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PAIRWISE, t-WISE, AND PARETO OPTIMALITIES

By Steven M. Goldman and Ross M. Starr[1]

An allocation is said to be t-wise optimal (for t a positive integer) if for every collection of t traders, there is no reallocation of their current holdings that will make some better off while making none worse off. The allocation is pairwise optimal if it is t-wise optimal for $t = 2$. A t-wise optimal allocation is the outcome of a trading process more decentralized than that of the Walrasian equilibrium. It represents the result of a variety of separate transactions in small groups without the (centralized) coordination provided by a single Walrasian auctioneer.

Necessary conditions and sufficient conditions on allocations for t-wise optimality to imply Pareto optimality are developed. These generally require sufficient overlap in goods holdings among traders to ensure the presence of common support prices. This is formalized as indecomposability of a truncated submatrix of the allocation matrix. A necessary and sufficient condition remains an open question.

0. INTRODUCTION

Our principal concern in this inquiry is with the decentralization of the trading process. The analysis departs from the familiar Arrow-Debreu general equilibrium framework to examine the efficiency of economies deprived of the coordinating function of the Walrasian price mechanism. The alternative, presented here, is to permit trade to take place only in small groups—say up to t traders in number. We envision an exchange economy wherein groups form and reform in order to barter—as individuals and as small coalitions. If *all* such small groups may form, then such a process might eventually converge to an equilibrium from which no reallocation involving t or fewer traders could result in a Pareto preferable allocation. That is, an allocation which is *t-wise optimal*. The dynamics of pairwise barter trade to achieve a pairwise optimal allocation is thoroughly studied in Feldman [**2**]. The corresponding analysis for trade in larger groups represents an open research topic, though we certainly expect Feldman's analysis to generalize.

It is by no means apparent that such a t-wise optimal allocation would be Pareto optimal. This reflects the difficulty of achieving a reallocation which is preferable for a large group through a sequence of weakly desirable small group trades. Since Pareto optimality is such an essential condition in welfare economics, it is useful to discover under what circumstances the two optimality concepts

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coincide. Below, we investigate necessary and sufficient conditions for the equivalence of t-wise and Pareto optimality.

A simple example can illustrate problems in relating the two optimality concepts. Consider a three-man, three-good economy with allocation

$$x^1 = \left(x_1^1, x_2^1, x_3^1\right) = (1, 1, 0),$$

$$x^2 = \left(x_1^2, x_2^2, x_3^2\right) = (0, 1, 1),$$

$$x^3 = \left(x_1^3, x_2^3, x_3^3\right) = (1, 0, 1),$$

and linear utility functions

$$u^1(x^1) = 2x_1^1 + x_2^1,$$

$$u^2(x^2) = 2x_2^2 + x_3^2,$$

$$u^3(x^3) = x_1^3 + 2x_3^3.$$

We can represent the allocation schematically by the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and preferences by the matrix of marginal utilities

$$P = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Rows represent traders and columns indicate commodities. The allocation is pairwise optimal but not Pareto (3-wise) optimal. Each agent holds one unit of his favorite good and his second most desired good. There is one other holder of the favorite good, and he regards the other good the first agent holds as worthless. Hence, there is no room for pairwise improvement. 3-wise improvement is possible, since each agent can be given more of his favorite in exchange for his second choice. In particular, the reallocation matrix

$$B = \begin{bmatrix} \epsilon & -\epsilon & 0 \\ 0 & \epsilon & -\epsilon \\ -\epsilon & 0 & \epsilon \end{bmatrix}$$

results in the Pareto preferable allocation

$$A + B = \begin{bmatrix} 1+\epsilon & 1-\epsilon & 0 \\ 0 & 1+\epsilon & 1-\epsilon \\ 1-\epsilon & 0 & 1+\epsilon \end{bmatrix}.$$

There is a utility increase of ϵ for each agent resulting from the reallocation.[2]

We are thus led to describe an allocation as pairwise optimal if for every possible pair of traders there is no reallocation of the pair's current holdings, between traders of the pair, that is weakly preferable to both traders with strict preference for at least one. Similarly, for a positive integer t, an allocation is said to be t-wise optimal if, for every group of t individuals, holdings are optimally allocated within the group. In an economy where trade takes place primarily among pairs or t-member groups of agents we expect the resulting allocation to be pairwise or t-wise optimal. It is then of interest to discover when pairwise or t-wise optimal allocations will be Pareto optimal as well.

As the example above suggests, corner solutions, the zeroes of the allocation matrix, play a pivotal role in the analysis. When traders' holdings have much in common with one another (i.e., when several positive entries coincide), the corresponding marginal rates of substitution will be equated across members of trading groups. Since the groups overlap, these MRS's will become common to all traders, hence leading to Pareto optimality. Conversely, if there is little overlap among agents' holdings the assurance of common MRS's is correspondingly weakened. Such a sparse overlap situation is likely to occur when there are many zeroes (corner solutions) in the allocation. This may lead, as in the example, to Pareto nonoptimality despite pairwise or t-wise optimality.

Conversely, sufficient overlap in traders' holdings ensures equivalence of t-wise optimality and Pareto optimality. If there is a universally held good, providing one point of overlap for all traders, then it acts like "money" to ensure the equivalence of pairwise, t-wise, and Pareto optimalities. Similarly, a single trader who holds positive amounts of all goods has complete overlap with all traders. He acts as a universal intermediary resulting again in the equivalence of pairwise, t-wise, and Pareto optimalities. The results are formalized as Theorems 1.1 and 1.2 below.

