

# The Emergence of Commodity Money as a Medium of Exchange

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## Abstract

This paper examines the endogenous emergence of a commodity money in a trading post economy. The commodity money is defined as the common medium of exchange and is determined by the equilibrium pattern of exchange. The long run equilibria are analyzed using an evolutionary style model. Agents follow a simple adaptive process, generating dynamics that are reduced to a Markov process. Examples are given where the economy spends almost all the time in one or more of the monetary equilibria. Properties that favor the selection of one good as the commodity money are high trading volume and low trading cost.

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## 1. Introduction

Most economic models take the existence (or non-existence) of money as given. These models are appropriate for economies in which the use of a particular money is legislated, such as the dollar in the United States or the pound in Great Britain. In other economies, legislation may simply have ratified the outcome of a spontaneous historical process, as in the adoption of gold and silver coins. However in some economies no medium of exchange is legislated, as in the use of the US dollar as a vehicle currency in the foreign exchange markets (Portes and Rey 1998), or the use of cigarettes as a medium of exchange in prison camp economies (Radford 1945). This paper examines the endogenous emergence and resulting stability of commodity money in such settings. We allow any commodity to be used as the common medium of exchange. Our analysis allows us to perform comparative statics that demonstrate that low trading costs and high initial trading volumes favor the use of a particular commodity as the medium of exchange.

Two kinds of model have been used to study the spontaneous emergence of a medium of exchange. One is the search theoretic approach used in Kiyotaki and Wright (1989), in which agents meet randomly and trade whenever it is mutually advantageous. Each agent chooses a trading plan to minimize his or her expected search cost for acquiring a desired good given the plans in use by the other agents. Agents consider both storage cost and salability when deciding on a medium of exchange. Rocheteau and Wright (2005) include a recent review of the search literature.

This paper takes an alternative approach, in which trade is structured through pre-existing trading posts, modeled, following Clower (1995), as places where agents can gather expecting that everyone will be interested in trading two specific goods. The advantages of trading posts are shown in Iwai (1996) and in Rocheteau and Wright's (2005) competitive search equilibrium. Unlike the search-theoretic models, trading post models build in the cost-reducing trading patterns that presumably precede the emergence of a commodity money.

Specifically, we consider a pure exchange economy with trading posts. Households gather at specific trading posts to exchange a given pair of goods. A household may trade directly for a desired consumption good. Alternatively the household may trade indirectly, first trading for an intermediate good and then trading that good for a desired consumption

good. We call the intermediate good a medium of exchange. When the trading posts have economies of scale, the models have multiple equilibria. These equilibria are characterized by the pattern of exchange. A barter equilibrium is an equilibrium where all households trade directly. A monetary equilibrium is defined as an equilibrium where all households use a common medium of exchange. There is one such monetary equilibrium for each possible good.

In pure-exchange trading post economies each agent goes only to the trading posts needed to trade his endowment for his desired consumption goods. Agents thus avoid search costs, but instead must cover the costs of operating the trading posts. Previous work, including Clower (1995), Starr and Stinchcombe (1999) and Starr (2003 a, b), addresses the existence of monetary and barter equilibria and their local stability but has little to say about equilibrium selection.

Equilibrium selection has been studied using computer simulations by Marimon, McGrattan and Sargent (1990) and by Howitt and Clower (2000) and through experiments by Brown (1996), Duffy and Ochs (1999) and Newhouse (2004). Similar to Johnson (1997), we study equilibrium selection with a model of adaptive learning in the style of Kandori, Mailath and Rob (1993), henceforth KMR. This approach provides a basis for assessing the stability and relative likelihood of barter and monetary equilibria in the long run, independent of initial trading conditions. Johnson demonstrates that a *fiat* monetary equilibrium is the most likely limit point of adaptive dynamics when the number of goods in an economy is large. We show analytically how the likelihood of a specific *commodity* monetary equilibrium is influenced by the costs of trading the various goods and the number of each type of agent. In other words we determine properties that favor the selection of a particular good as the medium of exchange in an economy.

