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Why Is There Money?

Endogenous Monetization of an Arrow-Debreu Economy

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WORK IN PROGRESS!

Full text at: http://econweb.ucsd.edu/[tilde]rstarr/RolfMantelLecture2017.pdf

The Hahn Problem: *In a general equilibrium model including money, where money does not enter preferences, can money be shown to have a positive equilibrium value?* Hahn (1965)

"The...challenge that...money poses to the theorist is this: the best developed model of the economy cannot find room for it. The best developed model is, of course, the Arrow-Debreu version of a Walrasian general equilibrium. " Hahn (1982)

Monetary Theory:

- Why does inherently useless fiat money command a positive price? Adam Smith (1776); George Knapp (1905[1924]); Abba Lerner (1947). Government makes taxes payable in fiat money. Then fiat money will have a positive price.

- Why does fiat money become the common medium of exchange? Thick market externality, Rey (2001).

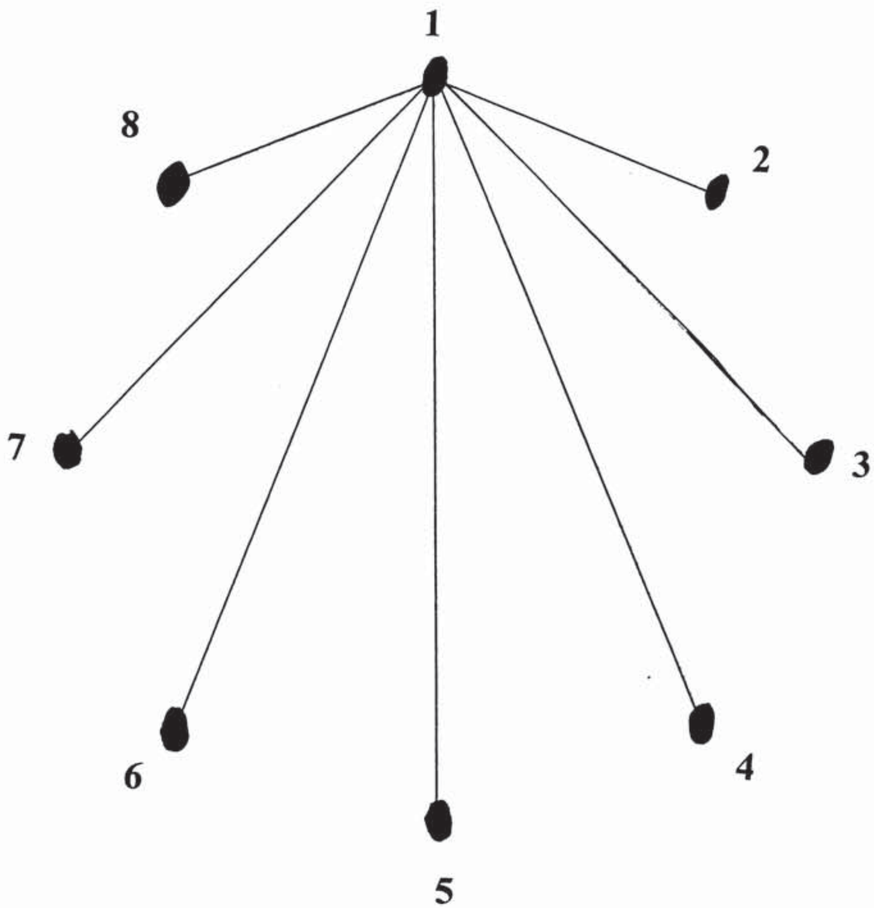
General Equilibrium Theory:
Existence of a market clearing equilibrium prices and allocation. Why do we care?

- Arrow-Debreu single comprehensive market.
- Pairwise trade at $\frac{1}{2}N(N-1)$ commodity-pair trading posts and $\frac{1}{2}N(N-1)$ budget constraints.
- Equilibrium = All trading posts clear.
- A **barter** equilibrium occurs when most trading posts are active in equilibrium — most goods trading directly for most other goods. A **monetary** equilibrium occurs if active trade is concentrated on a few trading posts, those trading the common medium of exchange against most other goods.

- Transaction costs: bid/ask price spread. Duncan Foley (1970).

