3.1 The Market, Commodities and Prices

N commodities

\( x = (x_1, x_2, x_3, ..., x_N) \in \mathbb{R}^N \), a commodity bundle

The market takes place at a single instant, prior to the rest of economic activity. commodity = good or service completely specified

description
location
date (of delivery)

A futures market: no reopening of trade.

Price system: \( p = (p_1, p_2, ..., p_N) \neq 0 \).

\( p_i \geq 0 \) for all \( i = 1, ..., N \).

Value of a bundle \( x \in \mathbb{R}^N \) at prices \( p \) is \( p \cdot x \).

4.1 Firms and Production Technology

\( F, \ j \in F, j = 1, ..., \#F. \)

Production technology: \( \mathcal{Y}_j \subset \mathbb{R}^N, y \in \mathcal{Y}_j \) (the script Y notation is to emphasize that \( \mathcal{Y}_j \) is bounded).

Negative co-ordinates of \( y \) are inputs; positive co-ordinates are outputs.

\( y \in \mathcal{Y}_j, y = (-2, -3, 0, 0, 1) \)

This is a more general specification than a production function. The relationship is \( f^j(x) = \max \{ w \mid (-x, w) \in \mathcal{Y}_j \} \).

4.2 The Form of Production Technology

P.II. \( 0 \in \mathcal{Y}_j \).

P.III. \( \mathcal{Y}_j \) is closed. (continuity)
P.VI. \( \mathcal{Y}^j \) is a bounded set for each \( j \in F \). (We'll dispense with this eventually)

P.III and P.VI \( \Rightarrow \mathcal{Y}^j \) is compact

Compactness of \( \mathcal{Y}^j \) is needed to be sure that profit maximization is well-defined, but P.VI is an ugly assumption: boundedness of a firm's attainable production possibilities should be communicated by the price system --- not by assumption. Chapter 8 of Starr's book weakens the assumption by showing that --- even when the firm's technology set is unbounded --- under weak assumptions, the set of attainable plans is bounded. Then circumscribe the unbounded technology set by a ball strictly containing the attainable plans. Apply the analysis of chaps. 4-7 to the artificially circumscribed production technology --- there will be an equilibrium (theorem 7.1) and an equilibrium is necessarily attainable, so the circumscribing ball is not a binding constraint in equilibrium. Then delete the artificial circumscribing ball; the prices and allocation remain an equilibrium. Conclusion: P.VI can be eliminated but it's a complex pain to do so.

4.3. Strictly Convex Production Technology

P.V. For each \( j \in F \), \( \mathcal{Y}^j \) is strictly convex.

Convexity implies no scale economies, no indivisibilities.

\( p \in R^N_+ \), \( p = (p_1, p_2, ..., p_N) \), \( p \neq 0 \).

\( \tilde{S}^j(p) \equiv \{y^{*j} \mid y^{*j} \in \mathcal{Y}^j, \ p \cdot y^{*j} \geq p \cdot y \ \text{for all} \ y \in \mathcal{Y}^j \} \).

**Theorem 4.1:** Assume P.II, P.III, P.V, and P.VI. Let \( p \in R^N_+ \), \( p \neq 0 \). Then \( \tilde{S}^j(p) \) is a well defined continuous point-valued function.

**Proof:**

- **Well defined:** \( \tilde{S}^j(p) = \text{maximizer of a continuous real-valued function on a compact set}. \)
- **Point-valued:** Strict convexity of \( \mathcal{Y}^j \), P.V. Point valued-ness implies that \( \tilde{S}^j(p) \) is a function.
- **Continuity:** Let \( p^v \in R^N_+ \); \( v = 1, 2, ...; p^v \neq 0 \), \( p^v \to p^o \neq 0 \). Show \( \tilde{S}^j(p^v) \to \tilde{S}^j(p^o) \).
Note: this is a consequence of the Maximum Theorem (see Berge, *Topological Spaces*), but we can provide a direct proof here, by contradiction. Suppose not. Then there is a cluster point of the sequence $\tilde{S}^j(p^v)$, $y^*$ so that $y^* \neq \tilde{S}(p^o)$ and $p^o \cdot \tilde{S}^j(p^o) > p^o \cdot y^*$ (why does this inequality hold? by definition of $\tilde{S}^j(p^o)$). That is there is a subsequence $p^v$ so that $\tilde{S}^j(p^v) \rightarrow y^*$. Note that $p^v \cdot \tilde{S}^j(p^o) \rightarrow p^o \cdot \tilde{S}^j(p^o)$. We have $p^v \cdot \tilde{S}^j(p^v) \rightarrow p^o \cdot y^*$ and $p^o \cdot \tilde{S}^j(p^o) > p^o \cdot y^*$. But the dot product is a continuous function of its arguments, so for $v$ large, $p^v \cdot \tilde{S}^j(p^o) > p^v \cdot \tilde{S}^j(p^o)$, a contradiction. Hence $\tilde{S}^j(p^v) \rightarrow \tilde{S}^j(p^o)$. Q.E.D.

**Lemma 1:** (homogeneity of degree 0) Assume P.II, P.III, and P.VI. Let $\lambda > 0$, $p \in R^N_+$. Then $\tilde{S}^j(\lambda p) = \tilde{S}^j(p)$.

$\tilde{S}(p) \equiv \sum_{j \in F} \tilde{S}^j(p)$

### 4.4 Attainable Production Plans

**Definition:** A sum of sets $\mathcal{Y}^j$ in $R^N$, is defined as $\mathcal{Y} = \sum_j \mathcal{Y}^j$ is the set

\[
\{ y \mid y = \sum_j y^j \text{ for some } y^j \in \mathcal{Y}^j \}. 
\]

**Aggregate technology set:**

\[
\mathcal{Y} \equiv \sum_{j \in F} \mathcal{Y}^j.
\]

Initial inputs to production $r \in R^N_+$

**Definition:** Let $y \in \mathcal{Y}$. Then $y$ is said to be **attainable** if $y + r \geq 0$.

$y \in \mathcal{Y}$ is attainable if $(y + r) \in [\mathcal{Y} + \{r\}] \cap R^N_+.$

Note that under this definition, and P.II, P.III, P.V, P.VI the attainable set of outputs is compact and convex.