7.1 Existence of Equilibrium

\[ P = \left\{ p \mid p \in R^N, \ p_k \geq 0, \ k = 1 \ldots, N, \ \sum_{k=1}^{N} p_k = 1 \right\} \]

\[ \tilde{Z}(p) = \sum_{i \in H} \tilde{D}^i(p) - \sum_{j \in F} \tilde{S}^j(p) - r \]

\[ = \sum_{i \in H} x^i - \sum_{j \in F} y^j - r \]

where \( x^i \) is household i's consumption plan, \( y^j \) is firm j's production plan and r is the resource endowment of the economy. \( \tilde{Z}(p) \) is the economy's excess demand function. Recall that all of the expressions in \( \tilde{Z}(p) \) are N-dimensional vectors.
Definition: \( p^0 \in P \) is said to be an equilibrium price vector if \( \tilde{Z}(p^0) \leq 0 \) (the inequality holds co-ordinatewise) with \( p^0_k = 0 \) for \( k \) such that \( \tilde{Z}_k(p^0) < 0 \). That is, \( p^0 \) is an equilibrium price vector if demand equals supply except for free goods, 
\[
\sum_{i \in H} \tilde{D}_i(p^0) \leq \sum_{j \in F} \tilde{S}_j(p^0) - r.
\]

Weak Walras' Law (Theorem 6.2): For all \( p \in P \), \( p \cdot \tilde{Z}(p) \leq 0 \). For \( p \) such that \( p \cdot \tilde{Z}(p) < 0 \), there is \( k = 1, 2, \ldots, N \), so that \( \tilde{Z}_k(p) > 0 \), assuming C.I - C.V, C.VII, C.VIII.

Continuity: \( \tilde{Z}(p) \) is a continuous function, assuming P.II, P.III, P.V, P.VI and C.I-C.V, C.VII-C.VIII (Theorem 4.1, Theorem 5.2, Theorem 6.1).
Theorem 7.1: Assume P.II, P.III, P.V, P.VI, and C.I-C.V, CVII-C.VIII. There is $p^* \in P$ so that $p^*$ is an equilibrium.

Proof: $T : P \rightarrow P$. For each $k=1,2,3, ..., N$.

$$T_k(p) \equiv \frac{p_k + \max \left[ 0, \tilde{Z}_k(p) \right]}{1 + \sum_{n=1}^{N} \max \left[ 0, \tilde{Z}_n(p) \right]} = \frac{p_k + \max \left[ 0, \tilde{Z}_k(p) \right]}{\sum_{n=1}^{N} \left\{ p_n + \max \left[ 0, \tilde{Z}_n(p) \right] \right\}}.$$  

By the Brouwer fixed point theorem there is $p^* \in P$ so that $T(p^*) = p^*$. But then for all $k = 1, ..., N$,

$$T_k(p_k^*) = p_k^* = \frac{p_k^* + \max \left[ 0, \tilde{Z}_k(p^*) \right]}{1 + \sum_{n=1}^{N} \max \left[ 0, \tilde{Z}_n(p^*) \right]}.$$
Thus, either $p_k^* = 0$ or

$$p_k^* = \frac{p_k^* + \max[0, \tilde{Z}_k(p^*)]}{1 + \sum_{n=1}^{N} \max\left[0, \tilde{Z}_n(p^*)\right]} > 0$$

**Case 1:** $p_k^* = 0 = \max\left[0, \tilde{Z}_k(p^*)\right]$. Hence $\tilde{Z}_k(p^*) \leq 0$.

**Case 2:** $p_k^* = \frac{p_k^* + \max[0, \tilde{Z}_k(p^*)]}{1 + \sum_{n=1}^{N} \max\left[0, \tilde{Z}_n(p^*)\right]} > 0$

