda f_L = 0 \tag{4.10}$$

$$\frac{\partial M}{\partial T^y} = g_T - \lambda f_T = 0.(corrected) \tag{4.11}$$

These are first-order conditions for technical efficiency in this model.

4.7 Firm x's marginal rate of technical substitution of L for T is defined as $MRTS^x_{LT} = \frac{f_T}{f_L}$. Show that technical efficiency requires that the firms' respective $MRTS$'s be equated. That is, show that at a technically efficient allocation of T and L, $\text{MRTS}^x_{LT} = \frac{f_T}{f_L} = \frac{g_T}{g_L} = MRTS^y_{LT}$.It is a well established result that at a competitive equilibrium
$(r/w) = \frac{f_T}{f_L} = \frac{g_T}{g_L}$,
where w is the wage rate on L and r is the rental rate on T. Thus you have just shown that a competitive equilibrium allocation is (or fulfills a necessary condition for being) technically efficient.

Suggested Answer: Restate 4.10 and 4.11 as
$g_L = \lambda f_L$
$g_T = \lambda f_T$
and divide through to get
$\frac{g_L}{g_T} = \frac{f_L}{f_T}$.

4.8 Let a typical household utility function be $u(x, y)$. u_x and u_y denote marginal utilities, partial derivatives of u with respect to x and y. The marginal cost of x at a competitive equilibrium is $(w/f_L) = (r/f_T)$. As usual in competitive equilibrium price equals marginal cost. Let p_x be the price of x, p_y be the price of y. We have $p_x = (w/f_L) = (r/f_T)$, $p_y = (w/g_L) = (r/g_T)$. The marginal rate of transformation of x for y (also known as the rate of product transformation of x for y) is $(g_L/f_L) = (g_T/f_T)$. It represents the (absolute value of the) slope of the production frontier — the additional volume of y that can be achieved by sacrificing a unit of x. From chapter 3 we have $(u_x/u_y) = (p_x/p_y)$ in competitive equilibrium. We established in chapter 2 (in the special case where $f_L = 1$; you may assume that it generalizes) that a necessary condition for Pareto efficiency is

$$(g_L/f_L) = (u_x/u_y) \tag{4.12}$$

— marginal rate of substitution equals marginal rate of transformation. Show that (4.12) is fulfilled in the competitive equilibrium of this model. Thus you've shown that competitive equilibrium in a two-good economy fulfills a necessary condition for Pareto efficiency.

Suggested Answer: $(u_x/u_y) = (p_x/p_y) = \frac{(w/f_L)}{(w/g_L)} = g_L/f_L$.

Suggested Answer, September 2004, Part 2

1. The first order condition is u_x / u_y = $p^x/p^y = 1$. So that is fulfilled at (50, 50). Yes the allocation is locally at marginal cost. Pareto efficiency is a bit tricky since the production conditions are not concave (there is increasing marginal product; concavity requires diminishing marginal product). We can do a quick check for efficiency by looking for a utility improvement at nearby points. (35, 75) is possible, where u(35, 75) = 2625 > 2500 = u(50, 50). So the allocation is not Pareto efficient.
2. Since the firms have diminishing marginal costs, as price takers, the firms will find increasing production in the region of diminishing costs, above 55 units, profitable.
3. No. The Second Fundamental Theorem requires convexity, and the scale economy depicted here is a non-convexity.
4. Yes. The first order condition in question 1 is still fulfilled.
5. By allocating all of *L* to producing good x, we can achieve an allocation of (145, 0) with *v*(145,0) = 145 > *v*(50,50) = 100. But as noted in 2, the allocation (50, 50) is not a competitive equilibrium, so the First Fundamental Theorem does not apply. There is no counterexample.

Suggested Answer, June 2014, Micro Qual, #3

Homotheticity and strict concavity make the problem particularly simple. There is just one set of preferences to deal with --- this could be Robinson Crusoe without production and his identical twin.

(a) Hard to tell if the question purposefully or carelessly omits the assumptions of continuity and nonnegativity. So assume u is a continuous function, that its domain is $\mathbf{R}^m_+$, and that $e^i \geq 0$ (co-ordinatewise). That's definitely sufficient to ensure existence of equilibrium.

(b) No trade requires that the two households have the same MRS's at endowment. But we know they have identical homothetic preferences, so it is sufficient that their endowments be linear multiples of each other. That is $e^1 = ke^2$ some $k > 0$.

(c) This is just a redistribution of endowment. The conditions in (a) are sufficient for existence of equilibrium and the First Fundamental Theorem of Welfare Economics applies, so the allocation is Pareto efficient.