Lecture Notes for January 27, 2010: Households

12.1 The structure of household consumption sets and preferences
Households are elements of the finite set $H$ numbered $1, 2, \ldots, \#H$. A household $i \in H$ will be characterized by its possible consumption set $X^i \subseteq \mathbb{R}^+_N$, its preferences $\succeq_i$, and its endowment $r^i \in \mathbb{R}^N_+$. 

12.2 Consumption sets
(C.I) $X^i$ is closed and nonempty.
(C.II) $X^i \subseteq \mathbb{R}^+_N$. $X^i$ is unbounded above, that is, for any $x \in X^i$ there is $y \in X^i$ so that $y > x$, that is, for $n = 1, 2, \ldots, N, y_n \geq x_n$ and $y \neq x$.
(C.III) $X^i$ is convex.

$$X = \sum_{i \in H} X^i.$$ 

12.2.1 Preferences
Each household $i \in H$ has a preference quasi-ordering on $X^i$, denoted $\succeq_i$. For typical $x, y \in X^i$, "$x \succeq_i y$" is read "$x$ is preferred or indifferent to $y$ (according to $i$)." We introduce the following terminology:

If $x \succeq_i y$ and $y \succeq_i x$ then $x \sim_i y$ ("$x$ is indifferent to $y$"),
If $x \succeq_i y$ but not $y \succeq_i x$ then $x \succ_i y$ ("$x$ is strictly preferred to $y$").

We will assume $\succeq_i$ to be complete on $X^i$, that is, any two elements of $X^i$ are comparable under $\succeq_i$. For all $x, y \in X^i$, $x \succeq_i y$, or $y \succeq_i x$ (or both). Since we take $\succeq_i$ to be a quasi-ordering, $\succeq_i$ is assumed to be transitive and reflexive.

The conventional alternative to describing the quasi-ordering $\succeq_i$ is to assume the presence of a utility function $u^i(x)$ so that $x \succeq_i y$ if and only if
$u^i(x) \geq u^i(y)$. We will show below that the utility function can be derived from the quasi-ordering. Readers who prefer the utility function formulation may use it at will. Just read $u^i(x) \geq u^i(y)$ wherever you see $x \succeq_i y$.

12.2.2 Non-Satiation

(C.IV) (Non-Satiation) Let $x \in X^i$. Then there is $y \in X^i$ so that $y \succ_i x$.

12.2.3 Continuity

We now introduce the principal technical assumption on preferences, the assumption of continuity.

(C.V) (Continuity) For every $x^o \in X^i$, the sets
\[ A^i(x^o) = \{x \mid x \in X^i, x \succeq_i x^o\} \]
\[ G^i(x^o) = \{x \mid x \in X^i, x^o \succeq_i x\} \]
are closed.

Example 12.1 (Lexicographic preferences) The lexicographic (dictionary-like) ordering on $\mathbb{R}^N$ (let’s denote it $\succeq_L$) is described in the following way. Let $x = (x_1, x_2, \ldots, x_N)$ and $y = (y_1, y_2, \ldots, y_N)$.
\[ x \succ_L y \text{ if } x_1 > y_1, \text{ or} \]
\[ \quad \text{if } x_1 = y_1 \text{ and } x_2 > y_2, \text{ or} \]
\[ \quad \text{if } x_1 = y_1, x_2 = y_2, \text{ and } x_3 > y_3, \text{ and so forth . . . .} \]
\[ x \sim_L y \text{ if } x = y. \]

$\succeq_L$ fulfills non-satiation, trivially fulfills strict convexity (C.VI(SC), introduced below), but does not fulfill continuity (C.V).

12.2.4 Attainable Consumption

Definition $x$ is an **attainable** consumption if $y + r \geq x \geq 0$, where $y \in \mathcal{Y}$ and $r \in \mathbb{R}_+^N$ is the economy’s initial resource endowment, so that $y$ is an attainable production plan.

Note that the set of attainable consumptions is bounded under P.VI.

12.2.5 Convexity of preferences

(C.VI)(C) (Convexity of Preferences) $x \succ_i y$ implies $((1 - \alpha)x + \alpha y) \succ_i y$, for $0 < \alpha < 1$. 
12.3 Representation of $\succeq_i$: Existence of a continuous utility function

(C.VI)(SC) (Strict Convexity of Preferences): Let $x \succeq_i y$, (note that this includes $x \sim_i y$), $x \neq y$, and let $0 < \alpha < 1$. Then $\alpha x + (1 - \alpha)y >_i y$.

