5. 1 Household Consumption Sets and Preferences

H, i= 1, 2, ..., #H
i ∈ H, Xi ⊆ R^N_+ , u^i: X^i → R (fully represents ⪰_i ), r^i ∈ R^N_+, 1 ≥ α^i j ≥ 0 for each j ∈ F.
x ∈ X^i , x=(x_1, x_2, ..., x_N)
Consumption Sets

(C.I) X^i is closed and nonempty.

(C.II) X^i ⊆ R^N_+ . X^i is bounded below and unbounded above. That is, x ∈ X^i and y ≥ x (the inequality holds co-ordinatewise) implies y ∈ X^i.

(C.III) X^i is convex.

X^i may be R^N_+ .
X = ∑_{i∈H} X^i .

Preferences

x, y ∈ X^i , "x ⪰_i y" is read "x is preferred or indifferent to y (according to i)."

Utility Function

Let u^h: X^h → R. Then u^h is a utility function.

**Definition:** We will say that the utility function u^h(·) represents the preference order ⪰_h if for all x, y ∈ X^h , u^h(x) ≥ u^h(y) if and only if x ⪰_h y. This implies that u^h(x) > u^h(y) if and only if x ⪰_h y and not [y ⪰_h x].

We will assume there is u^i: X^i → R so that u^i(·) represents ⪰_i .

Read u^i(x) ≥ u^i(y) wherever you see x ⪰_i y.

**Weak monotonicity**

(C.IV) (Weak Monotonicity) Let x, y ∈ X^i , with x >> y, (that is, x_i > y_i, i = 1, ..., N) . Then u^i(x) > u^i(y).
Continuity

(C.V) (Continuity) \( u'() \) is a continuous function. Equivalently, for every \( x^0 \in X^i \) the sets
\[ A^i(x^0) = \{ x \mid x \in X^i, x \succeq_i x^0 \} \] and \( G^i(x^0) = \{ x \mid x \in X^i, x^0 \succeq_i x \} \) are closed. That is, the inverse images of closed subsets of \( R \) under \( u(\bullet) \) are closed.

Continuity of \( u' \) allows us to use Corollary 2.2. What does the continuity assumption rule out? Lexicographic preferences provide an example of discontinuous preferences (which cannot be represented by a utility function; certainly not by a continuous utility function).

Example (Lexicographic preferences):
\[ \succ_L \ x = (x_1, x_2, \ldots, x_N), \ y = (y_1, y_2, \ldots, y_N). \]
\[ x \succ_L y \text{ if } x_1 > y_1, \text{ or } \]
\[ \text{if } x_1 = y_1, \text{ and } x_2 > y_2, \text{ or } \]
\[ \text{if } x_1 = y_1, \text{ and } x_2 = y_2, \text{ and } x_3 > y_3, \text{ and so forth ....} \]
\[ x \sim_L y \text{ if } x = y. \]

Strict Convexity of Preferences

(C.VII) (strict convexity of preferences):
\[ u'(x) \geq u'(y), x \neq y, 0 < \alpha < 1 \text{ implies } u'(\alpha x + (1-\alpha)y) > u'(y). \]

5.3 Choice and Boundedness of Budget Sets, \( \widetilde{B}^i(p) \)

Definition: \( x \) is an attainable aggregate consumption if \( y + r \geq x \geq 0 \) where \( y \in Y \) and \( r \in R^N_+ \) is the economy's initial resource endowment, so that \( y \) is an attainable production plan. Note that the set of attainable consumptions is bounded under P.II, P.III, P.V, P.VI.

Choose \( c \in R \), so that \( |x| < c \) (a strict inequality) for all attainable consumptions \( x \). Choose \( c \) sufficiently large that \( X^i \cap \{ x \mid x \in R^N, c \geq |x| \} \neq \emptyset \).

\( \widetilde{M}^i(p) \) represents \( i \)'s income as a function of \( p \). We do not need precisely to specify \( \widetilde{M}^i(p) \) at this point. When we do, income will be characterized as the value of the household endowment plus the value of the household share of firm profits = \( p \cdot r^i + \sum \alpha^i \pi^j(p) \).

\[ \widetilde{B}^i(p) = \{ x \mid x \in R^N, p \cdot x \leq \widetilde{M}^i(p) \} \]
\[ \cap \{ x \mid |x| \leq c \}. \]

\[ \widetilde{D}^i(p) \equiv \{ x \mid x \in \widetilde{B}^i(p) \cap X^i, x \text{ maximizes } u'(y) \text{ for all } y \in \widetilde{B}^i(p) \cap X^i \} \]
\[ \widetilde{D}(p) = \sum_{i \in H} \widetilde{D}^i(p). \]

Lemma 5.1: \( \widetilde{B}^i(p) \) is a closed and bounded (compact) set.

