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Equilibrium with Non-convex Transactions Costs: Monetary and Non-monetary Economies

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1. TRANSACTIONS COSTS IN A MONETARY ECONOMY

Transactions costs often possess the set-up property or some other form of diminishing marginal cost (i.e. a nonconvexity). The labour, time and other resources required to carry out an exchange involving a bushel of apples for money are not significantly greater than the resources required for the exchange of a single apple for money. Similarly, the resources needed to write a check for $1000 are considerably less than 1000 times those required for a $1 cheque. Such scale economies in the execution of transactions provide one of the main motives for holding inventories by households and firms. In a sequence economy where money acts as a medium of exchange, non-convex transactions costs of individuals will provide motivation for holding idle balances of money, i.e. inventories of the medium of exchange [15].[1] The willingness to hold idle balances is essential to obtain an equilibrium in an economy with a non-zero money supply and hence such behaviour is a cornerstone of monetary theory.

This poses a substantial problem in writing a general equilibrium theory of a monetary economy. The non-convexities involved generally imply that demand functions or correspondences will not fulfil the continuity or convexity conditions required for the application of the fixed point theorems used to prove the existence of equilibrium. There does not in general exist an equilibrium in the usual sense. Nevertheless, techniques developed in other contexts for the treatment of nonconvexity [1], [7], [13] allow us to demonstrate the existence of an approximate equilibrium. In an economy large and homogeneous enough, the approximation will be close enough to make the approximate equilibrium virtually indistinguishable from a true equilibrium. This is the principal result below. Non-convexities in transactions costs do not preclude the achievement of an allocation that is nearly a competitive equilibrium. The approximation theorem is surprising, since we allow the possibility of set-up costs to each individual transaction. Letting the number of agents become large does not, on the face of it, reduce the average number of transactions per agent. Although existence of an approximate equilibrium is proved, the effects of non-convexities are substantial; with economies of scale in transactions the equilibrium allocation will be characterized by a bunching of transactions into a few markets, and a corresponding bulge in inventory holdings.

The use of non-convex transactions costs is singularly appropriate to the study of costly transactions in a monetary economy since pure set-up cost (i.e. positive fixed cost with zero variable cost) permits cash transactions to be costly in such a way that these costs are

invariant under a change in units.   A currency reform that multiplies all currency values by 10^{-2} is unlikely also to multiply transactions costs by 10^{-2}.

The principal mathematical tool for treating non-convexity in this context is the theorem of Shapley and Folkman, Theorem A below, which says that the size of the non-convexity in the sum of a family of non-convex sets is bounded above in a fashion independent of the number of sets summed.   The standard technique for treating non-convexities then is to convexify the problem, achieve an equilibrium in the convexified economy, and use the Shapley-Folkman theorem to relate the allocation of the non-convex economy to the market clearing allocation of the convexified economy.   The non-convex economy is not too far from market clearing, and indeed the disequilibrium becomes negligible as the economy becomes large.

In studying non-convex preferences, the convexification used was to take the convex hull of " preferred or indifferent " sets [13].   In treating increasing returns in production, convex hulls of technology sets are used [1].   These approaches turn out to be equivalent to taking the convex hull of the household's or firm's demand or supply correspondences, and this is the essential point.   In the present model we might convexify the transactions technology, the household's budget set, or the household's demand correspondence.   The third alternative is the only one that allows convex approximation of the behaviour of the non-convex economy.[2]   However, the demand correspondences may not satisfy the necessary continuity properties.   We shall give a sufficient condition for demand correspondences arising out of non-convex opportunity sets to have the required continuity properties.

In a monetary economy, the supply of real outside money may enter essentially in the determination of the equilibrium.   If periods are short and a set-up cost of labour is required by the transactions technology, a household may find it difficult or impossible to perform both buying and selling transactions during the same period.   Purchases will then have to be paid for in money carried forward from previous periods.   The volume of transactions that can take place will depend then on the real outside money supply.   An increase in this quantity *ceteris paribus* may make additional transactions possible.

Non-convexities in transactions cost may provide a strong motive for making transactions and payments in large discrete amounts rather than smaller or, indeed, continuous amounts.   In actual economies, many contracts consist of an agreement to provide goods at a sequence of future dates in exchange for deliveries, usually less frequent, of money at future dates.   Such an exchange of futures contracts for futures contracts, all transacted at a single market date is an effect of non-convexities in transactions cost.   The model allows—indeed, encourages—transactions to be made in composites that have lower unit transactions costs than would their constituents transacted separately.   Long-term employment or rental agreements as well as the purchase of a shopping basket of goods together rather than each item separately are examples of exploitation of scale economies in transactions.

In the present model, trade takes place between individuals and the market, rather than between pairs of agents.   It is not possible, therefore, to study within the context of the model the use of shops, retailers and other intermediary agents.   It is worth while to note, however, that in a bilateral trade model the function of intermediary agents can be explained in part by scale economies on the size of transaction experienced by individual households. For example, if there are ten sellers of ten distinct commodities and twenty buyers each requiring some of each of the ten commodities, in the absence of intermediation 200 separate relatively small transactions will ensue.   If a single intermediary is used there will be only thirty large transactions.   The gross volume of trade is doubled (since each commodity is now traded twice), but if scale economies on transactions costs are present, the savings associated with the reduction in the number of separate transactions may more than compensate.

In Section 2 the model of a sequence economy with non-convex transactions costs is introduced. Section 3 develops the concept of local interiority needed for continuity of demand.

Section 4 gives further structure and contains a theorem establishing the existence of an equilibrium for a convexified version of the economy. In Section 5 we introduce the additional structure needed for a monetary version of the model, in particular to ensure positivity of the price of money. An equilibrium existence theorem is established for a convexified version of the monetary model. Section 6 uses the convexified equilibria from Sections 4 and 5 to establish approximate equilibria for the original non-convex economy. Proofs are gathered in Section 7.

2. MODEL OF A NON-MONETARY ECONOMY WITH NON-CONVEX TRANSACTIONS COSTS

The point of departure is the sequence economy of Heller [8] which is based on Kurz [11] and Hahn [6]. Thus, we are dealing with an economy with a sequence of markets: commodity i for delivery at date τ may be bought at date τ or $\tau-1$ or $\tau-2...$ or date $t < \tau$ where t is today's date. Moreover, the complete system of spot and futures markets is open at each date (although some markets may be inactive). We shall suppose that time ends at date T and that each of H households is alive at time 0 and cares nothing about consumption after T. There are n physical commodities. At each date and for each commodity, the household has available the current spot market, and futures markets for deliveries at all future dates. Spot and futures markets will also be available at dates in the future and prices on the markets taking place in the future are currently known. Thus in making his purchase and sale decisions, the household considers without price uncertainty whether to transact on current markets or to postpone transactions to markets available at future dates. There is a sequence of budget constraints, one for the market at each date. That is, for every date the household faces a budget constraint on the spot and futures transactions taking place at that date.

In addition to a budget constraint, the agent's actions are restricted by a transactions technology. This technology specifies for each complex of purchases and sales at date t what resources will be consumed by the process of transaction: labour time, paper and pens, gasoline, telephone services, and so forth. It is because transactions costs may differ between spot and futures markets for the same good that we consider the reopening of markets allowed by the sequence economy model.

Though we will take the individual's transactions technology as fixed for the purpose of this model, it should be recognized that unlike the production technology of the firm in standard competitive equilibrium models, an individual's transaction technology should be made to depend on the actions of others in the economy. Thus, the structure of the economy (including for example the legal system and contract enforcement) will affect an individual's transaction possibilities. A more general model would allow endogenous specification of the individual transaction technology.

We shall use the following notation. All of the vectors below are restricted to be non-negative.

$x_\tau^h(t)$ = vector of purchases for any purpose at date t by household h for delivery at date τ.

$y_\tau^h(t)$ = vector of sales analogously defined.

$z_\tau^h(t)$ = vector of inputs necessary to transactions undertaken at time t. The index τ again refers to date at which these inputs are actually delivered.

$\omega^h(t)$ = vector of endowments at t for household h.

$s^h(t)$ = vector of goods coming out of storage at date t.

$r^h(t)$ = vector of goods put into storage at date t.

$p_\tau(t)$ = price vector on market at date t for goods deliverable at date τ.

With this notation, $p_{i_t}(t)$ is the spot price of good i at date t, and $p_{i_t}(t)$ for $\tau > t$ is the futures price (for delivery at τ) of good i at date t.

The (non-negative) consumption vector for household h is

$$c^h(t) = \omega^h(t) + \sum_{\tau=1}^{t} (x_t^h(\tau) - y_t^h(\tau) - z_t^h(\tau)) + s^h(t) - r^h(t) \geqq 0 \quad (t = 1, ..., T). \qquad ...(2.1)$$

That is, consumption at date t is the sum of endowments plus all purchases past and present with delivery date t minus all sales for delivery at t minus transactions inputs with date t (including those previously committed) plus what comes out of storage at t minus what goes into storage. We suppose that households care only about consumption (and not about which market consumption comes from) and that preferences are convex, continuous and monotone. Thus, households maximize $U^h(c^h)$, where c^h is a vector of the $c^h(t)$'s, subject to constraint.