Equivalently, if preferences are characterized by a utility function $u^i(\cdot)$, then we can state C.VI(SC) as

$$u^i(x) \geq u^i(y), x \neq y, \text{ implies } u^i[\alpha x + (1 - \alpha)y] > u^i(y).$$

An immediate consequence of C.VI(C) is that $A^i(x^\circ)$ is convex for every $x^\circ \in X^i$.

12.3 Representation of $\succeq_i$: Existence of a continuous utility function

Definition Let $u^i: X^i \to \mathbb{R}$. $u^i(\cdot)$ is a utility function that represents the preference ordering $\succeq_i$ if for all $x, y \in X^i$, $u^i(x) \geq u^i(y)$ if and only if $x \succeq_i y$. This implies that $u^i(x) > u^i(y)$ if and only if $x >_i y$.

12.3.1 Weak Conditions for Existence of a Continuous Utility Function

Theorem 12.1 Let $\succeq_i, X^i$, fulfill C.I, C.II, C.III, C.V. Then there is $u^i: X^i \to \mathbb{R}, u^i(\cdot)$ continuous on $X^i$, so that $u^i(\cdot)$ is a utility function representing $\succeq_i$.

Proof See Debreu (1959, Section 4.6) or Debreu (1954). QED

12.3.2 Construction of a continuous utility function

Shortcut: use weak desirability, $X^i = R^N_+$ and a 45° line.

12.4 Choice and boundedness of budget sets, $\tilde{B}^i(p)$

Choose $c \in R_+$ so that $|x| < c$ (a strict inequality) for all attainable consumptions $x$. Choose $c$ sufficiently large that $X^i \cap \{x \mid x \in \mathbb{R}^N, c > |x|\} \neq \phi$;

$$\tilde{B}^i(p) = \{x \mid x \in \mathbb{R}^N, p \cdot x \leq \tilde{M}^i(p)\} \cap \{x \mid |x| \leq c\}.$$ 

$$\tilde{D}^i(p) \equiv \{x \mid x \in \tilde{B}^i(p) \cap X^i, x \succeq_i y \text{ for all } y \in \tilde{B}^i(p) \cap X^i\}$$

$$\equiv \{x \mid x \in \tilde{B}^i(p) \cap X^i, x \text{ maximizes } u^i(y) \text{ for all } y \in \tilde{B}^i(p) \cap X^i\}.$$ 

To characterize market demand let

$$\tilde{D}(p) = \sum_{i \in H} \tilde{D}^i(p).$$
Lemma 12.1 \( \bar{B}^i(p) \) is a closed set.

We will restrict attention to models where \( \bar{M}^i(p) \) is homogeneous of degree one, that is, where \( \bar{M}^i(\lambda p) = \lambda \bar{M}^i(p) \). It is immediate then that \( \bar{B}^i(p) \) is homogeneous of degree zero.

Lemma 12.2 Let \( \bar{M}^i(p) \) be homogeneous of degree 1. Let \( \bar{B}^i(p) \) and \( \bar{D}^i(p) \) \( \neq \emptyset \). Then \( \bar{B}^i(p) \) and \( \bar{D}^i(p) \) are homogeneous of degree 0.

\[
P = \left\{ p \mid p \in \mathbb{R}^N, p_n \geq 0, n = 1, 2, 3, \ldots, N, \sum_{n=1}^N p_n = 1 \right\}.
\]

12.4.1 Adequacy of income

(C.VII) For all \( i \in H, \bar{M}^i(p) > \inf_{x \in X^i \cap \{x| |x| \leq c\}} p \cdot x \) for all \( p \in P \).

Example 12.2 [The Arrow Corner]

\[
X^i = \mathbb{R}_+^2,
\]

\[
r^i = (1, 0),
\]

\[
\bar{M}^i(p) = p \cdot r^i.
\]

Let \( p^o = (0, 1) \). Then

\[
\bar{B}^i(p^o) \cap X^i = \{(x, y) \mid c \geq x \geq 0, y = 0\},
\]

the truncated nonnegative \( x \) axis. Consider the sequence \( p^{\nu} = (1/\nu, 1-1/\nu) \).

\[
p^{\nu} \to p^o.
\]

We have

\[
\bar{B}^i(p^{\nu}) \cap X^i = \left\{ (x, y) \mid p^{\nu} \cdot (x, y) \leq \frac{1}{\nu}, (x, y) \geq 0, c \geq |(x, y)| \geq 0 \right\},
\]

\((c, 0) \in \bar{B}^i(p^o)\), but there is no sequence \((x^{\nu}, y^{\nu}) \in \bar{B}^i(p^o)\) so that \((x^{\nu}, y^{\nu}) \to (c, 0)\). On the contrary, for any sequence \((x^{\nu}, y^{\nu}) \in \bar{B}^i(p^o)\) so that \((x^{\nu}, y^{\nu}) = \bar{D}^i(p^{\nu}), (x^{\nu}, y^{\nu})\) will converge to some \((x^*, 0)\), where \( 0 \leq x^* \leq 1 \). For suitably chosen \( \succeq_i \), we may have \((c, 0) = \bar{D}^i(p^o)\). Hence \( \bar{D}^i(p) \) need not be continuous at \( p^o \). This completes the example.