Lemma 5.2: Let \( \widetilde{M}^i(p) \) be homogeneous of degree 1. Then \( \widetilde{B}^i(p) \) and \( \widetilde{D}^i(p) \) are homogeneous of degree 0.
\[ P \equiv \{ p \mid p \in \mathbb{R}^N, p_i \geq 0, i = 1, 2, 3, \ldots, N, \sum_{i=1}^{N} p_i = 1 \} \]

**Positivity of Income**

\[(C.VIII) \quad \tilde{M}'(p) > \min_{x \in X \cap \{ y \mid y \in \mathbb{R}^N, c \geq y \}} p \cdot x \geq 0 \text{ for all } p \in P.\]

**Example (The Arrow Corner):** This example demonstrates the importance of (C.VIII). (C.VIII) is not fulfilled in the example resulting in discontinuous demand.

\[ X^i = \mathbb{R}_+^2 \]

\[ r^i = (1, 0) \]

\[ \tilde{M}'(p) = p \cdot r^i. \]

\[ p^0 = (0, 1). \]

\[ \tilde{B}'(p^0) \cap X^i = \{ (x, y) \mid c \geq x \geq 0, y = 0 \} \]

\[ p^v = \left( \frac{1}{v}, 1 - \frac{1}{v} \right). \quad p^v \rightarrow p^0. \]

\[ \tilde{B}'(p^v) \cap X^i = \left\{ (x, y) \mid p^v \cdot (x, y) \leq \frac{1}{v}, (x, y) \geq 0, c \geq |(x, y)| \geq 0 \right\}. \]

\[ (c, 0) \in \tilde{B}'(p^0) \] but there is no sequence \((x^v, y^v) \in \tilde{B}'(p^v)\) so that \((x^v, y^v) \rightarrow (c, 0)\). For any sequence \((x^v, y^v) \in \tilde{B}'(p^v)\) so that \((x^v, y^v) \rightarrow (c, 0)\) will converge to some \((x^*, 0)\) where \(0 \leq x^* \leq 1\). We may have \((c, 0) = \tilde{D}'(p^0)\). Hence \(\tilde{D}'(p)\) need not be continuous at \(p^0\). This completes the example.

**5.4 Demand behavior under strict convexity**

**Theorem 5.2:** Assume C.I - C.V, C.VII, C.VIII. Let \( \tilde{M}'(p) \) be a continuous function for all \( p \in P \). Then \( \tilde{D}'(p) \) is a well-defined, point-valued, continuous function for all \( p \in P \).

**Proof:** Well defined: Compactness of \( \tilde{B}'(p) \cap X^i \) and continuity of \( u'(\cdot) \).

Unique (point valued): Strict convexity of preferences, C.VII.

Continuous

\[ C.VIII \Rightarrow \tilde{M}'(p) > 0 \text{ for all } p \in P. \]
Let \( p^\nu \in P, \nu = 1, 2, 3, \ldots, \) \( p^\nu \to p^o. \) Show \( \tilde{D}^i(p^\nu) \to \tilde{D}^i(p^o). \) \( \tilde{D}^i(p^\nu) \) is a sequence in a compact set. Without loss of generality take a convergent subsequence, \( \tilde{D}^i(p^\nu) \to x^o. \) We must show that \( x^o = \tilde{D}^i(p^o). \) Proof by contradiction.

Define \( \hat{x} = \arg\min_{x \in X \cap \{ y : c \geq y \in R \}} p^o \cdot x \). \( p^o \cdot \tilde{D}^i(p^\nu) > p^o \cdot \hat{x} \) (by C.VIII).

Let \( \alpha^\nu = \min \left[ 1, \frac{\tilde{M}^i(p^\nu) - p^\nu \cdot \hat{x}}{p^\nu \cdot (\tilde{D}^i(p^o) - \hat{x})} \right] \). For \( \nu \) large, \( \alpha^\nu \) is well defined. \( 0 \leq \alpha^\nu \leq 1. \) \( \alpha^\nu \to 1. \) Let \( w^\nu = (1 - \alpha^\nu) \hat{x} + \alpha^\nu \tilde{D}^i(p^o). \) \( w^\nu \to \tilde{D}^i(p^o) \) and \( w^\nu \in \tilde{B}^i(p^\nu) \cap X^i. \) Suppose \( x^o \neq \tilde{D}^i(p^o). \) Then \( u^i(\tilde{D}^i(p^o)) > u^i(x^o) \). But for \( \nu \) large, \( u^i(w^\nu) > u^i(\tilde{D}^i(p^\nu)) \) by continuity of \( u^i \) and the convergence of \( w^\nu \to \tilde{D}^i(p^o), \tilde{D}^i(p^\nu) \to x^o. \) This is a contradiction, since \( \tilde{D}^i(p^\nu) \) maximizes \( u^i(\cdot) \) in \( \tilde{B}^i(p^\nu) \cap X^i. \) QED

**Lemma 5.3:** Assume C.I - C.V, C.VII, C.VIII. Then \( p \cdot \tilde{D}^i(p) \leq \tilde{M}^i(p). \) Further, if \( p \cdot \tilde{D}^i(p) < \tilde{M}^i(p) \) then \( |\tilde{D}^i(p)| = c. \)

**Proof:** Budget or length is a binding constraint --- if not budget, then length.