- Thick market externality: unique common medium of exchange implies very low transaction costs. Narrow bid/ask spread. Helene Rey (2001).

- Existence of general equilibrium can be sustained in the presence of (sufficiently continuous) external effects, Arrow & Hahn (1971).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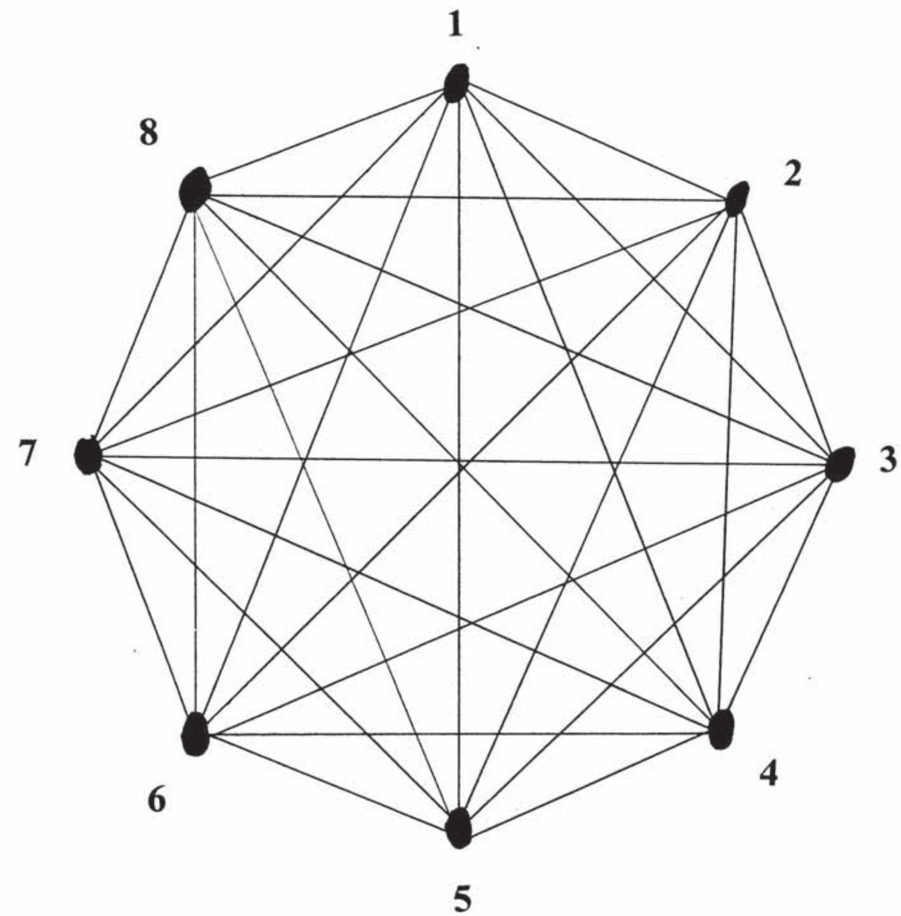
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Summary

- Demonstrate an example where useless fiat money has a positive equilibrium price and is endogenously determined to be the common medium of exchange.
- Demonstrate existence of general equilibrium in a trading post model with:

  transaction costs, bid/ask spread

  externality determining transaction technology

  separate budget constraint at each trading post

  endogenous medium of exchange function

  Of course, this result requires continuity and convexity everywhere, except that it admits a scale economy in transaction costs external to the individual firms.

- Bottom line: The Hahn Problem has a solution. The Arrow-Debreu general equilibrium model — suitably elaborated — admits a medium of exchange and fiat money.

General Equilibrium with a Bid/Ask Spread

Pure exchange economy with transaction costs. The only resource-using technology is the transaction process embodied in trading firms.

Commodity Space

N elementary commodities.

$\frac{1}{2}N(N-1)$ trading posts denoted $\{k, \ell\} \equiv \{\ell, k\}$

An equilibrium is monetary with a unique money if only N trading posts out of $\frac{1}{2}N(N-1)$ are active, those trading all goods against 'money.'

The General Equilibrium Model with Bid and Ask Prices

Foley (1970) has bid and ask prices, with the Arrow-Debreu style single budget constraint.

Two sets of prices, p^S and p^B in $R_+^{N(N-1)}$.

p^S wholesale or bid prices.

p^B retail or ask prices.