12.5 Demand behavior under strict convexity

Theorem 12.2 Assume C.I–C.V, C.VI(SC), and C.VII. Let \( \bar{M}^i(p) \) be a continuous function for all \( p \in P \). Then \( \bar{D}^i(p) \) is a well-defined, point-valued, continuous function for all \( p \in P \).
12.5 Demand behavior under strict convexity

Proof. \( \tilde{B}^i(p) \cap X^i \) is the intersection of the closed set \( \{ x \mid p \cdot x \leq \tilde{M}^i(p) \} \) with the compact set \( \{ x \mid |x| \leq c \} \) and the closed set \( X^i \). Hence it is compact. It is nonempty by C.VII. Because \( \tilde{D}^i(p) \) is characterized by the maximization of a continuous function, \( u^i(\cdot) \), on this compact nonempty set, there is a well-defined maximum value, \( u^* = u^i(x^*) \), where \( x^* \) is the utility-optimizing value of \( x \) in \( \tilde{B}^i(p) \cap X^i \). We must show that \( x^* \) is unique for each \( p \in P \) and that \( x^* \) is a continuous function of \( p \).

We will now demonstrate that uniqueness follows from strict convexity of preferences (C.VI(SC)). Suppose there is \( x' \in \tilde{B}^i(p) \cap X^i \), \( x' \neq x^* \), \( x' \sim_i x^* \). We must show that this leads to a contradiction. But now consider a convex combination of \( x' \) and \( x^* \). Choose \( 0 < \alpha < 1 \). The point \( \alpha x' + (1-\alpha)x^* \in \tilde{B}^i(p) \cap X^i \) by convexity of \( X^i \) and \( \tilde{D}^i(p) \). But C.VI(SC), strict convexity of preferences, implies that \( [\alpha x' + (1-\alpha)x^*] \sim_i x' \sim_i x^* \). This is a contradiction, since \( x^* \) and \( x' \) are elements of \( \tilde{D}^i(p) \). Hence \( x^* \) is the unique element of \( \tilde{D}^i(p) \). We can now, without loss of generality, refer to \( \tilde{D}^i(p) \) as a (point-valued) function.

To demonstrate continuity, let \( p^\nu \in P, \nu = 1, 2, 3, \ldots, p^\nu \rightarrow p^\circ \). We must show that \( \tilde{D}^i(p^\nu) \rightarrow \tilde{D}^i(p^\circ) \). \( \tilde{D}^i(p^\nu) \) is a sequence in a compact set. Without loss of generality take a convergent subsequence, \( \tilde{D}^i(p^\nu) \rightarrow x^\circ \). We must show that \( x^\circ = \tilde{D}^i(p^\circ) \). We will use a proof by contradiction.

Define

\[ \hat{x} = \arg \min_{x \in X^i \cap \{ y \mid y \in \mathbb{R}^N, c \geq |y| \}} p^\circ \cdot x. \]

The expression “\( \hat{x} = \arg \min_{x \in X^i \cap \{ y \mid y \in \mathbb{R}^N, c \geq |y| \}} p^\circ \cdot x \)” defines \( \hat{x} \) as the minimizer of \( p^\circ \cdot x \) in the domain \( X^i \cap \{ y \mid y \in \mathbb{R}^N, c \geq |y| \} \). \( \hat{x} \) is well defined (though it may not be unique) since it represents a minimum of a continuous function taken over a compact domain.

Now consider two cases. In each case we will construct a sequence \( w^\nu \) in \( X^i \cap \{ y \mid y \in \mathbb{R}^N, c \geq |y| \} \).

Case 1: If \( p^\circ \cdot \tilde{D}^i(p^\circ) < \tilde{M}^i(p^\circ) \) for \( \nu \) large \( p^\nu \cdot \tilde{D}^i(p^\nu) < \tilde{M}^i(p^\nu) \). Then let \( w^\nu = \tilde{D}^i(p^\nu) \).

Case 2: If \( p^\circ \cdot \tilde{D}^i(p^\circ) = \tilde{M}^i(p^\circ) \) then by (C.VII) \( p^\circ \cdot \tilde{D}^i(p^\circ) > p^\circ \cdot \hat{x} \).