As discussed above, the household is constrained by its transaction technology, T^h, which specifies, for example, how much leisure time and shoeleather must be used to carry out any transaction. Let $x^h(t)$ denote the vector of $x_\tau^h(t)$'s (and similarly for $y^h(t)$ and $z^h(t)$). We insist

$$(x^h(t), y^h(t), z^h(t)) \in T^h(t) \quad (t = 1, ..., T). \qquad ...(2.2)$$

Naturally, storage input and output vectors must be feasible, so

$$(r^h(t), s^h(t+1)) \in S^h(t) \quad (t = 1, ..., T). \qquad ...(2.3)$$

For convenience, suppose that $s^h(1) \equiv 0$. Both $S^h(t)$ and $T^h(t)$ are closed. They are not assumed to be convex. We postpone further discussion of the $T^h(t)$.

Households may transfer purchasing power forward in time by using futures markets and by storage of goods that will be valuable in the future. Purchasing power may be carried backward by using futures markets. But these may be very costly transactions if set-up costs are present. If outside money and bonds were present, then the household could either hold outside money as a store of wealth, or it could buy or sell bonds. We will postpone to Section 5 consideration of the monetary version of this model. The budget constraints for household h are then:

$$p(t)x^h(t) \leqq p(t)y^h(t) \quad (t = 1, ..., T). \qquad ...(2.4)$$

We need more notation at this point. Let $a^h(t) \equiv (x^h(t), y^h(t), z^h(t), r^h(t), s^h(t))$, and let a^h be a vector of the $a^h(t)$'s, and define x^h, y^h, z^h, r^h and s^h similarly. Define $B^h(p)$ as the set of a^h's which satisfy constraints (2.1)-(2.4), and let $B^{ht}(p)$ be the projection onto t of $B^h(p)$. Clearly the household maximizes $U^h(c^h)$ over $B^h(p)$. Denote the demand correspondence (i.e. the set of maximizing a^h's) by $\gamma^h(p)$.

3. LOCAL INTERIORITY[3]

In order to prove existence of equilibrium, we will need to show that demand correspondences are upper semi-continuous. To do this, we need the continuity of the budget correspondences $B^h(p)$. Discontinuities in the budget set may arise in the absence of convexity. For example, suppose there are free goods whose acquisition requires a set-up transactions cost. There may be a convergent sequence of price vectors such that a household cannot afford the set-up cost for any point in the sequence but that at the limit of the sequence it can precisely afford the set-up cost. The budget set then makes a discontinuous change at the limit. A sufficient condition to avoid this discontinuity is that the non-degenerate part of the budget set ($\hat{B}^h(p)$ below) be locally interior. Local interiority is defined below, and the continuity implication is proved in Section 7.

Let $\hat{B}^{ht}(p) = \{(x, y, z, r, s) \mid (x, y, z, r, s) \in B^{ht}(p) \text{ and } x - y - z \nleq 0\}$, where $u \nleq v$ means that if every component of u is less than or equal to v, then $u = v$. Thus, $\hat{B}^{ht}(p)$ consists of

points of $B^{ht}(p)$ which are not dominated by the no trade point. Note that $0 \in \hat{B}^{ht}(p)$. Also, $\hat{B}^{ht}(p)$ is not closed and not connected for the set-up cost case. We define $\hat{B}^h(p)$ to be the cross product of the $\hat{B}^{ht}(p)$'s. We show in Section 4 that (with monotonicity of preferences) there is no loss in restricting choices to $\hat{B}^h(p)$.

We shall now replace convexity of the sets T^h by local interiority and obtain the needed continuity of the correspondence $B^h(p)$. Recall how the Debreu proof [5] of lower semi-continuity of the convex budget correspondence goes. Given that $B^h(p)$ is convex and has an interior point a^* at p^0, one considers for any $a^0 \in B^h(p^0)$, the line segment, L, joining a^* and a^0. Since $B^h(p^0)$ is convex, this line segment also lies in the interior of $B^h(p)$ with the possible exception of a^0. Hence, for any $a \in L$, $a \neq a^0$, we have $a \in B^h(p)$ for p in some neighbourhood of p^0 (since $p^0 a > 0$ implies $pa > 0$, for all p near p^0). Therefore, for any sequence (p^k), $p^k \to p^0$, there exists $a^k \in L$ such that $a^k \in B^h(p^k)$, whenever k is sufficiently large. Moreover, a^k can be chosen to converge to a^0. For, if $a^0 \notin B^h(p^k)$ let $a^k \in L$ be on the boundary of $B^h(p^k)$, i.e. $p^k a^k = 0$. Then $p^0 \bar{a} = 0$ where $a^k \to \bar{a}$. But $\bar{a} \in L$ implies $\bar{a} \in \text{int } B(p^0)$ or $\bar{a} = a^0$. But $p^0 \bar{a} = 0$ means that $\bar{a} \notin \text{int } B^h(p^0)$. Hence $a^k \to a^0$, as was to be shown.
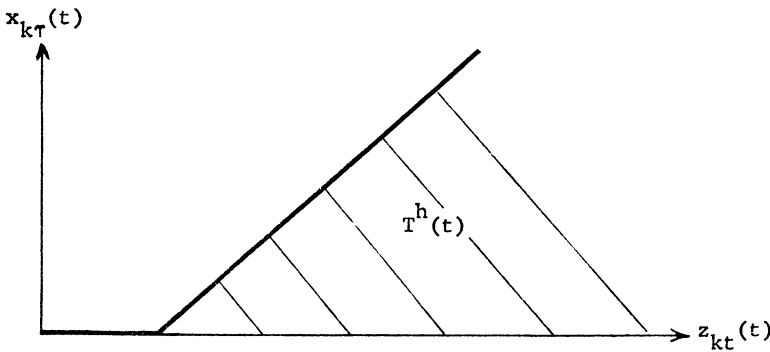

FIGURE 1

Notice how the argument above utilizes convexity only to show that if a line segment joins a point on the boundary with a point in the interior, then all its intermediate points lie in the interior as well. Suppose there exists a continuous curve having this same property in the case where a line segment does not. Close examination of the above argument reveals that lower semi-continuity holds in this case as well. This is the basic idea of this section.

Definition. $\hat{B}^h(p)$ is said to be *locally interior* if for each $a \neq 0$, $a \in \hat{B}^h(p)$ there is a^* so that

(i) $a^* \in \hat{B}^h(p)$.

(ii) a^* satisfies (2.4) with strict inequality, i.e. $p(t)(x^*(t) - y^*(t)) < 0$ for all t.

(iii) There is a continuous function $f:[0, 1] \to \hat{B}^h(p)$ so that $f(0) = a^*$, $f(1) = a$, and for all σ, $0 \leq \sigma < 1$, $f(\sigma)$ satisfies (2.4) with strict inequality for all t.

The reason for restricting local interiority to $\hat{B}^h(p)$ is that we can include the case of set-up costs, since local interiority on $B^h(p)$ would rule out the technology depicted schematically in Figure 1. The horizontal portion of $T^h(t)$ is not included in $\hat{B}^{ht}(p)$.

Geometrically, local interiority of a set S means that any point (except 0) " faces " the interior of the set in the sense that any point is arc-connected to some point in the interior. Actually, our condition is a bit weaker than that since we only require that points be interior to the half-space (in lower dimension) defined by (2.4).

Naturally, the classical case of convexity of T^h and S^h plus

(E) $\qquad\qquad\qquad\qquad\qquad \omega^h(t) \gg 0 \quad$ for all t and h

implies local interiority. For then $a^* \in$ int $\hat{B}^h(p)$ exists by (E) and $f(\sigma) = \sigma a^0 + (1-\sigma)a^*$ belongs to the interior of $\hat{B}^h(p)$ for $\sigma = 1$, by the usual arguments. Indeed, if T^h is strictly star-shaped with centre at $a^* = (\varepsilon, \varepsilon, \varepsilon, ..., \varepsilon)$ for some $\varepsilon > 0$, then $\hat{B}^h(p)$ is locally interior (cf. Arrow-Hahn for definition and discussion of strict star-shapedness). In this case, the function f is linear and a^* does not depend on a^0. Strictly star-shaped technologies may be of the U-shaped average cost curve type, although they cannot be of the set-up cost type.

What other economic conditions are sufficient for local interiority? If all transactions costs are purely set-up costs (i.e. no positive marginal transactions costs), if all transactions of the individual are strictly feasible using own resources, and storage technologies are convex, then local interiority also holds. For we then have that for all x, y there is a z such that $(x, y, z) \in T^h$ and $\sum_{\tau=1}^{t} z_t(\tau) \ll \omega^h(t)$. Hence, if a^1 and a^2 are both non-zero and both belong to $\hat{B}^h(p)$, for each σ there exists z such that $(\sigma x^1 + (1-\sigma)x^2, \ \sigma y^1 + (1-\sigma)y^2, \ z) \in T^h$ and $\sum_{\tau=1}^{t} z_t(\tau) \ll \omega^h(t)$. Thus, if $a(\sigma)$ is the convex combination of a^1 and a^2, $a(\sigma) \in$ int $\hat{B}^h(p)$ whenever $a^1 \in$ int $\hat{B}^h(p)$, since (2.1) and (2.4) are linear constraints. Hence, $\hat{B}^h(p)$ is locally interior (recall that although $0 \in \hat{B}^h(p)$, there is no requirement that 0 be connected to the interior of $\hat{B}^h(p)$ in the definition of local interiority).

Indeed, the argument used in the case of convexity can be used to show that $\hat{B}^h(p)$ is locally interior whenever T^h and S^h are locally convex and $\sum_{\tau=1}^{t} z_t(\tau) \ll \omega^h(t)$ for all $(x(t), y(t), z(t)) \in T^{ht}$. For in that case, there is easily seen to be a point interior to the constraint (2.4) and arbitrarily close to any $a \in \hat{B}^h(p)$. In fact, we can ask only that

$$\hat{T}^{ht} = \{(x(t), y(t), z(t)) \in T^{ht} \mid x(t) - y(t) - z(t) \nleqq 0\} \quad \text{(note that } 0 \notin T^h)$$

be locally convex, thereby admitting set-up costs with positive marginal cost after the set-up. See Figure 1 above.