The focus on corner solutions seems appropriate inasmuch as most individuals do not consume most goods. This is particularly true when we think of commodities as differentiated by date, location, quality, and design.

We have concentrated upon restrictions on the allocation matrix A rather than on the utility functions. This is motivated by the direct observability of allocations (as compared with preferences) and forms the structure for Feldman [**2**] and Rader [**9**, **10**] as well. Thus, it might be argued, statements of Pareto

[2] In the example above, the zeroes of the allocation matrix A are essential to the analysis. The zeroes of the marginal utility matrix P are inessential. They could be replaced by small positive numbers so that the optimality properties of the allocation are retained.

optimality based on allocation and trade structure (i.e., t-wise optimality) are grounded in potentially verifiable observations.

1. REPRESENTATION OF ALLOCATIONS AND PREFERENCES

We consider allocations in a pure exchange economy of M consumers and N commodities. An *allocation* is an $M \times N$ nonnegative matrix, $A = (a_{ij})$, where the ijth entry, a_{ij}, represents agent i's holding of good j. (Every row and column of A is assumed to have at least one positive entry.) Let the *utility functions* of the agents be represented by a vector of C^2, quasi-concave functions $u = (u^1, \ldots, u^M)$, where $u^i : R_+^N \to R$. The *linearized utility functions about* A form a vector $\bar{u} = (\bar{u}^1, \ldots, \bar{u}^M)$ where

$$p_{ij} = \frac{\partial u^i(a_{i1}, \ldots, a_{iM})}{\partial a_{ij}} \qquad \text{and} \qquad \bar{u}^i(x) = \sum_{j=1}^{N} p_{ij} x_j.$$

In the case (of considerable interest here) where $a_{ij} = 0$, we shall define

$$p_{ij} = \lim_{k \to 0+} \frac{\partial u^i(a_i + ke_j)}{\partial k}$$

where $a_i = (a_{i1}, \ldots, a_{iN})$ and e_j is the jth unit vector. *Marginal preferences* are then described by the $M \times N$ nonnegative matrix $P = (p_{ij})$. Every row and column of P is assumed to have at least one positive entry.

In what follows, we wish to characterize the efficiency of the economy (A, u) by that of its linear counterpart $(A, \bar{u})$. It is rather straightforward to establish that if $(A, \bar{u})$ is Pareto optimal then so is (A, u), since the optimality of $(A, \bar{u})$ implies the existence of a price vector $\xi \in R_+^N$ where $(A, \bar{u}, \xi)$ must be a competitive equilibrium. Then (A, u, ξ) must be a competitive equilibrium as well given the quasi-concavity and smoothness of u.

But for the Pareto optimality of (A, u) to imply optimality for $(A, \bar{u})$ is equivalent to requiring that (A, u) can be supported by a competitive—not merely compensated—equilibrium. Sufficient conditions for the existence of such a support have been extensively investigated (see McKenzie [7] or Arrow and Hahn [1]) and are generally stated in terms either of a minimum wealth constraint or resource relatedness and irreducibility.

Still an alternative condition deals with a weakened version of monotonicity. For a given consumption bundle if an agent's marginal utility for a good is zero, we shall suppose that reducing his consumption of that good—the remainder of the allocation held fixed—will not reduce utility. That is, a marginal utility of zero is assumed to remain zero after large finite variations in the quantity of the good.[3] Under this hypothesis, each of the following three statements implies the others: (i) (A, u) is Pareto optimal; (ii) $(A, \bar{u})$ is Pareto optimal; (iii) (A, u) can be

[3] Rader [10] assumes that $a_{ij} > 0$ if and only if $u_j^i(a_i) > 0$.

supported by a competitive equilibrium. This argument is present as Lemmas A.1 and A.2 and Theorem A.1 in the Appendix.

We will not specify a particular set of sufficient conditions here. Rather, we shall limit our further discussion to those economies where Pareto optima have competitive supports. Economies (A, u) fulfilling this condition can then be represented, without loss of generality, by their linear counterparts, $(A, \bar{u})$.

A *reallocation* is an $M \times N$ matrix Z so that $A + \epsilon Z$ is an allocation for some $\epsilon > 0$ and $\sum_i z_{ij} = 0$ for each j. A *pairwise reallocation* is a reallocation so that $z_i = 0$ for all but two of $i = 1, \ldots, M$. A *t-wise allocation* $(t = 2, \ldots, N)$ is a reallocation so that $z_i = 0$ for all but t of $i = 1, \ldots, M$. Z is a t-wise improvement if $p_i z_i \geqq 0$ for all $i = 1, \ldots, M$ with strict inequality for at least one i.

A *state* is represented by (A, P). The state is *t-wise* optimal if there is no t-wise reallocation Z constituting a t-wise improvement. The state is said to be *Pareto optimal* if it is M-wise optimal. We wish to investigate the relationship between pairwise, t-wise, and Pareto optimality. In particular we will establish sufficient conditions and necessary conditions for pairwise and t-wise optimality to imply Pareto optimality. The characterization of conditions that are both necessary and sufficient remains an open question.