This paper is organized as follows. Section 2 presents the model and explains how the Nash equilibrium used in this paper gives rise to properties we expect from general equilibrium theory. Section 3 explains how the long run equilibrium selection process developed by KMR is applied in this analysis. Section 4 presents two theorems that demonstrate properties that favor the use of a particular good as a medium of exchange. Section 5 gives detailed results for two specific examples of three good trading post economies, both of which result in long run monetary equilibria. Section 6 concludes.

## 2. The Model

The model discussed in this paper is adapted from Shapley and Shubik (1977); it is a multi-period, pure exchange, trading post model. Each period, adaptive agents receive endowments and then visit one or more trading posts to trade for their desired consumption goods, as explained below. This paper considers the simplest interesting case, the case of three, perfectly divisible consumption goods. Three is the minimum number of goods that allows both barter and monetary equilibria. Analyzing three goods limits both the number of agents' choices and the number of possible equilibria to provide clear analytical results.

### 2.1. Agents

Each agent is characterized by an endowment of one of the three goods each period and by a desire to consume a different good. In all there are six types of agents, one type for each permutation of three goods taken two at a time. These agents' types are labeled as  $X_{ij}$  for  $i, j \in \{a, b, c\}$  where  $X_{ij}$  is endowed with good  $i$  and desires to consume good  $j$ . An agent of type  $X_{ij}$  has a utility function equal to  $U_{ij}(j) = j$  (the amount of good  $J$  consumed by that agent). There are  $n_{ij}$  agents of each type  $X_{ij}$ . Each period, each agent receives 1 unit of his endowment good and then goes to one or more trading posts to trade for his desired consumption good. Agents can only trade goods with trading posts, not directly with each other.

### 2.2. Trading Posts

There is a unique trading post for each pair of consumption goods. The trading posts do not act strategically; each mechanically sets its bid-ask spread so that it will break even. This average cost pricing rule simplifies the accounting by eliminating monopoly profits and can be justified due to either potential entry or regulation.

Each trading post is characterized by the goods it trades, its fixed cost, and a cost-sharing rule. Trading post  $Y_{ij}$  trades consumption goods  $i$  and  $j$ . The order of the subscripts for posts does not matter; trading post  $Y_{ij}$  is the same as  $Y_{ji}$ . Trading post  $Y_{ij}$  must pay a fixed cost of  $F_{ij}$  each period that it operates. Post  $Y_{ij}$  also has a cost-sharing rule that

specifies what percentage of  $F_{ij}$  it will pay for with good  $i$  (denoted  $\alpha_{ij}^i$ ) and what percentage it will pay for with good  $j$  ( $\alpha_{ij}^j$ ) with  $\alpha_{ij}^i + \alpha_{ij}^j = 1$ . Trading posts follow a pricing rule where the quantity of  $I$  that  $Y_{ij}$  will give for 1 unit of  $J$  is denoted by  $q_{ij}$ ,

$$q_{ij} = \max \left[ \frac{I_{ij} - \alpha_{ij}^i F_{ij}}{J_{ij}}, 0 \right]$$

$I_{ij}$  refers to the total amount of good  $I$  brought to  $Y_{ij}$  and  $J_{ij}$  refers to the total amount of good  $J$  brought to  $Y_{ij}$ . Trading post  $Y_{ij}$  returns all units of good  $i$  that it receives less the amount it uses to cover its fixed cost. The amount it returns to each agent is proportional to the amount of good  $j$  that that agent brought to the trading post. The quantities  $q_{ij}$  and  $q_{ji}$  can be converted to bid and ask prices for good  $I$ . The bid price for good  $I$  at trading post  $Y_{ij}$  is the amount of good  $J$  that the firm will give to an agent for 1 unit of  $I$ , or simply  $q_{ji}$ . The ask price for good  $I$  at trading post  $Y_{ij}$  is the amount of  $J$  that the firm accepts for 1 unit of  $I$ , or  $1/q_{ij}$ .<sup>1</sup> The firm uses its bid-ask spread to cover its fixed cost.