Proofs of all lemmas and theorems below are found in Section 7.

Lemma 3.1. *Let $\hat{B}^h(p^0)$ be locally interior. Then $\overline{\hat{B}^h(p^0)}$ (the bar denotes closure) is lower semi-continuous at p^0.*

Lemma 3.2. $\overline{\hat{B}^h(p)}$ *is upper semi-continuous for all $p \in P$.*

Lemma 3.3. *Let $\hat{B}^h(p^0)$ be locally interior for all p^0. Then $\gamma^h(p^0)$ is upper semi-continuous at p^0.*

4. EXISTENCE OF EQUILIBRIUM FOR THE CONVEXIFIED NON-MONETARY ECONOMY

Because of possible non-convexities in demand correspondences arising from the non-convex transactions technology, it will not in general be possible to prove the existence of an equilibrium. Rather, we will adopt a now familiar technique due to Starr [13] using mathematical tools developed by Shapley and Folkman (reported in [13]) and extended by Starr [13] and Heller [7]. An economy with convex behaviour (" convexified economy ") is described bearing a close relationship to the non-convex economy. Standard fixed-point techniques are used to discover an equilibrium for the convexified economy. Then it is shown that there is an allocation for the non-convex economy that approximates the behaviour of the convexified equilibrium. That is, there are allocations for the non-convex economy that are nearly equilibria.

The correspondences $\gamma_t^h(p)$ are always homogeneous of degree zero in $p(t)$, as is easily seen from the definition of $B^h(p)$. We can therefore restrict the price space to the simplex. Let S^t denote the unit simplex of dimensionality, $n(T-t+1)$. Let $P = \underset{t=1}{\overset{T}{\times}} S^t$.

Lemma 4.1. *Let $a^h \in \gamma^h(p)$. Then $a^h \in \hat{B}^h(p)$.*

Proof. It is easy to show that if $a^h(t) \in B^{ht}(p)$, but $a^h(t) \notin \hat{B}^{ht}(p)$, then there exists $\hat{a}^h(t)$ such that $\hat{c}^h(t) > c^h(t)$. Hence, under monotonicity of preferences, a^h is dominated by $\hat{a}^h$.   QED

Therefore, we may restrict household decisions to $\hat{B}^h(p)$ without loss of generality. We now state formally the assumptions on the sets $T^h(t)$ and $S^h(t)$.

(A.1) $T^h(t)$ is a closed, but not necessarily convex set. The same holds for $S^h(t)$.

(A.2) (Free disposal.) If $(x^h(t), y^h(t), z^h(t)) \in T^h(t)$ then $x^{h'}(t) \leqq x^h(t)$, $y^{h'}(t) \leqq y^h(t)$ and $z^{h'}(t) \geqq z^h(t)$ imply $(x^{h'}(t), y^{h'}(t), z^{h'}(t)) \in T^h(t)$. Similarly, $S^h(t)$ exhibits free disposal.

(A.3) $0 \in T^h(t)$ and $0 \in S^h(t)$.

(A.4) (Local interiority.) For $p \in P$, $\hat{B}^h(p)$ is locally interior.

(A.5) (Boundedness.) $a^v \in T^h \times S^h$ and $\|a^v\| \to \infty$ implies $\|z^v\| \to \infty$.

Assumptions (A.1)-(A.3) are more or less self-explanatory and are not unreasonable.

Assumption (A.5) is needed to make " wash sales " (the buying and selling of the same good at the same time) inefficient for the household. Positive marginal transactions cost for all commodities would certainly imply (A.5). An alternative to (A.5) would be to simply require that households not make wash sales. In that case, conventional arguments for pure exchange economies would guarantee the boundedness of consumption plans and (therefore) purchase and sale plans.

Define $B_\rho^h(p)$ to be the intersection of $B^h(p)$ with a cube of the same dimensionality as $B^h(p)$, where each side of the cube is of length ρ. Let $\gamma_\rho^h(p)$ be the set of utility-maximizing elements of $\overline{B_\rho^h(p)}$. Then $\gamma_\rho^h(p)$ is clearly compact-valued and upper semi-continuous.

Let $\hat{\gamma}_\rho^h(p) = \text{con } \gamma_\rho^h(p)$, where " con " indicates convex hull. Then $\hat{\gamma}_\rho^h(p)$ is a convex, compact set for all $p \in P$ (since $\gamma_\rho^h(p)$ is compact). Moreover, $\hat{\gamma}_\rho^h(p)$ is an upper semi-continuous correspondence since its values consist of convex combinations of points belonging to values of another correspondence possessing a compact graph. We are now in a position to apply the Kakutani fixed point theorem to show the existence of prices and an allocation that act as an equilibrium for the convexified economy. So far this is a technical convenience rather than an economically meaningful result since the actual economy is non-convex, but it will be shown in Section 6 that the non-convex economy with many agents approximates the behaviour of the convexified economy.

Theorem 4.1 (Existence of Equilibrium for the Closed Convexified Non-monetary Economy). *Under (A.1)-(A.5), there is a price vector $p^* \in P$ and an allocation $\langle a^{h*} \rangle_h$ such that*

$$a^{h*} \in \text{con } \gamma^h(p^*)$$

$$\sum_{h=1}^{H} x^{h*} \leqq \sum_{h=1}^{H} y^{h*}$$

with $p_{it}^*(\tau) = 0$ for i, t, τ such that the strict inequality holds.

5. A MONETARY ECONOMY[4]

The model may now be trivially modified to incorporate fiat money and bonds by introducing money as a 0th commodity for which the household has no direct utility (cf. Heller [8] for more discussion of the [convex] monetary model). The non-trivial modification is to ensure the existence of equilibrium with a *positive* price of money in each period. A futures contract for delivery of money is a bond. For convenience, let $x_{0t}^h(t)$ (the zero-th component of $x_t^h(t)$) denote the total amount of spot money acquired by household h in the market at

date t. Similarly, let $y_{0t}^h(t)$ be the disbursement of spot money at t. Now if $\tau > t$, then $y_{0\tau}^h(t)$ is a commitment made at time t to deliver $y_{0\tau}^h(t)$ units of money at date τ. Suppose that by convention each bond promises one unit of money. Then $y_{0\tau}^h(t)$ is the number of bonds with maturity date τ sold by h. Similarly, $x_{0\tau}^h(t)$ is the number of bonds purchased by household h with maturity date τ.

One of the purposes of this section is to establish whether there are equilibria such that money can be exchanged for positive quantities of scarce goods. It would therefore be begging the question to make money the *numeraire*. Hence we say that spot money trades at a price of $p_{0t}(t)$ at date t (and $p_{0t}(t) = 0$ is thereby a possibility). The price of a bond maturing at τ is denoted $p_{0\tau}(t)$. With this convention we may write the budget constraints on the household as in (2.4), except that $p(t)$, $x(t)$, $y(t)$ all contain an added, zero-th component.[5]

The interpretation is that spot money on the market at t can be acquired by spot and futures sales of goods and bond sales. Similarly spot and future purchases of goods and bond purchases can be paid for in cash, goods, bonds or goods futures.[6]

What is to prevent a household from disbursing an unlimited amount of money $y_{0t}^h(t)$? The answer is to be found in constraint (2.1) for the zero-th component: " consumption " of money is a non-negative number. It is possible to disburse money for considerably more than one's current money holding without violating (2.1), but there must be corresponding receipts of money to balance out the discrepancy. This view of the budget constraint concentrates attention on the store of value rather than medium of exchange aspects of money.

Since we are dealing with fiat (rather than commodity) money, utility maximization will always imply zero consumption of money. We shall assume that there is a positive endowment of cash at least at the beginning and that at no time is there any input of cash being used up by the transaction process. Therefore, by (2.1),

$$r_0^h(t) - s_0^h(t) = \omega_0^h(t) + \sum_{\tau=1}^{t} \left[x_{0t}^h(\tau) - y_{0t}^h(\tau) \right].$$

Thus, net additions to storage of cash at t equals endowment plus total net acquisitions from the market place, where the total is taken over all previous transaction dates. Hence, a household needs to deliver cash only to the extent that his promises to others exceed the promises of others to him.

Transactions and storage technologies will also include money and bonds, since both of these are also costly to exchange. Some discussion of the interpretive difficulties of introducing money into the transaction technology is found in [8].

There is just one further constraint that the household must satisfy: the terminal condition that holdings of money at the end of time should be at least equal to the money endowment. Without this artificial requirement, no one would want to hold a positive money stock at the end. This would drive its terminal price to zero. But then no one would hold money at $T-1$, and so forth. The problem arises because of our initial artifice that $T < \infty$.

$$\sum_{\tau=1}^{T} \left[x_{0T}^h(\tau) - y_{0T}^h(\tau) \right] + s_0^h(T) \geq \sum_{\tau=1}^{T} \omega_0^h(\tau) \equiv M^h. \qquad \qquad ...(5.1)$$

Define $M = \Sigma_h M^h$.[7]

The constraint (5.1) says that the household is required to have at the economy's terminal date holdings of nominal money equal to its endowment thereof. One interpretation of this is that the government lends money at periods up to T and calls back its loan at the end of T. Because the model has a finite horizon, without this or a similar requirement the only equilibrium price of money would be zero [14], thereby demonetizing the economy. At any positive price there would be an excess supply of money in the final period leading to a fall in price in the last and all prior periods.[8] The artificial constraint against depletion of

money balances arises because of the artificial specification of a finite horizon. The finite horizon is required to avoid well-known difficulties (inapplicability of fixed point theorems) associated with infinite horizon models. If the finite horizon were removed, then the economic reason for depletion of money balances and the resultant demonetization would disappear.