For a given allocation A, we are interested in the sets of preferences for which (A, P) is t-wise or Pareto optimal. Specifically, let $\Pi^t(A) = \{P \mid (A, P) \text{ is } t\text{-wise optimal}\}$ and $\Pi^*(A) = \{P \mid (A, P) \text{ is Pareto optimal}\}$. $\Pi^t(A)$ is the set of preferences such that an endowment of A would be an allocation unblocked by any coalition of size t or less. For some A, if we have $\Pi^t(A) = \Pi^*(A)$ then for that allocation, preferences consistent with t-wise optimality and Pareto optimality coincide and A is said to exhibit the *t-wise equivalence property*. Hence, if the state (A, P) is pairwise optimal and if we know $\Pi^2(A) = \Pi^*(A)$ then the inference is that (A, P) is also Pareto optimal. We can restate the major straightforward results in this area then as the following theorems.

THEOREM 1.1 (Rader [**9**]): *Let A have a strictly positive row. Then*

$$\Pi^2(A) = \Pi^*(A).$$

THEOREM 1.2 (Feldman [**2**]): *Let A have a strictly positive column. Then*

$$\Pi^2(A) = \Pi^*(A).$$

The intuition for these results is that strict positivity of a row or column allows the immediate introduction of a price system which supports the allocation.

A strictly positive row (Theorem 1.1) represents a trader holding positive quantities of all goods. The corresponding price system is then the vector of this trader's marginal utilities. Pairwise optimality implies that all marginal rates of substitution of goods held in positive quantity coincide with those of the strictly positive row. Hence the price system established will support the allocation.

A strictly positive column represents a universally held good. If the good has positive marginal utility for any trader then pairwise optimality implies a positive marginal utility for all traders. Use the universal good as numeraire and establish prices for all other goods. For any two traders with two goods held by both, equality of their respective MRS's is guaranteed by pairwise optimality. For two goods held separately (one by each of two traders) that there should be a common supporting value of the MRS to use as a price ratio is guaranteed by the presence of the common good. That is, for commodities k and l, traders h and i holding positive quantities of the two goods respectively with good c in common, we have

$$\mathrm{MRS}^h_{k,l} \geqq \mathrm{MRS}^h_{k,c} \cdot \mathrm{MRS}^h_{c,l} \geqq \mathrm{MRS}^i_{k,c} \cdot \mathrm{MRS}^i_{c,l} \geqq \mathrm{MRS}^i_{k,l}.$$

Setting $p_k = \mathrm{MRS}^h_{k,c}$ and $p_l = \mathrm{MRS}^i_{l,c}$ gives the required supporting prices.

2. SYMMETRY

The equivalence, if it occurs, of t-wise (or pairwise) efficiency and Pareto efficiency is symmetric across goods and traders. That is, consider a transpose economy where the names of traders and commodities are substituted for one another. Then t-wise optimality implies Pareto optimality in the original economy if and only if it does so in the transpose economy. In the light of Theorems 1.1 and 1.2 this observation is not surprising. It says that these two separate theorems are really special cases of a single and more general result.

The intuition behind the symmetry argument is that efficient allocation may be considered alternatively as how a trader places his scarce purchasing power (when an efficient allocation is characterized by a market equilibrium) or how supplies of each good are allocated across traders. In the first case the allocation rule is to equate the marginal utility per dollar of expenditure across uses. In the second it is for each good to equate across traders the ratio of marginal utility of income to the marginal utility of that good. Let A' denote the transpose of A. Then we state the following theorem.

THEOREM 2.1: *Let A be an allocation. Then $\Pi^t(A) = \Pi^*(A)$ if, and only if, $\Pi^t(A') = \Pi^*(A')$.*

The proof appears in the Appendix.

3. GENERALIZATION OF THE EARLY RADER/FELDMAN RESULTS

The above cited results by Rader and Feldman may be generalized to the case of t-wise optimality.

THEOREM 3.1: *Let A be an allocation matrix with some row (or column) having $t-2$ or fewer zero entries. Then $\Pi^t(A) = \Pi^*(A)$.*

The proof appears in the Appendix.

Intuitively, if there is a single individual who consumes all but $t-2$ of the commodities then that agent's MRS's form "enough" of a price system to preclude cycles of size $t+1$ or larger. By virtue of Theorem 2.1 this argument is equally applicable to a commodity consumed by all but at most $t-2$ agents. Theorem 3.1 includes as special cases (for $t=2$) Theorems 1.1 and 1.2.

4. RADER'S CONDITION FOR SUFFICIENCY

Rader [**10**] advances a considerably more general condition for equivalence than found in Theorem 1.1 and Theorem 1.2 above. This gain is offered at the expense however of a further limitation on the allowable space of preferences. Specifically, attention is confined to those cases where each individual has a zero marginal utility for any commodity which does not appear in his allocation bundle. We will denote this restriction as *Condition R*. Essentially, Rader's condition requires that agents may be ordered in such a way that a supporting price system can be constructed from their MRS's by a process of extension. With considerable license, we shall restate these results here.

Let X be a symmetric $N \times N$ matrix $[x_{ij}]$ and $y \subset \{1, \ldots, N\}$, where y has n members. Then define $[X \operatorname{proj} y]$ as the $n \times n$ matrix $[\hat{x}_{ij}]$ where $\forall i, j \in y, \hat{x}_{ij} = x_{ij}$.

Starting from an allocation matrix A we are going to apply this matrix operation to the product matrix $A'A$. A positive ijth entry in $A'A$ indicates that it is possible to find a single agent who holds goods i and j. When $[A'A]_{ij}$ is positive, then pairwise optimality involves a direct determination of a supporting MRS for goods i and j. Now, for an N-vector a, let $a^+ \subset \{1, \ldots, N\}$ denote the set of indices i so that $i \in a^+ \leftrightarrow a_i > 0$. Consider $[A'A \operatorname{proj} a^+]$. A positive ijth entry here indicates that $[A'A]_{ij}$ is positive for some i and j for which the a entries are positive.