Let

\[ \alpha^\nu = \min \left[ 1, \frac{\tilde{M}^i(p^\circ) - p^\circ \cdot \hat{x}}{p^\circ \cdot (\tilde{D}^i(p^\circ) - \hat{x})} \right]. \]

For \( \nu \) large, the denominator is positive, \( \alpha^\nu \) is well defined (this is where C.VII enters the proof), and \( 0 \leq \alpha^\nu \leq 1 \). Let \( w^\nu = (1 - \alpha^\nu) \hat{x} + \alpha^\nu \tilde{D}^i(p^\nu) \). Note that \( \tilde{M}^i(p) \) is continuous in \( p \). The fraction in the definition of \( \alpha^\nu \) is
the proportion of the move from \( \hat{x} \) to \( \tilde{D}^i(p) \) that the household can afford at prices \( p' \). As \( \nu \) becomes large, the proportion approaches or exceeds unity.

Then in both Case 1 and Case 2, \( w^o \to \tilde{D}^i(p) \) and \( w^o \in \tilde{B}^i(p) \cap X^i \). Suppose, contrary to the theorem, \( x^o \neq \tilde{D}^i(p) \). Then \( u^i(x^o) < u^i(\tilde{D}^i(p)) \). But \( u^i \) is continuous, so \( u^i(\tilde{D}^i(p')) \to u^i(x^o) \) and \( u^i(w^o) \to u^i(\tilde{D}^i(p)) \). Thus, for \( \nu \) large, \( u^i(w^o) > u^i(\tilde{D}^i(p')) \). But this is a contradiction, since \( \tilde{D}^i(p') \) maximizes \( u^i(\cdot) \) in \( \tilde{B}^i(p') \cap X^i \). The contradiction proves the result. This completes the demonstration of continuity. QED

Theorem 12.2 gives a family of sufficient conditions for demand behavior of the household to be very well behaved. It will be a continuous (point-valued) function of prices if preferences are continuous and strictly convex and if income is a continuous function of prices and sufficiently positive.

What will household spending patterns look like? What is the value of household expenditures, \( p \cdot \tilde{D}^i(p) \)? There are two significant constraints on \( p \cdot \tilde{D}^i(p) \), budget and length: \( p \cdot \tilde{D}^i(p) \leq \tilde{M}^i(p) \) and \( |\tilde{D}^i(p)| \leq c \). In addition, of course, \( \tilde{D}^i(p) \) must optimize consumption choice with regard to preferences \( \succeq_i \) or equivalently with regard to the utility function \( u^i(\cdot) \). We have enough structure on preferences and the budget set to actually say a fair amount about the character of spending and where \( \tilde{D}^i(p) \) is located. This is embodied in

**Lemma 12.3** Assume C.I–C.V, C.VI(C), and C.VII. Then \( p \cdot \tilde{D}^i(p) \leq \tilde{M}^i(p) \). Further, if \( p \cdot \tilde{D}^i(p) < \tilde{M}^i(p) \) then \( |\tilde{D}^i(p)| = c \).

**Proof** \( \tilde{D}^i(p) \in \tilde{B}^i(p) \) by definition. However, that ensures \( p \cdot \tilde{D}^i(p) \leq \tilde{M}^i(p) \) and hence the weak inequality surely holds. Suppose, however, \( p \cdot \tilde{D}^i(p) < \tilde{M}^i(p) \) and \( |\tilde{D}^i(p)| < c \). We wish to show that this leads to a contradiction. Recall C.IV (Non-Satiation) and C.VI(C) (Convexity). By C.IV there is \( w^* \in X^i \) so that \( w^* \succ_i \tilde{D}^i(p) \). Clearly, \( w^* \notin \tilde{B}^i(p) \) so one (or both) of two conditions holds: (a) \( p \cdot w^* > \tilde{M}^i(p) \), (b) \( |w^*| > c \).

Set \( w' = \alpha w^* + (1 - \alpha)\tilde{D}^i(p) \). There is an \( \alpha(1 > \alpha > 0) \) sufficiently small so that \( p \cdot w' \leq \tilde{M}^i(p) \) and \( |w'| \leq c \). Thus \( w' \in \tilde{B}^i(p) \). Now \( w' \succ_i \tilde{D}^i(p) \) by C.VI(C), which is a contradiction since \( \tilde{D}^i(p) \) is the preference optimizer in \( \tilde{B}^i(p) \). The contradiction shows that we cannot have both \( p \cdot \tilde{D}^i(p) < \tilde{M}^i(p) \) and \( |\tilde{D}^i(p)| < c \). Hence, if the first inequality holds, we must have \(|\tilde{D}^i(p)| = c \).

QED