There is a variety of alternative more elaborate constructions that will achieve the same effect as (5.1) with no increase in analytic content, though some increase in plausibility. A bequest motive can generate sufficient terminal period demand for money to avoid demonetization. Agents alive at period T may have a derived utility from their stock of money at T reflecting their desire to make a bequest to their successors at $T+1$. Sufficiently exacting taxes payable in money can imply an arbitrary positive price of money [8], [12], [14]. If the present model were embedded in a temporary equilibrium treatment [1] with agents' preferences and expectations extending beyond the terminal date, this too would generate demand for terminal money balances if money were expected to be of positive value in succeeding periods.

The artificiality of these constructs means that we cannot conclude that money is useful simply because it has a positive equilibrium price.[9] Money will be useful only to the extent it enters into exchange. Money might not be traded in equilibrium, even when it has a positive price. This possibility seems unlikely if (1) money has lower transaction and storage costs than bonds, goods or futures and (2) households wish to make intertemporal transfers of purchasing power at equilibrium prices.

In the present model, as in all sequence economy models [6], [8], [11], the emphasis is on the role of money as a store of value, a means of transferring purchasing power forward and backward in time. Any durable good or futures contract can perform the function of shifting purchasing power forward or back, but transactions and storage costs associated with some commodities used for this purpose will be prohibitive. A distinguishing feature of money should be its low transactions and storage costs as compared to goods, bonds and futures contracts. Households may transfer purchasing power over time by accumulating and depleting their money balances and by the use of money futures contracts (loans). If the desired timing of purchase and sale transactions does not coincide, money acts as a carrier of purchasing power between the two transactions dates. If there are scale economies on the size of transactions (e.g. set-up transactions costs) the use of money will allow them to be exploited by concentrating in time transactions that might otherwise have to be more evenly spread. The smaller number of transactions should result in smaller total transactions cost. We now introduce assumptions on the monetary economy.

(M.1) (Durability of money.) For all t, τ, $z_{0\tau}^h(t) = 0$ is feasible for all vectors
$$(x^h(t), y^h(t), z^h(t)) \in T^h(t).$$
Storage of money is costless.

(M.2) (Transactions costs of sales.) For all $y(t) > 0$, $(0, y(t), z(t)) \in T^h(t)$ implies $z_\tau(t) \neq 0$, for some $\tau \geqq t$.

(M.3) (Low transactions costs of money acquisitions.) For all t, there exists some h' and $(x^{h'}, y^{h'}, z^{h'})$ such that $(x^{h'}(t), y^{h'}(t), z^{h'}(t)) \in T^{h'}(t)$ with $x_{0t}^{h'}(t)$ as large as we please and $z_\tau^{h'}(t) \ll \omega^h(\tau)$.

(M.4) (Net productivity of terminal period monetary trade.) For all h there is
$$(x^h, y^h, z^h) \in T^h(T)$$
so that $y^h(T)$ is positive only in its 0th component and $x^h - y^h - z^h$ can assume any non-negative value.

Assumptions (M.1) and (M.2) are more or less self-explanatory and not unreasonable. (M.3) says that there is always at least one household with endowment great enough to cover the transactions costs of very large acquisitions of money. Clearly we could dispense

with (M.3) (or rather, it would be trivially fulfilled) if money had zero transactions costs. (M.4) says that in the terminal period it is technologically possible for the household to make any arbitrary net purchase, with money the only means of payment. Together, (M.3) and (M.4) imply that if the price of money falls low enough in any preterminal period (the price in the terminal period being supposed positive) there is at least one household that can use this as an opportunity to acquire large stocks of money to be held until the terminal period when they can be converted to large net purchases of goods. If there is sufficient substitutability between preterminal and terminal period consumption (assumption (S) below), at least one household will exploit the opportunity. The result is to keep the equilibrium price of money positive in all periods.

(S)   For some household h satisfying (M.3), any consumption vector c^h and for any date t, let $\bar{c}^h(t)$ be arbitrarily specified. Then there exists $\bar{c}^h(T)$ such that

$$U^h(c^h(1), ..., c^h(t-1), \bar{c}^h(t), c^h(t+1), ..., \bar{c}^h(T)) > U^h(c^h).$$

Property (S) is satisfied, for example, by any additive separable utility function which is unbounded above.

As before, $\gamma^h(p)$ is the demand correspondence of household h (but with the additional zero-th components corresponding to money and bonds). The correspondences $\gamma_t^h(p)$ are again homogeneous of degree zero in $p(t)$. Let S^t denote the unit simplex of dimensionality $(n+1)(T-t+1)$. Let S_α^T denote the unit simplex of dimensionality $n+1$ with the restriction that $p(T) \in S_\alpha^T$ implies $p_{0T}(T) = \alpha$, where $0 < \alpha < 1$. Define

$$P^\alpha = \{(p(1), p(2), ..., p(T)) \mid p(t) \in S^t, p(T) \in S_\alpha^T\}.$$

As in the previous section $\hat{\gamma}_\rho^h(p)$ is the convex hull of the demand correspondence of the economy bounded by ρ. By the same reasoning as in Sections 3 and 4, $\hat{\gamma}_\rho^h(p)$ is convex, compact and upper semi-continuous.

We can now establish the existence of an equilibrium with a positive price of money for the convexified monetary economy. As in the corresponding Theorem 4.1 for the non-monetary economy the result is in itself merely a convenience. However, it is the basis of establishing in Section 6 approximate equilibria for the nonconvex economy.

Theorem 5.1 (Existence of equilibrium for the convexified monetary economy). *Under* (A.1)-(A.5), (M.1)-(M.4) *and* (S), *for any* α, $0 < \alpha < 1$, *there is a price vector* $p^* \in P^\alpha$ *and an allocation* $\langle a^{h*} \rangle_h$ *such that*

$$a^{h*} \in \text{con } \gamma^h(p^*),$$

$$\sum_{h=1}^{H} x^{h*} \leqq \sum_{h=1}^{H} y^{h*}$$

with $p_{it}^*(\tau) = 0$ *for* i, t, τ *such that the strict inequality holds, and* $p_{0t}^*(t) \neq 0$ *for all* t.

The multiplicity of equilibria established by the theorem is really a case of the indeterminacy of the price of money. This occurs because of the absence of wealth effects (constraint (5.1)). For more discussion, see [8].

6. APPROXIMATE EQUILIBRIUM

An approximate equilibrium is generally defined as a price vector, p^*, and two allocations, w^* and $w^\dagger$. One, $w^\dagger$, is the allocation desired by households and firms at those prices, which may not clear the market. The other, w^*, is an allocation obeying a materials balance (market clearing) condition although it need not represent agents' optimizing behaviour. The equilibrium is approximate of modulus A if some suitably chosen norm of the difference between two allocations is no larger than A. The desired allocation represents an approximate equilibrium in the sense that the failure to clear the market at those prices is bounded by A. Conversely, should the economy be forced to the market clearing allocation the

disparity between this allocation and a desired allocation is bounded by A. The outcome one has in mind in this formulation is that agents in the economy will seek *ex ante* the optimizing allocation. Since actual sales must equal actual purchases, *ex post*, the allocation must be market-clearing. It is of interest then that the discrepancy between intentions (*ex ante*) and realization (*ex post*) be small, so that incentive to change behaviour and hence upset the realized allocation be small.[10] In the results below, A is independent of the number of households in the economy. Thus, as the economy becomes large, disequilibrium becomes small relative to the size of the economy.

The only requirement of equilibrium in the usual sense that $w^\dagger$ fails to fulfil is the most essential, market clearing. In all other respects, particularly inclusion in the production and consumption possibility sets of firms and households respectively, $w^\dagger$ is well behaved. The same cannot generally be said of w^*, the market clearing allocation. w^* may require firms to produce technologically impossible input-output combinations or require households to have negative consumptions of some commodities. Restricting w^* to avoid these difficulties is possible under some assumptions. In an economy with non-convex preferences and production technologies and without transactions cost there exists an approximate equilibrium with a bound on the expression $|\Sigma w^{*i} - \Sigma w^{\dagger i}|$ where the index i runs over households and firms. At the allocation w^* firms are not restricted to remain in their technology sets. The bound is not on the amount by which individual agent actions in w^* depart from their desired actions $w^\dagger$ but rather on the disparity in aggregate behaviour between the two allocations. This bound is a measure of the extent to which markets fail to clear at $w^\dagger$, the desired allocation. Arrow and Hahn refer to such p^*, w^*, $w^\dagger$ as a social approximate equilibrium.

The social approximate equilibrium represents only part of the desired near equilibrium concept. The existence of such a price-allocation combination assures us that a norm of the aggregate excess demands is relatively small. By itself, this is only an interesting piece of arithmetic. What is actually required is a prescription of individual agents' behaviour that takes advantage of this smallness to achieve an allocation that is not only near by to agents' choices but also fulfils material balance and non-negativity constraints. If these constraints are not fulfilled then we are considering an allocation that cannot be a candidate for a worthwhile approximate equilibrium since it violates the most elementary conditions defining an achievable allocation.