Finally, we consider whether $[A'A \operatorname{proj} a^+]$ is reducible (or decomposable), that is, whether by identical rearrangements of rows and columns that matrix can be represented as block diagonal. If not, it is said to be irreducible.

What does the irreducibility of $[(A'A) \operatorname{proj} a^+]$ indicate? For any two goods in a^+ it means that there is a finite sequence of consumers in A so that the first and last elements of the sequence each hold one of the two goods and successive members of the sequence are related to common holdings of other goods in a^+.[4] Hence, if we augment A by a, no independent determination of an MRS would be added to the system.

Conversely, reducibility indicates that there are at least two goods for which there is no implicit determination of an MRS using only goods in a^+.

LEMMA 4.1: *Let A be an allocation matrix such that $\Pi^2(A) = \Pi^*(A)$ and let a be an individual allocation vector such that $[(A'A) \operatorname{proj} a^+]$ is irreducible. Then, if*

[4]The matrix $[(A'A) \operatorname{proj} a^+]$ is analogous to the transition matrix of a Markov process. There, irreducibility allows the transition between two states with positive probability after finitely many steps (see, for example, Kemeny and Snell [**5**]). Here, irreducibility permits the inference of an implicit MRS between two commodities from a sequence of agents.

preferences are restricted as per condition R above, the allocation A augmented by a, say D, also displays the pairwise equivalence property, i.e., $\Pi^2(D) = \Pi^*(D)$ *and D is called an R-extension of A. The proof is a special case of Lemma* 5.1.

THEOREM 4.2 (Rader **[10]**): *Assuming Condition R, a matrix which may be constructed through successive R-extensions from a single initial row will satisfy the pairwise equivalence property.*

The proof follows immediately from Lemma 4.1.

Below we propose an extension of these results to the case of t-wise efficiency and with the removal of the Condition R. Heuristically, we shall propose a condition for extension such that the new individual will be supported by any price system supporting the previous allocation. Then, observing from the earlier symmetry condition that both rows and columns may be added to an economy, we shall present the appropriate generalization of 4.2. The removal of Condition R is accomplished by modifying the extension rule slightly to provide that new individuals must not desire unconsumed commodities too much.

5. A GENERALIZATION OF RADER'S SUFFICIENCY CONDITION

LEMMA 5.1: *Let A be an allocation matrix such that* $\Pi^t(A) = \Pi^*(A)$ *and let a be an individual allocation vector such that* $[(A'A)^{t-1}\text{proj}\, a^+ \cup \{k\}]$ *for all* $k \leqq N$, *is irreducible. Then A augmented by a, denoted* A^+, *exhibits the t-equivalence property.*

In the above Lemma, the new allocation A^+ is called an extension of A.

The proof is offered in the Appendix.

THEOREM 5.1: *An allocation matrix which may be constructed by successive extensions of the* 1×1 *matrix* $A = (1)$ *by rows and/or columns will satisfy the t-wise equivalence property.*

The proof follows immediately from Lemma 5.1.

Irreducibility of $[(A'A)^{t-1}\text{proj}\, a^+ \cup \{k\}]$ indicates that for any two entries in a^+ or an entry in a^+ and k there is a sequence of traders related by common holdings along the sequence so that the first and last traders have holdings of the specified goods. The common goods relating adjacent members of the sequence are in $a^+ \cup \{k\}$ with the possible exception of subsequences of length no greater than $t-1$. Thus augmentation of A by a^+ will involve no independent determination of an MRS not already implied in A.

6. NECESSITY

All of the results offered thus far have dealt with sufficiency. We shall now direct our attention toward necessary conditions for equivalence. There is a bootstrap property to the relationship between t-wise and Pareto optimality. For an allocation A to exhibit the equivalence property, there must be a coincidence of all intermediate degrees of optimality. Now it is immediately apparent that $\Pi^t(A) \supset \Pi^*(A)$. Indeed, this is a simple consequence of the definition. We will prove that, in essence, if A is not t-equivalent then there exists some preference ordering which is t-wise optimal for A but allows for a Pareto improving trade among $t+1$ agents.

THEOREM 6.1: *$\Pi^t(A) = \Pi^*(A)$ if, and only if, $\Pi^t(A) = \Pi^{t+1}(A)$.*

The proof is contained in the Appendix.

Theorem 6.1 can be illustrated by a simple example for $t = 2$. If (A, P') is pairwise optimal but not Pareto optimal then the theorem assures us that there is P'' so that (A, P'') is pairwise but not 3-wise optimal. The following example gives us precisely this case:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \qquad P' = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \qquad P'' = \begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

We have shown in Theorem 5.1 that the extension process will always create an allocation with the equivalence property. Further we will argue that a necessary condition for A to have the equivalence property is that A be an extension of the (sub)allocation consisting of A with *any* row or column deleted.

THEOREM 6.2: *Suppose that $\Pi^t(A) = \Pi^*(A)$ and that A^- denotes A with some row vector, say a for agent v, removed. If $(A^{-\prime}A^-)$ is irreducible, then for all $k \leqq N$, $[(A^{-\prime}A^-)^{t-1}\text{proj}\, a^+ \cup \{k\}]$ is irreducible.*

The proof is contained in the Appendix.

