The Arrow-Debreu Model of General Competitive Equilibrium

The Market, Commodities and Prices

N commodities

\( x = (x_1, x_2, x_3, \ldots, x_N) \in \mathbb{R}^N \), a commodity bundle

The market takes place at a single instant, prior to the rest of economic activity. commodity = good or service completely specified
description
location
date (of delivery)

A futures market: no reopening of trade.

**Price system**: \( p = (p_1, p_2, \ldots, p_N) \neq 0 \).
\( p_i \geq 0 \) for all \( i = 1, \ldots, N \).
Value of a bundle \( x \in \mathbb{R}^N \) at prices \( p \) is \( p \cdot x \).

**Firms and Production Technology**

\( F, \quad j \in F, \ j = 1, \ldots, \#F. \)

Production technology: \( Y^j \subset \mathbb{R}^N \), \( y \in Y^j \) (the script Y notation is to emphasize that \( Y^j \) is bounded).

Negative co-ordinates of \( y \) are inputs; positive co-ordinates are outputs.
\( y \in Y^j, \ y = (-2, -3, 0, 0, 1) \)

This is a more general specification than a production function. The relationship is \( f^j(x) \equiv \max \{ w \mid (-x, w) \in Y^j \} \).

**The Form of Production Technology**

P.II. \( 0 \in Y^j \).

P.III. \( Y^j \) is closed. (continuity)
P.V.I 

$\mathcal{Y}^j$ is a bounded set for each $j \in F$. (We'll dispense with this eventually)

P.III and P.VI $\Rightarrow \mathcal{Y}^j$ is compact

Compactness of $\mathcal{Y}^j$ is needed to be sure that profit maximization is well-defined, but P.VI is an ugly assumption: boundedness of a firm's attainable production possibilities should be communicated by the price system --- not by assumption. Chapter 15 of Starr's book weakens the assumption by showing that --- even when the firm's technology set is unbounded --- under weak assumptions, the set of attainable plans is bounded. Then circumscribe the unbounded technology set by a ball strictly containing the attainable plans. Apply the analysis of chaps. 11 - 14 to the artificially circumscribed production technology --- there will be an equilibrium (theorem 14.1) and an equilibrium is necessarily attainable, so the circumscribing ball is not a binding constraint in equilibrium. Then delete the artificial circumscribing ball; the prices and allocation remain an equilibrium. Conclusion: P.VI can be eliminated but it's a complex pain to do so.

**Strictly Convex Production Technology**

P.V. For each $j \in F$, $\mathcal{Y}^j$ is strictly convex.

Convexity implies no scale economies, no indivisibilities.

$p \in \mathbb{R}^N_+, p = (p_1, p_2, \ldots, p_N), p \neq 0.$

$\tilde{S}(p) = \{ y^* | y^* \in \mathcal{Y}^j, \quad p \cdot y^* \geq p \cdot y \text{ for all } y \in \mathcal{Y}^j \}.$

**Theorem 11.1:** Assume P.II, P.III, P.V, and P.VI. Let $p \in \mathbb{R}^N_+, p \neq 0$. Then $\tilde{S}(p)$ is a well defined continuous point-valued function.

**Proof:**

Well defined: $\tilde{S}(p) = \text{maximizer of a continuous real-valued function on a compact set}.$

Point-valued: Strict convexity of $\mathcal{Y}^j$, P.V. Point valued-ness implies that $\tilde{S}(p)$ is a function.

Continuity: Let $p^v \in \mathbb{R}^N_+; v = 1, 2, \ldots; p^v \neq 0$, $p^v \to p^o \neq 0$. Show $\tilde{S}(p^v) \to \tilde{S}(p^o)$. 

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Note: this is a consequence of the Maximum Theorem (see Berge, Topological Spaces), but we can provide a direct proof here, by contradiction. Suppose not. Then there is a cluster point of the sequence $\tilde{S}^j(p^v)$, $y^*$ so that $y^* \neq \tilde{S}(p^o)$ and $p^o \cdot \tilde{S}^j(p^o) > p^o \cdot y^*$ (why does this inequality hold? by definition of $\tilde{S}^j(p^o)$). That is there is a subsequence $p^v$ so that $\tilde{S}^j(p^v) \to y^*$. Note that $p^v \cdot \tilde{S}^j(p^o) \to p^o \cdot \tilde{S}^j(p^o)$. We have $p^v \cdot \tilde{S}^j(p^v) \to p^o \cdot y^*$ and $p^o \cdot \tilde{S}^j(p^o) > p^o \cdot y^*$. But the dot product is a continuous function of its arguments, so for $v$ large, $p^v \cdot \tilde{S}^j(p^o) > p^v \cdot \tilde{S}^j(p^v)$, a contradiction. Hence $\tilde{S}^j(p^v) \to \tilde{S}^j(p^o)$. Q.E.D.

**Lemma 1:** (homogeneity of degree 0) Assume P.II, P.III, and P.VI. Let $\lambda > 0$, $p \in R^N_+$. Then $\tilde{S}^j(\lambda p) = \tilde{S}^j(p)$.

\[ \tilde{S}(p) \equiv \sum_{j \in F} \tilde{S}^j(p) \]

### 4.4 Attainable Production Plans

**Definition:** A sum of sets $Y^j$ in $R^N$, is defined as $Y = \sum_{j} Y^j$ is the set \( \{y \mid y = \sum_{j} y^j \ \text{for some} \ y^j \in Y^j \} \).

**Aggregate technology set:**

\[ Y \equiv \sum_{j \in F} Y^j. \]

Initial inputs to production $r \in R^N_+$

**Definition:** Let $y \in Y$. Then $y$ is said to be **attainable** if $y + r \geq 0$.

$y \in Y$ is attainable if \( (y + r) \in [Y + \{r\}] \cap R^N_+. \)

Note that under this definition, and P.II, P.III, P.V, P.VI the attainable set of outputs is compact and convex.