Consider the case of the household depicted in Figure 2. The opportunity set determined by transactions technology and budget constraint is the kinked line. The household is indifferent between the two consumption choices $c^\dagger$ and ω (the endowment point), but neither of these is consistent with material balance of the market as a whole. Typically some convex combination of ω and $c^\dagger$ will fulfil material balance, c^{**} for example. But c^{**} is not in the household's opportunity set because c^{**} requires the household to transact outside the transactions technology. Is it possible to find an allocation near to the optimizing point fulfilling material balance and transactions technology requirements? In the case of the figure the answer is " yes "; $c^{\dagger\dagger}$ achieves these objectives by acceptance of a small reduction in consumption. Such a reduction is not always possible. If $c^\dagger$ were on the boundary of the quadrant, reduction of consumption of some commodities would require negative consumptions, another violation of constraint.

In a pure exchange economy with non-convex preferences and without transactions costs there is an approximate equilibrium with a bound on $(\Sigma | w^{*i} - w^{\dagger i}|^2)^{1/2}$. The sum over households of the squared disparities between the allocations is bounded. In large economies the two allocations for a typical individual will be nearly identical. Arrow and Hahn call such an allocation an individual approximate equilibrium. It is possible, in fact, to find w^{*h} non-negative, so that no household is required to consume a negative amount of any commodity.[11]

In the analysis below we distinguish between social approximate equilibrium, individual approximate equilibrium, attainable individual approximate equilibrium, and attainable
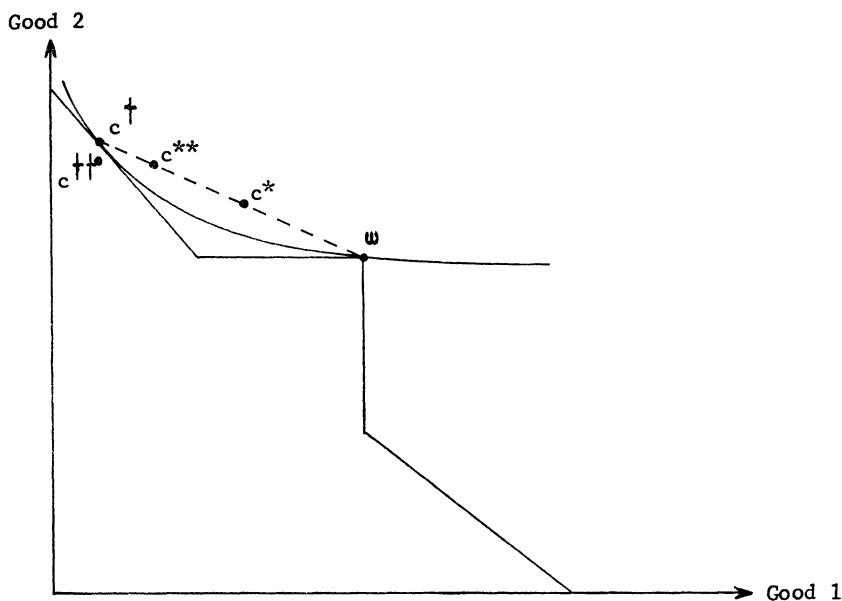
Figure 2

individual uniformly approximate equilibrium (the uniformity is on the Euclidean length of disappointment among households). The last two concepts are the more interesting since they describe achievable allocations. The first two are used mainly as technical aids to proving the existence of the latter.

The approximate equilibrium allocation $w^\dagger$ involves a choice of behaviour among households so that aggregate behaviour is approximately convex though individual behaviour is non-convex. The smoothing of aggregate behaviour comes from balancing one agent's behaviour off against another's. Thus identical households may have very different actions (though nearly equal utilities) specifically chosen so that the sum of their actions is well balanced. A household may be indifferent among making a given transaction on any of five days of the week but with set-up costs, making 20 per cent of the transaction on each day would be inferior because of the increased transactions costs incurred. To approximate convexified behaviour the allocation will prescribe that of five such households each make its transaction on a different day. Thus the approximate equilibrium implies a theory of the distribution of individual behaviour partially independent of the distribution of individual characteristics. It is often observed that only one of the many possible actions among which an agent is indifferent at given prices is consistent with equilibrium allocation. In the present model, the Shapley-Folkman theorem chooses one or several constellations of such indifferent points to limit the extent of disequilibrium.

Definition. The inner distance of a set S is defined as

$$\rho(S) = \sup_{x \in \text{con } S} \inf_{y \in S} |x - y|.$$

Theorem A (Shapley-Folkman). *Let $S_1, \dots, S_m$ be compact sets in E^n and let $x \in \text{con } S$, $S = \sum_{i=1}^m S_i$. Then there are points $b_i \in S_i$, $i = 1, \dots, m$, such that $|x - \sum_{i=1}^m b_i| \leq R$. Let R be the sum of the n largest $\rho(S_i)$.*

Proof. See [7].

The limits supplied by the Shapley-Folkman theorem depend on the dimensionality of the space. In the present models, for each commodity there are $[T(T-1)]/2$ spot and futures

markets on which it may be traded. Further, each commodity and contract is subject to five separate decisions: buy, sell, incur a transactions cost, place in storage, remove from storage. There are n commodities and money. Thus the dimensionality of the spaces involved is

$$N = \begin{cases} \frac{5}{2}T(T-1)n & \text{for the non-monetary economy} \\ \frac{5}{2}T(T-1)(n+1) & \text{for the monetary economy.} \end{cases}$$

Analytically identical results are developed below for both monetary and non-monetary economies. To avoid repetition, N will be used to represent the dimensionality in both cases.

Definition. A social approximate equilibrium of modulus A is a price vector $p^* \in P$ (for the non-monetary economy) or $p^* \in P^\alpha$ (for the monetary economy) and two allocations a^{h*} and $a^{h\dagger}$ such that

(a) $\sum_h x^{h*} \leq \sum_h y^{h*}$,

(b) $p_{it}^*(\tau) = 0$ if $\sum_h x_{it}^{h*}(\tau) < \sum_h y_{it}^{h*}(\tau)$,

(c) $a^{h\dagger} \in \gamma^h(p^*)$,

(d) a^{h*} satisfies (2.4) at p^* all t,

(e) $| \sum_h a^{h*} - \sum_h a^{h\dagger} | \leq A$.

The usual technique for demonstrating the existence of an approximate equilibrium is to take convex hulls of the preference and technology sets, apply conventional techniques suitable for analysing the newly convex economy, and finally bound the disparity between the convexified economy and the non-convex economy by use of the Shapley-Folkman theorem. Following the approach successful in other contexts one would be led to convexify the household transactions technologies. The result, however, would be a serious oversight, since the household optimizing behaviour in an economy thus convexified is very different from behaviour in the actual economy. Households facing convexified transactions technologies will have no set-up cost induced behaviour such as bunching of purchases and sales with the resultant inventory holding. Convexification of the household demand correspondence is far more helpful, since it convexifies the behaviour, not the initial conditions, of the non-convex economy.

Let $R(p) = \sup_h \rho(\gamma_h(p))$. $R(p)$ is the maximum over all households at prices p of the inner distance of the household demand correspondence $\rho(\gamma^h(p))$. Recall that p^* is the convexified equilibrium price vector. Is it reasonable to suppose that $R(p^*)$ be finite? Since the number of households is finite, the only way this condition can fail to hold is if some household's demand correspondence at prices p^* includes points arbitrarily far from one another and excludes intermediate points. Such a condition could arise only if (i) there were unbounded scale economies in the transactions technology, and (ii) the household were indifferent among isolated bundles arbitrarily far apart all consistent with budget constraint (hence requiring free goods) and transactions technology. The possibility seems remote enough to exclude. We can then approximate points in the convex hull as the average of points in the original non-convex sets.

We may wish to consider the behaviour of the economy as the number of agents becomes large. How will $R(p^*)$ react? The addition of new endowments, tastes, and technologies may of course cause the convexified equilibrium price vector, p^*, to change. Even for p^* constant, new tastes and technologies may cause $R(p^*)$ to increase. To take the simplest case, if the economy grows by replication $R(p^*)$ will remain unchanged since tastes, technologies, and convexified equilibrium prices remain constant. Addition of other households depending on their endowments, tastes and technologies may make $R(p^*)$ increase or

decline. To the extent that the growth is in households similar to those already in the economy, there is a presumption that $R(p^*)$ will remain relatively constant. For expository purposes below we will suppose that $R(p^*)$ is independent of the number of households in the economy.

Theorem 6.1. *Let p^* be an equilibrium price vector for the convexified economy. Let $R(p^*)$ exist, and be finite. Then at p^* there is a social approximate equilibrium of modulus $NR(p^*)$.*

Theorem 6.1 says that if $R(p^*)$ exists and is finite, for p^* the convexified equilibrium price vector, then there is a social approximate equilibrium at prices p^* of modulus $NR(p^*)$. Thus, there is a bound on the difference between aggregate demand and supply. Further, the bound is independent of the number of households in the economy so that in a large economy the disequilibrium is proportionately small. However, Theorem 6.1 provides no guarantee that the discrepancy between $a^{h\dagger}$ and a^{h*} be small for the households involved. The subsequent Theorem 6.2 uses the bound on aggregate disequilibrium established in Theorem 6.1 to show existence of the individual approximate equilibrium, an approximate equilibrium with a bound on the sum of the lengths of individual discrepancies. As the number of households increases the average discrepancy becomes correspondingly small.