The necessity of the irreducibility condition posited in Theorem 6.2 can be directly illustrated for $t = 2$. If irreducibility of $[(A^{-\prime}A^-)\text{proj}\, a^+]$ is not fulfilled then, reordering rows and columns, we can represent A as

v	$+$	$+$	0
	$\neq 0$	0	$\neq 0$
	0	$\neq 0$	$\neq 0$

.

If we then choose P to look like

2	1	0
1	0	2
0	2	1

,

we have an expanded version of the familiar three-man-three-good example, so that $P \in \Pi^2(A)$ but $P \notin \Pi^*(A)$.

Theorems 5.1 and 6.2 suggest that the irreducibility of $[(A^{-\prime}A^{-})^{t-1}\operatorname{proj} a^{+} \cup \{k\}]$ is very nearly a necessary and sufficient condition for t-equivalence. The gap between the necessary and sufficient conditions is those allocation matrices that fulfill the irreducibility property (necessity) but cannot be constructed by the irreducible extension process (sufficiency). Hence a typical counterexample to the conjecture that the condition is both necessary and sufficient would be

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

a multiple version of the three-man-three-good example we started with. $A^{-\prime}A^{-}$ is strictly positive for any deleted row so the irreducibility condition is trivially fulfilled. Nevertheless, as in the original example, pairwise optimality does not imply Pareto optimality.

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APPENDIX: PROOFS

REPRESENTATION OF A CONVEX ECONOMY BY ITS LINEAR COUNTERPART

LEMMA A.1: *Suppose* (i) (A, u) *is Pareto optimal, and* (ii) *if* $p_{ij} = 0$ *then* $u^i(a_i) = u^i(b_i)$ $\forall b_i$, $b_{ik} = a_{ik}$, $\forall k \neq j$. *Then* $(A, \bar{u})$ *is Pareto optimal and* $\exists \xi \in R^N_+$ *such that, taking* a_i *as i's endowment,* $(A, \bar{u}, \xi)$ *is a competitive equilibrium.*

PROOF OF LEMMA A.1: We shall first prove, by contradiction, that $\exists \xi$ such that $(A, \bar{u}, \xi)$ is a competitive equilibrium. Suppose not. Then, from Gale [3], there exists a "super self sufficient" subset of agents, S, and complement, S', where (a) $\forall s \in S$, $\forall s' \in S'$, $p_{sj} > 0 \rightarrow a_{s'j} = 0$; (b) $\exists s \in S$, $\exists j \in \{1, \ldots, N\}$, $\forall t \in S$ $a_{sj} > 0$ but $p_{tj} = 0$. Now $p_{sj} = 0$ implies, by hypothesis (ii), that a reduction in a_{sj} will leave agent s unaffected. But since $\exists s' \in S'$ such that $p_{s'j} > 0$, then a small transfer of good j from s to s' would be preferred by s'. Thus (A, u) would not be Pareto optimal. Therefore $\exists \xi$, such that $(A, \bar{u}, \xi)$ is a competitive equilibrium, and then $(A, \bar{u})$ is Pareto optimal. *Q.E.D.*

LEMMA A.2: *Suppose* $(A, \bar{u})$ *is Pareto optimal. Then* (A, u) *is also Pareto optimal.*

PROOF OF LEMMA A.2: $\bar{u}$ satisfies condition (ii) in Lemma A.1 above. Therefore, there exists ξ such that $(A, \bar{u}, \xi)$ is a competitive equilibrium.

Then (A, u, ξ) is also a competitive equilibrium. Suppose not. Then $\exists i \in \{1, \ldots, M\}$ and $z_i \in R^N$ such that $\xi z_i \leqq 0$, $u^i(a_i + z) > u^i(a_i)$.

By, quasi-concavity of u^i, $\sum_{j=1}^{N} p_{ij} z_{ij} > 0$ but since $\xi z_i \leqq 0$ this contradicts competitive equilibrium of $(A, \bar{u}, \xi)$ and, as argued above, also the Pareto optimality of $(A, \bar{u})$.

Thus, (A, u, ξ) is a competitive equilibrium and (A, u) is Pareto optimal. *Q.E.D.*

THEOREM A.1: *Let u^i, $i = 1, \ldots, M$, fulfill condition* (ii) *of Lemma A.1. Then (A, u) is a Pareto optimum if and only if $(A, \bar{u})$ is a Pareto optimum. (A, u, ξ) is a competitive equilibrium if and only if $(A, \bar{u}, \xi)$ is a competitive equilibrium.*

PROOF OF THEOREM A.1: The theorem follows directly from Lemmas A.1 and A.2.

PROVING THEOREM 2.1

We will find the following lemma useful:

LEMMA A.3: *Let $U \in \Pi^t(A)$; then $U' \in \Pi^t(A')$.*

PROOF OF LEMMA A.3: Let $U \in \Pi^t(A)$, $V = U'$; suppose contrary to the assertion of the theorem $V \notin \Pi^t(A')$. A proof by contradiction will be used in order to avoid consideration of all t-member subsets. t-optimality of (U, A) means that each t-member subeconomy in (U, A) is Pareto efficient. $V \notin \Pi^t(A')$ implies that there is some t-element subeconomy of (V, A') whose allocation is inefficient and, hence, for which there are no supporting prices. Let the t traders of that subeconomy be $J \subset \{1, \ldots, N\}$. By Lemma A.4 there is a t-good reallocation representing a Pareto improvement for J. Let the t goods be the t-element subset $I \subset \{1, \ldots, M\}$. The contradiction will be established by showing that the presence of supporting prices for I in (U, A) imply support prices for J in (V, A').