To avoid repeated tedious notation, let

$$0 < \alpha = \frac{1}{1 + \sum_{n=1}^{N} \max\left[0, \tilde{Z}_n(p^*)\right]} \leq 1$$
We have
\[ p_k^* = \alpha p_k^* + \alpha \max[0, \tilde{Z}_k(p^*)] \]
\[ (1 - \alpha)p_k^* = \alpha \max[0, \tilde{Z}_k(p^*)] \]

Multiplying through by \( \tilde{Z}_k(p^*) \),

\[ (* \ (1 - \alpha)p_k^* \tilde{Z}_k(p^*) = \alpha (\max[0, \tilde{Z}_k(p^*)]) \tilde{Z}_k(p^*) \]

Restating the Weak Walras' Law,

\[ 0 \geq p^* \cdot \tilde{Z}(p^*) = \sum_{k \in \text{Case } 1} p_k^* \tilde{Z}_k(p^*) + \sum_{k \in \text{Case } 2} p_k^* \tilde{Z}_k(p^*) \]
\[ = 0 + \sum_{k \in \text{Case } 2} p_k^* \tilde{Z}_k(p^*) = \sum_{k \in \text{Case } 2} p_k^* \tilde{Z}_k(p^*) \]

or
\[ 0 \geq \sum_{k \in \text{Case 2}} p_k^* \tilde{Z}_k(p^*) \]

Multiplying through by \((1-\alpha)\), and substituting (*) we have

\[ 0 \geq (1 - \alpha) \sum_{k \in \text{Case 2}} p_k^* \tilde{Z}_k(p^*) \]
\[ = \alpha \sum_{k \in \text{Case 2}} (\max[0, \tilde{Z}_k(p^*)]) \tilde{Z}_k(p^*). \]

But this means that \(\tilde{Z}_k(p^*) \leq 0\), for all \(k\) in case 2.

But then, there is no \(k\), either in case 1 or 2, so that \(\tilde{Z}_k(p^*) > 0\). But the Weak Walras' Law says that if \(p^* \cdot \tilde{Z}(p^*) < 0\), it follows that there is \(k\) so that \(\tilde{Z}_k(p^*) > 0\). Hence we must have \(p^* \cdot \tilde{Z}(p^*) = 0\). Thus for \(k\) so that \(\tilde{Z}_k(p^*) < 0\), it follows that \(p_k^* = 0\). This completes the proof.

Q.E.D.
Theorem 7.1 is a proof of the consistency of the competitive model of chapters 4-7. It is possible to find prices, \( p^* \in \mathcal{P} \) so that competitive markets clear. When economists talk about competitive market prices finding their own level, they are not necessarily speaking vacuously. Under the hypotheses above, there is a competitive equilibrium price system.
Lemma 7.1: Assume P.II, P.III, P.V, P.VI, and C.I-C.V, CVII-C.VIII. Let \( p^* \) be an equilibrium. Then \( |\tilde{D}^i(p^*)| < c \) where \( c \) is the bound on the Euclidean length of demand, \( \tilde{D}^i(p) \). Further, in equilibrium, Walras' Law holds as an equality, \( p^* \cdot \tilde{Z}(p^*) = 0 \).

Proof: Since \( \tilde{Z}(p^*) \leq 0 \) (co-ordinatewise), we know that 
\[
\sum_{i \in H} \tilde{D}^i(p^*) \leq \sum_{j \in F} S^j(p^*) + \sum_{i \in H} r^i, \text{ co-ordinatewise.}
\]
But that implies that the aggregate consumption \( \sum_{i \in H} \tilde{D}^i(p^*) \) is attainable, so for each household \( i \), \( |\tilde{D}^i(p^*)| < c \) where \( c \) is the bound on demand, \( \tilde{D}^i(p) \).

We have for all \( p \), \( p \cdot \tilde{Z}(p) \leq 0 \). In equilibrium, at \( p^* \), we have \( \tilde{Z}(p^*) \leq 0 \) with \( p^*_k = 0 \) for \( k \) so that \( \tilde{Z}_k(p^*) < 0 \). Therefore \( p^* \cdot \tilde{Z}(p^*) = 0 \). QED