Definition. An individual approximate equilibrium of modulus A is a price vector $p^* \in P$ (for the non-monetary economy) or $p^* \in P^\alpha$ (for the monetary economy) and two allocations a^{h**}, $a^{h\dagger}$ such that

(a) $\sum_h x^{h**} \leqq \sum_h y^{h**}$,

(b) $p^*_{it}(\tau) = 0$ if $\sum_h x^{h**}_{it}(\tau) < \sum_h y^{h**}_{it}(\tau)$,

(c) $a^{h\dagger} \in \gamma^h(p^*)$,

(d) a^{h**} fulfils (2.4) at p^*,

(e) $(\sum_h | a^{h**} - a^{h\dagger} |^2)^{1/2} \leqq A$.

Theorem 6.2. *Let p^* be an equilibrium price vector for the convexified economy. Let finite $R(p^*)$ exist. Then at p^* there is an individual approximate equilibrium of modulus $NR(p^*)$.*

The individual approximate equilibrium specifies for each agent an action a^{h**} that results in fulfilment of the material balance requirement and (on average across households) imposes a relatively small disparity between the agents' desired and market clearing actions. If the economy becomes large, the disparity for a typical household converges to zero. Because of non-convexity, a^{h**} may not be in the household's transactions technology, and as such does not represent an attainable allocation. Theorem 6.3 provides conditions under which it is possible to find individual approximate equilibria that are also elements of the household transactions technologies and provide for non-negative consumptions. Again, in a large economy, the difference between the two allocations for each household is correspondingly small.

Definition. An attainable individual approximate equilibrium of modulus A is a price vector $p^* \in P$ (for the non-monetary economy) or $p^* \in P^\alpha$ (for the monetary economy) and two allocations $a^\dagger$, $a^{\dagger\dagger}$, such that:

(a) $\sum_h x^{h\dagger\dagger} \leqq \sum_h y^{h\dagger\dagger}$,

(b) $c^{h\dagger\dagger} \geqq 0$,

(c) $a^{h\dagger} \in \gamma_h(p^*)$ all h,

(d) $a^{h\dagger\dagger}$ fulfils (2.4) at p^*,

(e) $a^{h\dagger\dagger} \in T^h \times S^h$,

(f) $(\sum_h \mid a^{h\dagger} - a^{h\dagger\dagger} \mid^2)^{1/2} \leqq A$.

Figure 2 shows typical consumption choices available to a household. ω is the endowment point, c^* is the consumption associated with a^*, c^{**} with a^{**}, $c^\dagger$ with $a^\dagger$, and $c^{\dagger\dagger}$ with $a^{\dagger\dagger}$. The dotted line represents the convex hull of the set of chosen consumptions. Households are indifferent between the two separated chosen points, ω, $c^\dagger$; no point between them is achievable and so cannot be chosen. A large number of such households distributed among the chosen points can approximate in their average behaviour any point in the convex hull of the chosen points.

In seeking an attainable individual approximate equilibrium we have to find an allocation near to $a^\dagger$ that is market clearing, leaves each household within its transactions technology, and assures it of non-negative consumption. The defect of $a^\dagger$ is that the allocation implies shortages; intended purchases of some commodities are greater than intended sales. The solution is then either to increase sales or reduce purchases. Increasing sales, even by a small amount, may cause a discrete increase in transactions cost, moving the allocation even farther from market clearing. A reduction in purchases will not increase transactions costs but if the purchases so reduced are of inputs to the transactions technology large changes in purchases and sales may be made necessary by small reductions in inputs to transactions. Again, this may result in a large move away from equilibrium. A reduction in purchases of goods going to consumption can have no such disruptive effect, but the reduction must not be so large as to cause households to have negative consumption. The conclusion is that if the goods in short supply at $a^\dagger$ are consumed at $a^\dagger$ from purchases (not from endowment) in amounts larger than the shortage, then there is an attainable individual approximate equilibrium.

This is the significance of the inequality in Theorem 6.3 below. $(x^{h\dagger} - y^{h\dagger} - z^{h\dagger})$ is the household's purchases net of sales and transactions costs. $(r^{h\dagger} - s^{h\dagger} - w^h)$ is the net input to storage not coming from endowment. The difference between the two is the portion of purchases going to consumption (rather than into storage or transactions costs). The inequality requires that the sum over households of purchases intended for consumption exceed the shortage of goods implied by the allocation $a^{h\dagger}$.

Theorem 6.3. *Let* p^*, $a^{h\dagger}$, a^{h**} *be an individual approximate equilibrium of modulus A and let* $[\Sigma((x^{h\dagger} - y^{h\dagger} - z^{h\dagger})^+ - (r^{h\dagger} - s^{h\dagger} - \omega^h)^+)] \geqq \Sigma(x^{h\dagger} - y^{h\dagger})$. *Then there is an attainable individual approximate equilibrium of modulus A.*

We may wish to consider attainable individual approximate equilibrium where the discrepancy between the attainable allocation and the desired allocation is spread evenly across households, so that no household bears a disproportionately large burden of the disequilibrium. An attainable approximate equilibrium with a uniform bound on Euclidean length of the disparity between the two allocations is defined below.

Definition. An attainable individual *uniformly* approximate equilibrium of modulus ε is a price vector $p^* \in P$ (for the non-monetary economy) or $p^* \in P^\alpha$ (for the monetary economy) and two allocations $a^\dagger$, $a^{\dagger\dagger\dagger}$ such that

(a) $\sum_h x^{h\dagger\dagger\dagger} \leqq \sum_h y^{h\dagger\dagger\dagger}$,

(b) $a^{h\dagger} \in \gamma_h(p^*)$ all h,

(c) $a^{h\dagger\dagger\dagger}$ fulfils (2.4) at p^*,

(d) $a^{h\dagger\dagger\dagger} \in T^h \times S^h$,

(e) $c^{h\dagger\dagger\dagger} \geqq 0$,

(f) $| a^{h\dagger} - a^{h\dagger\dagger\dagger} | \leqq \varepsilon$ for all h.

Theorem 6.4. *Let* p^*, $a^\dagger$, $a^{\dagger\dagger}$ *be an individual attainable approximate equilibrium of modulus A and suppose there is for each household h, $e^h \in R^N$, so that*

(i) $| e^h | \leqq \varepsilon$,

(ii) $\sum_h e^h = (\sum_h x^{h\dagger} - \sum_h y^{h\dagger})^+$

(iii) $(x^{h\dagger} - y^{h\dagger} - z^{h\dagger})^+ - (r^{h\dagger} - s^{h\dagger} - \omega^h)^+ \geqq e^h \geqq 0$.

Then there is $a^{\dagger\dagger\dagger}$ so that p^*, $a^\dagger$, $a^{\dagger\dagger\dagger}$ is an attainable individual uniformly approximate transactions equilibrium of modulus ε.

As in Theorem 6.3 above, the conditions of Theorem 6.4, particularly (ii) and (iii), ensure that the sum over households of purchases intended for consumption exceed shortages at allocation $a^{h\dagger}$. Condition (i) requires, further, that for each commodity in short supply there is a substantial number of households with purchases in excess of those required for storage or transactions costs. Thus the implied adjustment in consumption need not be large for any household. (i) may require that relatively large numbers of agents have positive levels of purchases on each market. Such a requirement is contrary to the tendency toward specialization in agents' actions that is implied by increasing returns in the transactions technology.

Under the assumptions of Sections 4 and 5, and if $R(p^*)$ is finite, we have:

(i) there exists at p^* a social approximate equilibrium of modulus $NR(p^*)$ and an individual approximate equilibrium of the same modulus;

(ii) if intended consumption out of purchases is large enough at p^* there exists an attainable individual approximate equilibrium of modulus $NR(p^*)$;

(iii) if all households have large enough intended consumption out of purchases at p^* there is an attainable individual uniformly approximate equilibrium of modulus ε where ε varies inversely with the number of households.

7. PROOFS

Lemma 3.1. *Let $\hat{B}^h(p^0)$ be locally interior. Then $\overline{\hat{B}^h(p^0)}$ is lower semi-continuous at p^0.*

Proof. Since $0 \in \hat{B}^h(p)$ for all p, we know that $\hat{B}^h(p^0)$ is lower semi-continuous at 0.

By local interiority, there is a continuous function f such that (for $\sigma < 1$) $f(\sigma)$ strictly satisfies the budget constraints (2.4) at p^0 and lies in $\hat{B}^h(p^0)$. It follows that for each $\sigma < 1$, $f(\sigma)$ also satisfies (2.4) with strict inequality for all p in some neighbourhood U_σ of p^0. Thus $f(\sigma) \in \hat{B}^h(p)$ for $p \in U_\sigma$.

For convenience, we make use of an alternative definition of l.s.c.: For all p^0 and open sets G such that $\hat{B}^h(p^0) \cap G \neq \varnothing$ there exists a neighbourhood $U(p^0)$ such that $p \in U(p^0)$ implies $\hat{B}^h(p) \cap G \neq \varnothing$.

Let $a^0 \in \hat{B}^h(p) \cap G$. Let f be a continuous function such that $f(1) = a^0$. Thus $f(\sigma) \in G$ for σ close to one, and $f(\sigma) \in \hat{B}^h(p)$ for $p \in U_\sigma$. Thus, for such σ and p, $f(\sigma) \in \hat{B}^h(p) \cap G$.