$\Delta =$ unit simplex in $R^{2N(N-1)}$. $p^B \geq p^S$ co-ordinatewise.

$\pi \equiv p^B - p^S$.

$p \equiv (p^S, \pi) \in \Delta$.

A single co-ordinate will typically be denoted $p^S(k,\ell), \pi(k,\ell), x^{iS}(k,\ell)$. This is to be read as price or quantity of commodity k at trading post $\{k,\ell\}$ where it is traded for $x^{iB}(\ell,k)$. There is no entry $x(k,k)$.

Example: Equilibrium with a Thick Market Externality and a Unique Medium of Exchange

Households Let $N \geq 3$. Ω denotes the greatest integer $\leq (N-1)/2$. Begin with a population of $[10 \times N \times \Omega]$ households. Let the households $i \in H$ be enumerated in the following way:

$a.m.n$ where

$a = 1, 2, \ldots, N$,

$m = 1, 2, ..., 10$,

$n = (a+1)(\text{mod } N), (a+2)(\text{mod } N), ..., (a+\Omega+1)(\text{mod } N)$

The typical household $a.m.n$ is endowed with good a, in quantity A, prefers good n, and there are 10 identically situated households denoted by m.

Household $a.m.n$'s utility function is

$u^{a.m.n}(x) = x_n$

That is, household $a.m.n$ values good n only and gladly trades his endowed good a for n.

Transaction Costs and Monetary Equilibrium

The fraction $\frac{\Sigma_i\, x^{iS}(k,\ell)}{10 \times \Omega \times A}$ represents the fraction of total possible household offers of commodity k taking place at trading post $\{k, \ell\}$.

Consider pure trading technology, Y^j, with a 'iceberg' style transaction cost.

At near-zero offer volume, $0 < y^{jB}(k,\ell) = -\ \frac{1}{2} y^{jS}(k,\ell)$. More generally,

$0 < y^{jB}(k,\ell) = -\ \frac{1}{\delta}\ \ y^{jS}(k,\ell)$ where $\delta = 1 + \gamma$,

$$\gamma = \max[1 - \frac{\Sigma_i\, x^{iS}(k,\ell)}{\rho\, \Sigma_{i \in H}\, \Sigma_{m=1,\ldots,N}\, r^i(k,m)}\ , 0] + 0.001\ .$$

Claim: There is an equilibrium in an economy with this technology. Further, there is an equilibrium with transactions concentrated on a single intermediary commodity.

EXAMPLE 6.2 Let the population and transaction technology be as above. Let $\rho = 1$. Choose $\tilde{\mu} \in \{1, 2, \ldots, N\}$. Set $1 = p^S(\tilde{\mu}, \ell), 1.001 = p^B(\tilde{\mu}, \ell)$, for all $\ell = 1, 2, \ldots, N$. For all $k \in \{1, 2, \ldots, N\}, k \neq \tilde{\mu}$, set $1 = p^S(k, \ell), 2 = p^B(k, \ell)$, for all $\ell = 1, 2, \ldots, N$. Set $x^{a.m.nS}(a, \tilde{\mu}) = -A, x^{a.m.nB}(\tilde{\mu}, a) = 0.999A$. $x^{a.m.nS}(\tilde{\mu}, n) = -0.999A, x^{a.m.nB}(n, \tilde{\mu}) = 0.998A$. Denote a typical firm $\tilde{j}$ as a market maker in the trading posts $\{\tilde{\mu}, k\}, k = 1, 2, \ldots, N$.

Let $\Sigma_{\tilde{j}}\, y^{\tilde{j}S}(\tilde{\mu}, k) = -10 \times \Omega \times A = \Sigma_{\tilde{j}}\, y^{\tilde{j}S}(k, \tilde{\mu})$; $\Sigma_{\tilde{j}}\, y^{\tilde{j}B}(k, \tilde{\mu}) = 9.98 \times \Omega \times A$; $\Sigma_{\tilde{j}}\, y^{\tilde{j}B}(\tilde{\mu}, k) = 9.99 \times \Omega \times A$. At this trading volume, with the specified external economy, $\tilde{j}$ breaks even with $\pi(\tilde{\mu}, k) = \pi(k, \tilde{\mu}) = 0.001$.