Consider the traders I in (U, A). $U \in \Pi^t(A)$ implies that there is $p \in R_+^N$ supporting A.

Let $\lambda_i = \max_{j=1,\ldots,N} u_{ij}/p_j$. We know $u_{ij}/p_j < \lambda_i \Rightarrow a_{ij} = 0$. Or, for all i, j so that $a_{ij} > 0$ we have $u_{ij}/p_j = \lambda_i$ and $u_{ij}/\lambda_i = p_j$.

Now consider the t traders J in (V, A'), and the t goods I which can be advantageously redistributed. We have for each $j \in J$, $v_{ji}/\lambda_i = u_{ij}/\lambda_i = p_j$ for all goods i so that $a_{ij} > 0$. Thus, $\lambda = (\lambda_i)_{i \in I}$ is a vector of support prices for the allocation $(a_{ij})_{j \in J, i \in I}$ of goods I among traders J. But the presence of these prices implies that $(a_{ij})_{j \in J, i \in I}$ is a Pareto efficient allocation, which is a contradiction.

t-wise optimality of (U, A) implies the presence of support prices for the allocation of every t-member subeconomy in (V, A'). Hence (V, A') is t-optimal; $V \in \Pi^t(A')$. *Q.E.D.*

LEMMA A.4: $\Pi^M(A) = \Pi^N(A)$.

PROOF OF LEMMA A.4: Two simple proofs are available. (i) Madden notes that if there is any blocking coalition, there is one of size N (Madden [**6**, Theorem 2]), or equivalently Graham, Jennergen, Peterson, and Weintraub argue that N-wise optimality implies Pareto optimality (Graham, Jennergen, Peterson, and Weintraub [**4**, Corollary 2]).

(ii) Equivalently, Lemma A.5 implies that an allocation that is blocked, is blocked by coalitions of size no greater than N.

LEMMA A.5: *Let Z be a t-improving allocation for (A, P). Then Z can be expressed as the sum of no more than tN t-improving reallocation matrices Z^h where Z^h has no more than one positive entry in each row and column.*

PROOF OF LEMMA A.5: Ostroy-Starr [**8**, Lemma 2].

We now have sufficient machinery to prove Theorem 2.1.

PROOF OF THEOREM 2.1: From Lemma A.4 we have $\Pi^M(A) = \Pi^N(A)$. By Lemma A.1, $U \in \Pi^N(A) = \Pi^M(A)$ if and only if $U' \in \Pi^N(A')$. So $U \in \Pi^M(A)$ if and only if $U' \in \Pi^N(A')$.

By Lemma A.3 $U \in \Pi^t(A)$ if and only if $U' \in \Pi^t(A')$. Thus, $\Pi^t(A) = \Pi^M(A)$ if and only if $\Pi^t(A') = \Pi^N(A')$. But $\Pi^*(A) \equiv \Pi^M(A)$. *Q.E.D.*

PROOF OF THEOREM 3.1: Let A denote those traders who hold m, B denote those who do not. Suppose the theorem is false. Then there is an improving reallocation Z. Without loss of generality (by Theorem 6.1, Lemma A.5) we can take Z to have: (a) precisely $t+1$ non-zero rows; (b) each non-zero row has precisely two non-zero entries; (c) each non-zero column has precisely two non-zero entries (summing to zero). There are at least three elements of A among the rows with non-zero entries in Z, but not all of Z's non-zero rows correspond to elements of A (by Theorem 1.2). Let 1 be an element of A who receives, under Z, a good from some element of B denoted 4. That is, $z_{1n} > 0$, $z_{4n} < 0$; $1 \in A$, $4 \in B$.

Consider the following enumeration procedure in Z. Starting at $z_{1n} > 0$ find the entry in z_1, $z_{1n'} < 0$. Follow this column to its non-zero entry $z_{in'} > 0$, find the other non-zero entry in the row $z_{in''} < 0$, and so forth. This procedure will eventually return to z_{1n}. Before it does so denote as 2 the final element of A in the procedure. $z_{2n^*} < 0$, $z_{3n^*} > 0$ for some element $3 \in B$. Without loss of generality, we may take 4 to be the sole net beneficiary of Z. Denote the set of elements of B between 3 and 4, i.e., 3, 4 and those elements of B not previously enumerated as B^*.

Define λ_j, $j = 1, \ldots, N$, as $\lambda_j = \max_{i \in A}(p_{ij}/p_{im})$. $\lambda \equiv (\lambda_1, \ldots, \lambda_N)$. Note that $\lambda_j = p_{ij}/p_{im}$ for all $i \in A$, $a_{ij} > 0$.

Construct the matrix Y as follows. Let $y_{1m} = z_{1n'}$, $\lambda_{n'} = -y_{2m}$, $y_{1n} = z_{1n}$, $y_{2n^*} = z_{2n^*}$ (N.B.: in the case where n, n^* is in the m and n, n^* entry areas). Let $y_{ij} = 0$ for all $i \in A$, $i \neq 1, 2$. Let $y_{ij} = z_{ij}$ for $i \in B^*$. Then Y is an improving reallocation and there are only $u \leqq t$ non-zero rows in Y. This is a contradiction. *Q.E.D.*

Proof of Lemma 5.1: If the ijth element of $(A'A)^{t-1}$ is positive, then there is a chain of no more than $t-1$ individuals linking commodities i and j.

