Selected Problems

Chapter 2, Exercises 2.4, 2.9, 2.14.

2.4. Let \( K = [0,1) \cup (1,2] \subset \mathbb{R} \). Then \( \overline{K} = [0, 2] \), a convex set. But \( K \) is non-convex since it does not include its midpoint. That is, \( 1 \not\in K \).

2.9. Closed subsets of \( \mathbb{R} \): \( S = \{ 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{\nu}, \ldots \mid \nu = 1, 2, 3, \ldots \} \). \( S \) is closed inasmuch as it contains its cluster points.

\( T = [0, 1] \), the closed interval between 0 and 1.

Closed subsets of \( \mathbb{R}^N \): \( A = \{ x = (x_1, x_2, \ldots, x_N) \mid x_2 = x_3 = \ldots = x_N = 0, x_1 \in \mathbb{R} \} \). \( A \) is the \( x_1 \) co-ordinate axis. \( A \) is a closed set since it includes all its cluster points.

\( B = \{ x \mid x \in \mathbb{R}^N, |x| \leq 10 \} \). \( B \) is the closed ball of radius 10, centered at the origin.

2.14. • Show that \( A \cap B \) is convex.

Let \( x, y \in A \cap B \). Then \( x, y \in A \) and \( B \). Then by convexity of \( A \) and \( B \) we have that \( \alpha x + (1-\alpha)y \in A \) and \( B \). Then \( \alpha x + (1-\alpha)y \in A \cap B \).

• Show that \( A+B \) is convex.

\( x, y \in A+B \) means that there are \( x^a, y^a \in A \), and \( x^b, y^b \in B \) so that \( x^a + x^b = x \in A+B \) and \( y^a + y^b = y \in A+B \). Then

\[ \alpha x + (1-\alpha)y = \alpha x^a + \alpha x^b + (1-\alpha)y^a + (1-\alpha)y^b = \alpha x^a + (1-\alpha)y^a + \alpha x^b + (1-\alpha)y^b. \]

But \( \alpha x^a + (1-\alpha)y^a \in A \), \( \alpha x^b + (1-\alpha)y^b \in B \), by convexity of \( A \) and \( B \). So \( \alpha x + (1-\alpha)y \in A+B \).

• Show that \( \overline{A} \) is convex.

Let \( x, y \in \overline{A} \). We wish to show that \( \alpha x + (1-\alpha)y \in \overline{A} \). The only difficulty arises if \( x \) or \( y \not\in A \). Suppose \( x \not\in A \). But then \( x \) is the limit of a sequence \( x^\nu \in \overline{A} \). Then the sequence \( \alpha x^\nu + (1-\alpha)y \in \overline{A} \) and approaches \( \alpha x + (1-\alpha)y \) as its limit, so \( \alpha x + (1-\alpha)y \in \overline{A} \).