Fiat Money

Government, agent G, sells tax receipts, the $N+1^{\text{st}}$ good. It also sells good $N+2$, an intrinsically worthless instrument, (latent) fiat money, that agent G undertakes to accept in payment of taxes, that is, in exchange for $N+1$.

Government, agent G, uses its revenue to purchase a variety of goods $n = 1, ..., N$, in the amount x_n^G.

EXAMPLE 6.4

Start from example 6.2. Let $\tilde{\mu} = N + 2$. Let each household $a.m.n$ have a designated tax bill $\tau^{a.m.n} = \tau^{\circ} = \rho A > 0$.

Set $1 = p^S(k, \ell), 2 = p^B(k, \ell)$, for all $k, \ell = 1, 2, \ldots, N, N{+}1; k \neq \ell$. This expression says that good k can be traded directly for good ℓ including $\ell = N + 1$, tax receipts. But it is priced for low volume. So transaction costs are high, $\pi(k, \ell) = \pi(\ell, k) = 1 = \pi(N + 1, k)$.

$u^{a.m.n}(x) = x_n - 2[\max[(\tau^{a.m.n} - x^{a.m.n}_{N+1}), 0]]$.

Set $1 = p^S(N + 2, \ell), 1.001 = p^B(N + 2, \ell)$, for all $\ell = 1, 2, \ldots, N$. $p^B(\ell, N + 2) = 1.001, p^S(\ell, N + 2) = 1$.

$p^S(N{+}1, N{+}2) = 1, p^B(N{+}1, N{+}2) = 1.001$. Let $x^G(n, N{+}2) = 10 \cdot (0.999)N\Omega\tau^{\circ}$ and $x^G(N + 2, n) = -10N\Omega\tau^{\circ}$.

$x^G(N + 2, N + 1) = 10.01N\Omega\tau^{\circ}$. $x^G(N + 1, N + 2) = -10N\Omega\tau^{\circ}$.

For all $a.m.n$,

$x^{a.m.nS}(N+2,n) = -.999(A-\tau^\circ), x^{a.m.nB}(n,N+2) = 0.998\cdot(A-\tau^\circ)$.

$x^{a.m.nS}(N+2,N+1) = -1.001\tau^\circ, x^{a.m.nB}(N+1,N+2) = \tau^\circ$.

set $x^{a.m.nB}(N+2,a) = .999A, x^{a.m.nS}(a,N+2) = -A$. In this example, $N+2$ fiat money, becomes the sole common medium of exchange.

Fiat money equilibrium is NOT unique.

Existence of Equilibrium in a Formal Trading Post General Equilibrium Model

Firms

$j \in F$. $(y^{jS}, y^{jB}, w^j) \in Y^j \subseteq R^{3N(N-1)}$. $\quad Y \equiv \Sigma_j Y^j$.

Positive co-ordinates of y^{jB}, y^{jS} indicate sales, negative co-ordinates indicate purchases.

$w \geq 0$ (co-ordinatewise) indicates inputs to the trading transaction costs.

y^{jS} is the vector of j's purchases and sales , at bid (wholesale) prices.

y^{jB} is the vector of j's purchases and sales at buying (retail) price.

Both y^{jS} and y^{jB} can have both positive and negative co-ordinates.

The budget constraint on firm transactions is for each two commodities $k, \ell, = 1, 2, \ldots, N$.

$$p^S(k,\ell) \cdot y^{jS}(k,\ell) + p^B(k,\ell) \cdot y^{jB}(k,\ell)$$
$$+p^B(\ell,k) \cdot y^{jB}(\ell,k) + p^S(\ell,k) \cdot y^{jS}(\ell,k) \geq 0 \quad (\mathcal{B}')$$

Equivalently,

$$p^S(k,\ell) \cdot [y^{jS}(k,\ell) + y^{jB}(k,\ell)] + p^S(\ell,k) \cdot [y^{jB}(\ell,k) + y^{jS}(\ell,k)]$$
$$+\pi(k,\ell) \cdot y^{jB}(k,\ell) + \pi(\ell,k) \cdot y^{jB}(\ell,k) \geq 0 \quad (\mathcal{B}')$$

The Trading Sector and Attainable Trades

Lemma 1 Assume (d.1) through (d.5) (continuity, convexity, no-free marketing, Y^j is a convex cone).