If the matrix formed from $(A'A)^{t-1}$ by deleting the rows and columns for which a is zero is irreducible, then for all i and j such that $a_i, a_j > 0$ there exists a chain of individuals linking i and j, which "returns" to the set of commodities for which a is positive at least every $t-1$ steps.

For every $P^+ \in \Pi^t(A^+)$, let P denote P^+ restricted to agents $1, \ldots, M$. (i) $P \in \Pi^t(A)$; (ii) $\exists p$ such that (A, P, p) is a competitive equilibrium. Claim: (A^+, P^+, p) is a C.E.

Since $[(A'A)^{t-1}a^+ \cup \{i\}]$ is irreducible, there is a chain in the first M agents connecting j to i (for all j where $a_{M+1,j} > 0$), which returns to the set of goods that $M+1$ holds every $t-1$ steps or less. Call these links.

In the following argument we will show (a) that $P_{M+1,x}/P_{M+1,y} \geqq p_x/p_y$ for any $x, y \in [1, \ldots, N]$ where $a_{M+1,x} > 0$ and $[(A'A)^{t-1}]_{xy} > 0$, and (b) that $P_{M+1,j}/P_{M+1,i} \geqq p_j/p_i$ for any $i, j \in [1, \ldots, N]$ where $a_{M+1,j} > 0$. (a) Now, if x, y are two commodities with $a_{M+1,x} > 0$ and $[(A'A)^{t-1}]_{xy} > 0$ then there is a sequence of $t-1$ agents, $m_1, \ldots, m_{t-1}$, and a sequence of t goods, $n_1, \ldots, n_t$, where $n_l = x$ and $n_t = y$ such that $\forall l \in [0, t-1]$

$$a_{m_l n_l}, a_{m_l n_{l+1}} > 0.$$

Then $P_{m_l n_l}/P_{m_l, n_{l+1}} = p_{n_l}/p_{n_{l+1}}$, for all $l \in [0, t-1]$. Suppose $P_{M+1,x}/P_{M+1,y} < p_x/p_y$. Then pick Z so that $z_{m_l, n_l} = \epsilon/p_{n_l}$, $z_{m_l, n_{l+1}} = -\epsilon/p_{n_{l+1}}$, and $z_{M+1,x} = -\epsilon/p_x$, $z_{M+1,y} = \epsilon/p_y$.

The change in m_l's utility is given by

$$\epsilon\left[\frac{P_{m_l n_l}}{p_{n_l}} - \frac{P_{m_l n_{l+1}}}{p_{n_{l+1}}}\right] = 0.$$

The change in $M+1$'s utility, however, is

$$\epsilon\left[\frac{P_{M+1,y}}{p_y} - \frac{P_{M+1,x}}{p_x}\right] > 0.$$

Thus the trade vector Z improves a group of size t;

Therefore $P_{M+1,x}/P_{M+1,y} \geqq p_x/p_y$.

(b) Take $i, j \in [1, \ldots, N]$ where $a_{M+1,j} > 0$. Since $[(A'A)^{t-1}\text{proj}\, a^+ \cup \{i\}]$ is irreducible, there exists a sequence $n_1, \ldots, n_Q \in [1, \ldots, N]$ where (i) $(A'A)^{t-1}_{n_q n_{q+1}} > 0$; (ii) $n_1 = j$ and $n_Q = i$; (iii) $a_{M+1,n_q} > 0$ for $q \in [1, \ldots, Q-1]$.

By (1) above,

$$\frac{P_{M+1,n_q}}{P_{M+1,n_{q+1}}} = \frac{p_{n_q}}{p_{n_{q+1}}} \quad \text{and} \quad \frac{P_{M+1,n_{Q-1}}}{P_{M+1,n_Q}} \geqq \frac{p_{n_{Q-1}}}{p_{n_Q}}.$$

Thus $P_{M+1,j}/P_{M+1,i} \geqq p_j/p_i$ and p supports the $M + 1^{\text{st}}$ agent as well.

Therefore (A^+, P^+, p) is a competitive equilibrium and $\Pi^t(A^+) = \Pi^*(A^+)$. *Q.E.D.*

PROOF OF THEOREM 6.1: $\Pi^t(A) \neq \Pi^{t+1}(A)$ implies trivially $\Pi^t(A) \neq \Pi^M(A)$. Thus, $\Pi^M(A) = \Pi^t(A) \Rightarrow \Pi^t(A) = \Pi^{t+1}(A)$. We must show

$$\Pi^t(A) = \Pi^{t+1}(A) \Rightarrow \Pi^t(A) = \Pi^M(A).$$

To do this we will prove

$$\Pi^t(A) \neq \Pi^M(A) \Rightarrow \Pi^t(A) \neq \Pi^{t+1}(A).$$

Search $\Pi^t(A)\backslash\Pi^M(A)$ for P^* with the smallest possible blocking coalition S to A. $|S| = t + k$, $k \geqq 1$. If $k = 1$, we are done. Suppose $k > 1$. Let Z be a $t + k$-improving reallocation on S involving $t + k$ goods with no more than one positive entry in each row and column (A can be so restricted without loss of generality by Lemma A.5).

