Problem Set 1

Feel free to collaborate with classmates on problem sets.

This problem set deals with a Robinson Crusoe economy with two factors of production and two commodities.

Let there be two factors: land denoted T, and labor denoted L. The resource endowment of T is T₀; the resource endowment of L is L₀.

Let there be two goods, x and y.

Robinson has a utility function u(x, y). There is no utility from leisure.

The prevailing wage rate of labor is w, and the rental rate on land is r.

Good x is produced in a single firm by the production function
\[ f(L^x, T^x) = x, \]
where \( L^x \) is L used to produce x, \( T^x \) is T used to produce x.

\[ f(L^x, T^x) \geq 0 \text{ for } L^x \geq 0, T^x \geq 0; f(0,0)=0. \]

Good y is produced in a single firm by the production function
\[ g(L^y, T^y) = y \]
where \( L^y \) is L used to produce y, \( T^y \) is T used to produce y.

\[ g(L^y, T^y) \geq 0 \text{ for } L^y \geq 0, T^y \geq 0; g(0,0)=0. \]

The price of good x is \( p^x \). The price of good y is \( p^y \). Profits of firm x are \( \Pi^x = p^x f(L^x, T^x) - wL^x - rT^x \). Profits of firm y are \( \Pi^y = p^y g(L^y, T^y) - wL^y - rT^y \).

Robinson's income then is \( wL + rT + \Pi^x + \Pi^y \).

Assume f, g, u, to be strictly concave, differentiable. Assume all solutions are interior solutions. Subscripts denote partial derivatives. That is, \( u_x = (\partial u/\partial x) = \) marginal utility of x, ..., \( f_L = (\partial f/\partial L) = \) marginal product of labor in x, ....

The production frontier consists of those x - y combinations that efficiently and fully utilize \( L^0 \) and \( T^0 \) in producing x and y. The marginal rate of transformation of x for y, \( MRT_{x,y} \) is defined as \(-dy/dx\) along this frontier. \( MRT_{x,y} \) is the additional y available from efficiently reallocating inputs of T and L to producing y while sacrificing one unit of x. At a technically efficient (efficient in allocation of inputs on the production side) allocation, we have
\[ -(dy/dx) = MRT_{x,y} = (\partial y/\partial L^y)/(\partial x/\partial L^x) = g_L/f_L. \]

The marginal rate of transformation of x for y equals the ratio of marginal products.

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A (Pareto) efficient allocation in the economy is characterized by maximizing $u(x,y)$ subject to the technology and resource constraints. Thus a Pareto efficient allocation corresponds to values of $x,y,L^x,L^y,T^x,T^y$ maximizing the Lagrangian, $\Lambda$, with Lagrange multipliers $a, b, c, d$:

$$\Lambda = u(x,y) + a(x-f(L^x,T^x)) + b(y-g(L^y,T^y)) + c(L^o-L^x-L^y) + d(T^o-T^x-T^y) \quad (1)$$

1. Differentiate $\Lambda$ with respect to $x$, $y$, $T^x$, $T^y$, and set the derivatives equal to 0. That gives first order conditions for an extremum of $\Lambda$, a Pareto efficient allocation. Let (2) be your first order condition with respect to $x$, (3) with respect to $y$, (4) with respect to $T^x$, (5) with respect to $T^y$.

2. Show that Pareto efficiency requires that the marginal rate of substitution of $x$ for $y$ be the marginal rate of transformation (as computed with respect to $T$). That is, Pareto efficiency requires that

$$\frac{u_x}{u_y} = \frac{g_T}{f_T} \quad (6)$$

Hint: You can demonstrate (6) by combining (2), (3), (4) and (5) appropriately. Explain in words what (6) means. Why does it make sense as an efficiency condition?

3. Differentiate $\Lambda$ with respect to $L^x$, $L^y$, to characterize first order conditions for a Pareto efficient allocation of labor.

4. Repeat exercise 1 with respect to $L$. That is, show that Pareto efficiency requires that $\frac{u_x}{u_y} = \frac{g_L}{f_L}$. 

5. Show that Pareto efficiency requires that marginal rates of technical substitution of $L$ for $T$ are the same for both firms. That is, Pareto efficiency requires $\frac{g_L}{g_T} = \frac{f_L}{f_T}$. Explain in words what this expression means.

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6. First order conditions for profit maximization and for utility maximization subject to budget constraint are:

\[ w = p_x f_L = p_y g_L; \quad (7) \]
\[ r = p_x f_T = p_y g_T; \quad (8) \]
\[ \frac{p_x}{p_y} = \frac{u_x}{u_y} \quad (9). \]

These conditions (7), (8), (9), will be fulfilled in a competitive equilibrium. Show that these equilibrium conditions lead to fulfillment of the efficiency conditions in 2, 3, 4, and 5.