The l.s.c. of $\overline{\hat{B}^h(p)}$ follows immediately from that of $\hat{B}^h(p)$.   QED

Lemma 3.2. *$\overline{\hat{B}^h(p)}$ is upper semi-continuous for all $p \in P$.*

Proof. Let $p^v \to p^0$. Let $a^v \in \hat{B}^h(p^v)$, $a^v \to a^0$. We seek to show that $a^0 \in \overline{\hat{B}^h(p^0)}$, that is, that a^0 fulfils (2.1)-(2.4) at p^0 and that $x^0 - y^0 - z^0$ is not less than zero in every component. This last follows from continuity of inequalities. (2.1) is fulfilled by the continuity of the

addition operation. (2.2) and (2.3) follow from closedness of $S^h(t)$, $T^h(t)$. (2.4) follows from continuity of the dot product and addition.   QED

Lemma 3.3. Let $\hat{B}^h(p^0)$ be locally interior for all p^0. Then $\gamma^h(p^0)$ is upper semi-continuous at p^0.

Proof. Obvious in view of the previous lemmas, the fact that utility functions are continuous, and the theorem ([3], p. 116) that sets of maximizing points of a continuous function constrained by a continuous correspondence are upper semi-continuous.   QED

Theorem 4.1 (Existence of equilibrium for the convexified non-monetary economy). Under (A.1)-(A.5), there is a price vector $p^* \in P$ and an allocation $\langle a^{h*} \rangle_h$ such that

$$a^{h*} \in \text{con } \gamma^h(p^*),$$

$$\sum_{h=1}^H x^{h*} \leqq \sum_{h=1}^H y^{h*}$$

with $p_{it}^*(\tau) = 0$ for i, t, τ such that the strict inequality holds

Proof. Define $a(t) = \langle a^h(t) \rangle_{h=1}^H$, $a = \langle a(t) \rangle_{t=1}^T$, and $\hat{\gamma}_\rho(p) = \sum_{h=1}^H \hat{\gamma}_\rho^h(p)$. Let

$$\mu_t(a(t)) = \{p(t) \in S^t \mid p(t)(x(t) - y(t)) \text{ is maximal in } S^t\}$$

and $\mu_T(a(T)) = \{p(T) \in S^T \mid p(T)(x(T) - y(T)) \text{ is maximal in } S^T\}$ and define

$$\mu(a) = \overset{T}{\underset{t=1}{\text{X}}} \mu_t(a(t)).$$

Consider the transformation $\Phi(a, p) = \hat{\gamma}_\rho(p) \times \mu(a)$.

As we have seen, this transformation satisfies all the properties required to apply the Kakutani Theorem and so there is a fixed point

$$(a^*, p^*) \in \hat{\gamma}_\rho(p^*) \times \mu(a^*).$$

Thus, $a^* \in \hat{\gamma}_\rho(p^*)$ is such that $(a^* = (x^*, y^*, z^*, r^*, s^*))$

$$0 \geqq p^*(t)(x^*(t) - y^*(t)) \geqq p(t)(x^*(t) - y^*(t)), \text{ for all } p(t) \in S^t. \qquad \ldots(7.1)$$

For $t \leqq T$, successively use the inequalities (7.1) for the vectors

$$p^i(t) = (0, \ldots, 0, 1, 0, \ldots, 0) \in S^t$$

where unity occurs in the ith component. Thus, $x^*(t) \leqq y^*(t)$ for $t \leqq T$.

All that remains to be shown is that an equilibrium in the ρ-economy is an equilibrium in the unbounded economy, for sufficiently large ρ. Let a^ρ denote an equilibrium in the economy with ρ as an upper bound. If $\|a^\rho\| \to \infty$ as $\rho \to \infty$, then $\|z^\rho\| \to \infty$, by our boundedness assumption (A.5). But, as is easy to verify, $c + z \leqq \omega$ in any equilibrium, a contradiction. Thus, the constraint that a^h lie in the cube defined by ρ is not binding for sufficiently large ρ.   QED

Theorem 5.1 (Existence of equilibrium for the convexified monetary economy). *Under* (A.1)-(A.5), (M.1)-(M.4), and (S), for any α, $0 < \alpha < 1$, there is a price vector $p^* \in P^\alpha$ and an allocation $\langle a^{h*} \rangle_h$ such that

$$a^{h*} \in \text{con } \gamma^h(p^*),$$

$$\sum_{h=1}^H x^{h*} \leqq \sum_{h=1}^H y^{h*}$$

with $p_{it}^*(\tau) = 0$ for i, t, τ such that the strict inequality holds, and

$$p_{0t}^*(t) \neq 0 \text{ for all } t.$$

Proof. As in the proof of Theorem 4.1, we obtain a fixed point $(a^*, p^*) \in \hat{\gamma}_\rho(p^*) \times \mu(a^*)$, where now $\mu_T(a(T)) = \{p(T) \in S_\alpha^T \mid p(T)(x(T) - y(T)) \text{ is maximal in } S_\alpha^T\}$. From this we again obtain the inequalities (7.1) except that $p^*(T)$ and $p(T)$ belong to S_α^T rather than S^T.

For $t < T$, successively use the inequalities (7.1) for the vectors

$$p^i(t) = (0, \ldots, 0, 1, 0, \ldots, 0) \in S^t$$

where unity occurs in the ith component. Thus, $x^*(t) \leqq y^*(t)$ for $t < T$. We cannot use this argument for $t = T$ because $p_{0T}(T) \geqq \alpha > 0$, for $p(T) \in S_\alpha^T$.

Now, if $p_{kt}^*(v) = 0$, we claim that $y_{kt}^{h*}(v) = 0$. For, we know that $y_{kt}^{h*}(v)$ is a convex combination of utility-maximizing vectors for each h. But any $y_{kt}^h(v) > 0$ is transactions-costly, so that it is better that excess amounts of good (k, t) be freely disposed of (through the storage-technology) if its marginal utility is negative. Hence $y_{kt}^*(v) = 0$ because a convex combination of zeroes is still zero. It follows that $x_{kt}^*(v) < y_{kt}^*(v)$ is impossible, because then $p_{kt}^*(v) = 0$ (by the definition of $\mu_t(\cdot)$) and so $x_{kt}^*(v) < 0$, a contradiction. In particular, we can now say that $x_{kT}^*(v) = y_{kT}^*(v)$, for $v < T$.

However, we can still show that $x_{0T}^*(T) = y_{0T}^*(T)$. Because of constraint (5.1), the fact that $c_0^*(T) = 0$, and $x_{0T}^*(v) = y_{0T}^*(v)$ (for $v < T$), we have $(M = \Sigma_h M^h)$

$$0 = c_0^*(T) = \omega_0(T) + s_0^*(T) + \sum_{v=1}^{T-1} [x_{0T}^*(v) - y_{0T}^*(v)] + x_{0T}^*(T) - y_{0T}^*(T) - M$$

$$= \omega_0(T) + s_0^*(T) - M + x_{0T}^*(T) - y_{0T}^*(T). \qquad \ldots(7.2)$$

But, $\omega_0(T) + s_0^*(T) = M$, because of (5.1), the durability of money (M.1) and because there is equilibrium in all periods prior to T. Thus, $x_{0T}^*(T) = y_{0T}^*(T)$.

This means that

$$0 \geqq p^*(T)(x^*(T) - y^*(T)) = \sum_{k=1}^{n} p_{kT}^*(T)(x_{kT}^*(T) - y_{kT}^*(T))$$

$$\geqq \sum_{k=1}^{n} p_{kT}(T)(x_{kT}^*(T) - y_{kT}^*(T))$$

for all $p(T) \in S_\alpha^T$. But $(\alpha, 0, \ldots, 0, 1-\alpha, 0, \ldots, 0) \in S_\alpha^T$ and so $x^*(T) \leqq y^*(T)$.

To show the positivity of the spot price of money, we argue the contrapositive. If $p_{0t}^*(t) = 0$, for some t, then we will argue that large positive values of $x_{0t}(t)$ are possible for some household. This is not trivial because there are transactions costs to positive $x_{0t}(t)$ and there may be only zero prices for goods the household owns. However, (M.3) guarantees that some household (h', let us say) can acquire arbitrarily large $x_{0t}(t)$ using his own resources. Hence, household h' may acquire arbitrarily large amounts of cash at t and store it until T whereupon, by assumption (M.4), he may trade it at price $p_{0T}^*(T) = \alpha > 0$, thereby acquiring arbitrarily large amounts of goods at T. By assumption (S) (used repeatedly for $t, t+1, \ldots, T-1$, if necessary), this is always preferable to any consumption plan with $x_{0t}^h(t) = 0$. But $x_{0t}^*(t) = y_{0t}^*(t) = 0$, if $p_{0t}^*(t) = 0$, as was shown earlier. Yet, $x_{0t}^{*h'}$ is the convex combination of maximizing plans. But any maximizing plan must have positive $x_{0t}^{*h'}(t)$ and so $x_{0t}^{*h'}(t) > 0$, a contradiction.

All that remains to be shown is that an equilibrium in the ρ-economy is an equilibrium in the unbounded economy, for sufficiently large ρ.