Denote the elements of S by $i_1, i_2, \ldots, i_{t+k}$ and the goods for which they have negative (i.e., supply) entries in Z by $j_1, j_2, \ldots, j_{t+k}$ respectively. For further notational convenience order the elements so that

$$z_{i_1 j_1} < 0, \qquad z_{i_1 j_{t+k}} > 0,$$

$$z_{i_2 j_2} < 0, \qquad z_{i_2 j_1} > 0,$$

$$z_{i_3 j_3} < 0, \qquad z_{i_3 j_2} > 0, \qquad \text{and so forth.}$$

We may, without loss of generality, arrange the magnitudes of the trade so that agent i_1 receives all of the benefits and

(i) $$P_{i_l j_l} z_{i_l j_l} + P^*_{i_l j_{l-1}} z_{i_l j_{l-1}} = 0 \qquad \text{for} \qquad l \neq 1.$$

Consider $P^*_{i_3 j_1}$, i_3's marginal utility for good j_1 (the good that i_2, not i_3, receives under Z). Now if $P^*_{i_3 j_1} z_{i_2 j_1} + P^*_{i_3 j_3} z_{i_3 j_3} \geqq 0$, then let $\bar{Z}$ be identical with Z except for $\bar{z}_{i_3 j_1} = z_{i_2 j_1}$ and $\bar{z}_{i_2 j_1} = \bar{z}_{i_2 j_2} = 0$.

But then $\bar{Z}$ is a $t + k - 1$ improving reallocation for P^*, a contradiction. Therefore $P^*_{i_3 j_1} z_{i_2 j_1} + P^*_{i_3 j_3} z_{i_3 j_3} < 0$.

Let $\bar{P}$ be P^* with $\bar{P}_{i_3 j_1}$ replacing $P^*_{i_3 j_1}$, where

(ii) $$\bar{P}_{i_3 j_1} z_{i_2 j_1} + P^*_{i_3 j_3} z_{i_3 j_3} = 0.$$

Since $\bar{Z}$ would be a $t + k - 1$ improving reallocation for $\bar{P}$—a contradiction—then $\bar{P} \notin \Pi^t(A)$. Thus, there is Y a t-improving reallocation for $(\bar{P}, A)$, where $y_{i_3 j_1} > 0$, and for some other good j_3', $y_{i_3 j_3'} < 0$.

Without loss of generality, we may scale Y so that

(iii) $$y_{i_3 j_1} = z_{i_2 j_1}$$

and arrange the trade so that

(iv) $$\bar{P}_{i_3 j_1} y_{i_3 j_1} + P^*_{i_3 j_3'} y_{i_3 j_3'} = 0.$$

Consider $\bar{Y}$ identical to Y except that $\bar{y}_{i_3 j_1} = 0$,

$$\bar{y}_{i_3 j_2} = z_{i_3 j_2} = -\bar{y}_{i_2 j_2} = -z_{i_2 j_2}, \qquad \bar{y}_{i_2 j_1} = y_{i_3 j_1} = z_{i_2 j_1}.$$

Now, from (i) (for $l = 3$) and (ii)

(v) $$P^*_{i_3 j_2} z_{i_3 j_2} = \bar{P}_{i_3 j_1} z_{i_2 i_1}$$

and by (v) and (iii)

(vi) $$P^*_{i_3j_2} z_{i_3j_2} = \bar{P}_{i_3j_1} y_{i_3j_1}$$

and by (vi) and (iv)

$$P^*_{i_3j_2} z_{i_3j_2} + P^*_{i_3j_3'} y_{i_3j_3'} = 0.$$

Since $\bar{y}_{i_3j_2} = z_{i_3j_2}$ and $\bar{y}_{i_3j_3'} = y_{i_3j_3'}$, then

$$P^*_{i_3j_2} \bar{y}_{i_3j_2} + P^*_{i_3j_3'} \bar{y}_{i_3j_3'} = 0,$$

so i_3 neither benefits nor loses by $\bar{Y}$.

Since $\bar{y}_{i_2j_1} = y_{i_3j_1} = z_{i_2j_1}$ and

$$\bar{y}_{i_2j_2} = z_{i_2j_2},$$

then by (i) for $l = 2$

$$P^*_{i_2j_1} \bar{y}_{i_2j_1} + P^*_{i_2j_2} \bar{y}_{i_2j_2} = 0$$

and i_2 neither benefits nor loses by $\bar{Y}$.

The remaining trades of $\bar{Y}$ are the same as Y, a t-improving reallocation for $(\bar{P}, A)$. But P^* and $\bar{P}$ are the same except for $P_{i_3j_1}$ and $\bar{y}_{i_3j_1} = 0$. Therefore, $\bar{Y}$ is a $t + 1$ improving reallocation for (P^*, A). This is a contradiction and implies $k = 1$. *Q.E.D.*

Proof of Theorem 6.2: Suppose not. Then there are two commodities i and j at least one of which is positive in a (say j) such that the ijth element of $[(A^{-\prime} A^{-})^{t-1} \text{proj}\, a^{+} \cup \{i\}]^x$ equals zero for all $x > 0$. Assign utility weights as follows. Marginal utilities equal 1 for all agents other than v and all commodities. For agent v, marginal utility equals 2 for good i and any other good k which v holds for which there exists some x such that the ikth element of $[(A^{-\prime} A^{-})^{t-1} \text{proj}\, a^{+} \cup \{i\}]^x > 0$. Consider a chain of agents (not v) connecting i and j (possible by assumption that A^{-} is irreducible). The chain involves at least t agents since $[(A^{-\prime} A^{-})^{t-1} \text{proj}\, a^{+} \cup \{i\}]_{ij} = 0$. Consider a transfer around the chain from commodity j. Transfer a unit of i from the last one in the chain to v and a unit of j from v to the first in the chain. The trade is Pareto improving involving at least $t + 1$ traders. *Q.E.D.*

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