Then the set of attainable elements $(y^S, y^B, w) \in Y$ is bounded.

And for each $j\prime \in F$, the set of $(y^{j\prime S}, y^{j\prime B}, w^{j\prime}) \in Y^{j\prime}$ attainable in $Y^{j\prime}$ is bounded.

Let C denote a strict upper bound on the length of an attainable output in Y^j for all $j \in F$.

Households

There is a finite set of households H with typical element $i \in H$.

The household i possible consumption set is $W^i \subseteq R^{2N(N-1)}$.

i has a preference ordering $\succeq_i$ on W^i. i makes trades $x^i \in R^{2N(N-1)}$.

$x^i = (x^{iS}, x^{iB})$ reflects $x^{iB} \geq 0, x^{iB} \in R^{N(N-1)}$, the vector of i's purchases, and $x^{iS} \leq 0, x^{iS} \in R^{N(N-1)}$ the vector of i's sales.

A single co-ordinate will typically be denoted $x^{iS}(k, \ell), x^{iB}(k, \ell)$. This is to be read as commodity k at trading post $\{k, \ell\}$ where it is traded for $x(\ell, k)$.

The budget constraint on household transactions is for each two commodities $k, \ell, = 1, 2, \ldots, N$.

$$p^S(k,\ell) \cdot x^{iS}(k,\ell) + p^B(k,\ell) \cdot x^{iB}(k,\ell)$$
$$+p^B(\ell,k) \cdot x^{iB}(\ell,k) + p^S(\ell,k) \cdot x^{iS}(\ell,k) \leq 0 \quad (\mathcal{B})$$

Equivalently

$$p^S(k,\ell) \cdot [x^{iS}(k,\ell) + x^{iB}(k,\ell)] + p^S(\ell,k) \cdot [x^{iB}(\ell,k) + x^{iS}(\ell,k)]$$
$$+\pi(k,\ell) \cdot x^{iB}(k,\ell) + \pi(\ell,k)x^{iB}(\ell,k) \leq 0 \quad (\mathcal{B})$$

Lemma 6 (Walras's Law): Let $(p^S, \pi) \in \Delta, \pi = p^B - p^S$.

Let $(x^{iS}, x^{iB}) \in D^i(p^S, p^B, \mathcal{Y})$ and let $(y^{jS}, y^{jB}) \in S^j(p^S, p^B)$.

Then $p^S \cdot [\Sigma_i\, x^{iS} - \Sigma_j\, y^{jS}] + p^B \cdot [\Sigma_i\, x^{iB} - \Sigma_j\, y^{jB}] \leq 0$.

Equivalently,
$p^S \cdot [\Sigma_i\, x^{iS} - \Sigma_j\, y^{jS} + \Sigma_i\, x^{iB} - \Sigma_j\, y^{jB}] + \pi \cdot [\Sigma_i\, x^{iB} - \Sigma_j\, y^{jB}] \leq 0$

MEDIUM OF EXCHANGE

In competitive equilibrium, let $x^{iS}(k, \ell) < 0, x^{iB}(k, m) > 0$, for some ℓ, m. Then k is a medium of exchange.

EXTERNAL EFFECTS

(f.1) For each $j \in F, Y^j = \phi^j(\Sigma_i(x^{iS}, x^{iB}))$.

(f.2) For each $j \in F, \phi^j$ is a continuous (upper and lower hemicontinuous) convex-valued correspondence. $\phi^j : R^{2N(N-1)} \to R^{3N(N-1)}$.

(f.3) For each $j \in F$, the following set is bounded:

$$\bigcup_{(x^{iS}, x^{iB}) \in R^{2N(N-1)}} \{(y^{jS}, y^{jB})\text{attainable in } Y^j = \phi^j(\sum_i (x^{iS}, x^{iB}))\}$$

Theorem 2: Assume (a.1), (a.2), (b.1), (b.2), (b.3), (c.1), (d.1), (d.2), (d.3), (d.4), (e.1), (e.2), (f.1), (f.2), (f.3) (continuity, convexity, no free marketing, possibility of inaction, positivity of endowment, technology is a convex cone) .

Then the economy has a competitive equilibrium.