Let a^ρ denote an equilibrium in the economy with ρ as an upper bound. If $\| a^\rho \| \to \infty$ as $\rho \to \infty$, then $\| z^\rho \| \to \infty$, by our boundedness assumption (A.5). But, as is easy to verify, $c + z \leqq \omega$ in any equilibrium, a contradiction. Thus, the constraint that a^h lie in the cube defined by ρ is not binding for sufficiently large ρ. $\qquad$ QED

Theorem 6.1. *Let p^* be an equilibrium price vector for the convexified economy. Let $R(p^*)$ exist, and be finite. Then at p^* there is a social approximate equilibrium of modulus $NR(p^*)$.*

Proof. We have $a^{h*} \in \text{con } \gamma^h(p^*)$, $\Sigma x^{h*} < \Sigma y^{h*}$, $p_{it}^*(\tau) = 0$ if $\sum_h x_{it}^{h*}(\tau) < y_{it}^{h*}(\tau)$. This gives properties (a), (b), (d). Applying Theorem A gives the existence of $a^{h\dagger}$ satisfying (c) and (e).   QED

Theorem 6.2. *Let p^* be an equilibrium price vector for the convexified economy. Let finite $R(p^*)$ exist. Then at p^* there is an individual approximate equilibrium of modulus $NR(p^*)$.*

Proof. Let a^{h*} be the equilibrium allocation for the convexified economy, and let $a^{h\dagger}$ be a social approximate equilibrium of modulus $NR(p^*)$. Choose $e^h \in R^N$ so that

(i) $\sum_h e^h = \sum_h a^{h*} - \sum_h a^{h\dagger}$

(ii) no coordinate of e^h is opposed in sign to the corresponding coordinate of

$$\sum_h a^{h*} - \sum_h a^{h\dagger};$$

(iii) e^h fulfils (2.4) at p^*.

These conditions are compatible since a^{h*}, $a^{h\dagger}$ fulfil (2.4).

Let $a^{h**} = a^{h\dagger} + e^h$.

$$\sum_h x^{h**} = \sum_h x^{h\dagger} + \sum_h (x^{h*} - x^{h\dagger}) = \sum_h x^{h*}.$$

$$\sum_h y^{h**} = \sum_h y^{h\dagger} + \sum_h (y^{h*} - y^{h\dagger}) = \sum_h y^{h*}:$$

Since p^*, a^{h*}, $a^{h\dagger}$ is a social approximate equilibrium, we have $\Sigma x^{h*} \leq \Sigma y^{h*}$ and $p_{it}^*(\tau) = 0$ if $\Sigma_h x_{it}^{h*}(\tau) < \Sigma_h y_{it}^{h*}(\tau)$. But since $\Sigma x^{h**} = \Sigma x^{h*}$, $\Sigma y^{h**} = \Sigma y^{h*}$, we have

(a) $\sum_h x^{h**} \leq \sum_h y^{h**}$,   and

(b) $p_{it}^*(\tau) = 0$ if $\sum_h x_{it}^{h**}(\tau) < \sum_h y_{it}^{h**}(\tau).$

Since p^*, a^*, $a^\dagger$ is a social approximate equilibrium

(c) $a^{h\dagger} \in \gamma^h(p^*).$

This gives that $a^{h\dagger}$ fulfils (2.4) at p^*, as does e^h by (iii). Hence

(d) $a^{h**}(= a^{h\dagger} + e^h)$ fulfils (2.4) at p^*.

(i) and (ii) give $(\Sigma \mid e^h \mid^2)^{1/2} \leq \mid \Sigma a^{h*} - \Sigma a^{h\dagger} \mid$.

But $\mid \Sigma a^{h*} - \Sigma a^{h\dagger} \mid \leq NR(p^*)$ and $e^h = a^{h**} - a^h$, so

(e) $(\Sigma \mid a^{h**} - a^{h\dagger} \mid^2)^{1/2} \leq NR(p^*).$ QED

Theorem 6.3. *Let p^*, $a^{h\dagger}$ a^{h**} be an individual approximate equilibrium of modulus A and let $[\Sigma((x^{h\dagger} - y^{h\dagger} - z^{h\dagger})^+ - (r^{h\dagger} - s^{h\dagger} - \omega^h)^+)] \geq \Sigma(x^{h\dagger} - y^{h\dagger})$. Then there is an attainable individual approximate equilibrium of modulus A.*

Proof. Choose $\alpha^h \in R^N$ so that

(1) $\Sigma\alpha^h = (\Sigma(x^{h\dagger} - y^{h\dagger}))^+,$

(ii) $x^h \geq \alpha^h \geq 0$, all h,   and

(iii) $(x^{h\dagger} - y^{h\dagger} - z^{h\dagger})^+ - (r^{h\dagger} - s^{h\dagger} - \omega^h)^+ \geq \alpha^h$ all h.

The existence of such α^h is guaranteed by hypothesis. Let

$$a^{h\dagger\dagger} = (x^{h\dagger} - \alpha^h, y^{h\dagger}, z^{h\dagger}, r^{h\dagger}, s^{h\dagger}).$$

$a^{h\dagger} \in \gamma^h(p^*)$ implies $a^{h\dagger} \in T^h \times S^h$. Thus by free disposal $a^{h\dagger\dagger} \in T^h \times S^h$. $a^{h\dagger\dagger}$ satisfies (2.4) since $a^{h\dagger}$ fulfils (2.4) and $a^{h\dagger\dagger}$ merely reduces the smaller side of the inequality.

$$\sum_h x^{h\dagger\dagger} = \sum_h x^{h\dagger} - \sum_h \alpha^h = \sum_h x^{h\dagger} - (\sum_h (x^{h\dagger} - y^{h\dagger}))^+ \leq \sum_h y^{h\dagger} = \sum_h y^{h\dagger\dagger}$$

so $a^{h\dagger}$ fulfils (a). $a^{h\dagger} \in \gamma^h(p^*)$ so $c^{h\dagger} \geq 0$, but $c^{h\dagger} \geq \alpha^h \geq 0$ so $c^{h\dagger\dagger} = c^{h\dagger} - \alpha^h \geq 0$. p^*, $a^{h\dagger}$, a^{h**} is an individual approximate equilibrium, so $a^{h\dagger} \in \gamma^h(p^*)$ and (c) and (d) are ful lled. To satisfy (f), note that p^*, $a^{h\dagger}$, a^{h**} is an individual approximate equilibrium of modulus A. But since

$$(\sum_h |a^{h\dagger} - a^{h**}|^2)^{1/2} \leq A \quad \text{and} \quad \sum_h x^{h**} \leq \sum_h y^{h**},$$

$$|\sum_h x^{h\dagger} - \sum_h y^{h\dagger}| \leq A$$

so

$$(\sum_h |\alpha^h|^2)^{1/2} \leq A.$$

But

$$a^{h\dagger\dagger} - a^{h\dagger} = (\alpha^h, 0, 0, 0, 0)$$

so

$$(\sum_h |a^{h\dagger\dagger} - a^{h\dagger}|^2)^{1/2} \leq A,$$

fulfilling (f).   QED

Theorem 6.4. *Let p^*, $a^\dagger$, $a^{\dagger\dagger}$ be an individual attainable approximate transactions equilibrium of modulus A and suppose there is for each household h, $e^h \in R^N$, so that*

(i) $|e^h| \leq \varepsilon$,

(ii) $\Sigma e^h = (\Sigma x^{h\dagger} - \Sigma y^{h\dagger})^+$,

(iii) $(x^{h\dagger} - y^{h\dagger} - z^{h\dagger})^+ - (r^{h\dagger} - s^{h\dagger} - \omega^h)^+ \geq e^h \geq 0$.

Then there is $a^{\dagger\dagger\dagger}$ so that p^*, $a^\dagger$, $a^{\dagger\dagger\dagger}$ is an attainable individual uniformly approximate transactions equilibrium of modulus ε.

Proof. Let $a^{h\dagger\dagger\dagger} = a^{h\dagger} - (e^h, 0, 0, 0, 0)$. Since $a^{h\dagger} \in T^h \times S^h$, $a^{h\dagger\dagger\dagger} \in T^h \times S^h$ by free disposal. (ii) implies (a). Since $a^{h\dagger}$ fulfils (2.4), the reduced purchases of $a^{h\dagger\dagger\dagger}$ must also fulfil (2.4). (iii) ensures (e) and (f) follow from (i).   QED

NOTES

1. Non-convex transactions costs are not the only possible motivation for holding stocks of money. Because of transactions costs (convex or non-convex), holding stocks of money may be the least costly way of transferring purchasing power from one period to the next, or such stocks may be held for speculative gains due to expected decline in the price level.

2. In a continuum model the issue of individual convexification does not arise since convexification in aggregate is assured by the Lyapunov theorem [2].

3. Readers not interested in development of the mathematical structure may skip this section without loss of continuity; they should note only Lemma 3.3, that local interiority of the budget correspondence is a sufficient condition for upper semi-continuity of the demand correspondence. Extensions and further discussion of the ideas in this section are contained in [9].

4. Readers not interested in the introduction of money to this model may skip this section and proceed to Section 6 without loss of continuity.

5. Note that these constraints imply that forward contracts require payment today and delivery in the future, rather than an exchange of both money and goods in the future. Also, bonds here are not annuities, they are discounted notes.

6. For an analysis of a model similar to this one, but where (as emphasized in [4]) goods cannot buy goods, see [10].

7. Constraint (5.1) could be replaced by the requirement that $r_0^h(T) \geqq M^h$, where

$$\Sigma_h M^h = M = \Sigma_h \Sigma_i \omega_0^h(t).$$

8. The price need not fall strictly to 0 but merely to a positive value so low that no monetary trade takes place because gains from trade are swamped by transactions costs.

9. Though the converse does hold; if its price is zero, money is surely useless.

10. If we suppose that consumers, producers, and price formation are insensitive to disequilibria small as a proportion of total activity, then an approximate equilibrium of sufficiently small modulus (bound on the discrepancy between supply and demand) would be virtually indistinguishable from a true equilibrium. If, on the contrary, agents and markets are very sensitive to proportionally small discrepancies between supply and demand, then the distinction between a true equilibrium and an approximate equilibrium will be of major importance and the latter will be a very poor substitute for the former.

11. Arrow and Hahn spend several pages demonstrating this point without explicitly announcing it as a result.

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