Proof: Note compactness of the attainable set, convexity, continuity.

Recall the price space is $\Delta \equiv$ unit simplex in $R^{2N(N-1)}$.

Let Π indicate multiple Cartesian product,

$p = (p^S, \pi) \in \Delta$ be prevailing price vector,

$X^{\#H} \in \Pi_{i\in H}\tilde{D}^i(p, \mathcal{Y}) \subseteq R^{\#H2N(N-1)}$ be the complex of household demands,

$\mathcal{Y} \in \Pi_{j\in F}\tilde{Q}^j(p, Y^j) \subseteq R^{\#F3N(N-1)}$ be the complex of firm plans,

$Y^{Agg} \in \Pi_{j\in F}\phi^j(\Sigma_{i\in H}\, x^i) \subseteq R^{\#F3N(N-1)}$ be the complex of (endogenously determined) transaction technologies,

$z \in \tilde{Z}(p, \mathcal{Y}) \subseteq R^{2N(N-1)}$ be the vector of excess demands, $z = \Sigma_{i \in H} x^i - \Sigma_{j \in F} y^j$

Let $\Gamma(z)$ be the price adjustment correspondence.

$$\Gamma(z) \equiv \{ \text{ argmax}_{(p^S,\pi) \in \Delta}[p^S \cdot (z^S + z^B) + \pi \cdot z^B]\}.$$

Let $\widehat{\mathcal{T}}(p, X^{\#H}, \mathcal{Y}, Y^{Agg}, z)$

$$\equiv \Gamma(z) \times \Pi_{i \in H} \tilde{D}^i(p, \mathcal{Y}) \times \Pi_{j \in F} \tilde{Q}^j(p, Y^j) \times \Pi_{j \in F} \phi^j(\Sigma_{i \in H} x^i) \times \tilde{Z}(p, \mathcal{Y})$$

Note that $\Gamma, \tilde{D}^i, \tilde{Q}^j, \phi^j,$ and $\tilde{Z}(p, \mathcal{Y})$ are all well defined, upper hemi-continuous, and convex-valued. The $\tilde{.}$ notation indicates restriction to a compact ball with radius $\leq C$. Then by the Kakutani Fixed Point Theorem, there is a fixed point of $\widehat{\mathcal{T}}$, $(p^\circ, x^{\#H\circ}, \mathcal{Y}^\circ, Y^{Agg\circ}, z^\circ)$.

Now to demonstrate that the fixed point is a market-clearing equilibrium.

By the Walras's Law, Lemma 6, $p^\circ \cdot z^\circ \leq 0$, but $p^\circ \geq 0$ and p° is $\text{argmax}_{(p^S,\pi)\in\Delta}[p^S \cdot (z^S + z^B) + \pi \cdot z^B]$ so $z^\circ \leq 0$.

$z^\circ = \Sigma_{i\in H}\, x^{i\circ} - \Sigma_{j\in F}\, y^{j\circ}$ where $x^{i\circ} \in \tilde{D}^i(p^\circ, \tilde{\mathcal{Y}}^\circ)$ and $y^{j\circ} \in \tilde{S}^j(p^\circ)$.

But $\Sigma_{i\in H}\, x^{i\circ} \leq \Sigma_{j\in F}\, y^{j\circ}$, so $x^{i\circ}, i \in H$ is attainable, so $|x^{i\circ}| < C$.

But $|x^{i\circ}| < C$ and $x^{i\circ} \in \tilde{D}^i(p^\circ, \tilde{\mathcal{Y}}^\circ)$ implies that the length constraint to C is not binding, so $x^{i\circ} \in D^i(p^\circ, \tilde{\mathcal{Y}}^\circ)$.

SUMMARY

- Demonstrate an example where useless fiat money has a positive equilibrium price and is endogenously determined to be the common medium of exchange.
- Demonstrate existence of general equilibrium in a trading post model with:

  transaction costs, bid/ask spread

  externality determining transaction technology

  separate budget constraint at each trading post

  endogenous medium of exchange function

  Of course, this result requires continuity and convexity everywhere, except that it admits a scale economy in transaction costs external to the individual firms.

- Bottom line: The Hahn Problem has a solution. The Arrow-Debreu general equilibrium model — suitably elaborated — admits a medium of exchange and fiat money of positive value.