### 2.3. Decisions

The agents in this model follow a simple adaptive learning rule. Each agent can choose from two possible trading plans, direct and indirect trade. An agent  $X_{ij}$  that chooses direct trade simply trades good  $i$  for good  $j$ . An agent  $X_{ij}$  that chooses indirect trade first trades  $i$  for  $k$ , then trades  $k$  for  $j$ . Good  $k$  is referred to as the medium of exchange for such an agent. When all agents choose direct trade, the system is (possibly) at the barter equilibrium. When agents trade through a common medium of exchange, the system is at a monetary equilibrium. At the monetary equilibria, agents who are endowed with or consume the monetary good trade directly and all other agents trade indirectly. Although indirect trade requires the use of two trading posts (and paying a share of each post's fixed cost through the bid-ask spreads), it will be desirable if the combined average cost of using those posts is below the average cost of using the other post.

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<sup>1</sup> For instance assume  $F_{ab} = 2$  and  $\alpha_{ab}^a = \alpha_{ab}^b = 0.5$ , and that 10 units of good  $A$  and 9 units of good  $B$  are brought to post  $Y_{ab}$ . The amount of  $A$  that this post will give for 1 unit of  $B$  is  $q_{ab} = (10 - 1)/9 = 1$  unit of  $A$  and similarly  $q_{ba} = (9 - 1)/10 = 0.9$  units of  $B$ . The bid-ask spread for good  $A$  is  $1/q_{ab} - q_{ba} = 1 - 0.9 = 0.1$  units of good  $B$ . The bid-ask spread for good  $B$  is  $1/q_{ba} - q_{ab} = 1.11 - 1 = 0.11$  units of good  $A$ .

The agents' risk preferences do not need to be modeled because only one agent is allowed to change actions each period and that agent knows what the population distribution was during the previous period. Agents face no uncertainty when they choose actions. The amount of  $j$  (based on that agent's choice of direct or indirect trade) is given by,

$$j(\text{direct}) = \max\left[\frac{J_{ij} - \alpha_{ij}^j F_{ij}}{I_{ij}}, 0\right]$$

$$j(\text{indirect}) = \left(\max\left[\frac{K_{ik} - \alpha_{ik}^k F_{ik}}{I_{ik}}, 0\right]\right) \left(\max\left[\frac{J_{jk} - \alpha_{jk}^j F_{jk}}{K_{jk}}, 0\right]\right)$$

The first term for the amount of  $j$  received through indirect trade gives the amount of good  $k$  that the agent receives for 1 unit of  $i$ . The product of the 2 terms then gives the total amount of good  $j$  that the agent receives.<sup>2</sup> An example of these calculations is given in Appendix A for an economy with 10 agents of each type.

This model differs from a standard Walrasian model in that there is no single centralized market. The agents face a balanced budget constraint at each trading post, in that the costs of their purchases each period must equal the proceeds of their sales at every post. In the first period, it is assumed that the agents' actions are drawn from a uniform distribution. In all additional periods, a single randomly selected agent will be given the option to choose a new strategy. This agent, with high probability, chooses the best response to the population's play in the previous period. (The agent will choose direct trade in the case of a tie.) However, in the style of Kandori, Mailath and Rob (1993), the agent has a small probability of making an error and playing something other than the best response to the previous period's play. Because agents in this model have only two choices, an agent that makes an error plays the lower utility strategy (indirect trade in case of ties). All other agents' actions remain the same as in the previous period. These agents have no choice of actions for the current period. This assumption introduces enough friction to rule out cycles and ensures that the population will converge to an equilibrium.

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<sup>2</sup> As an approximation, if each market has a relatively balanced amount of goods brought to it from each side ( $I_{ij} \approx J_{ij}$ ,  $I_{ik} \approx K_{ik}$  and  $J_{jk} \approx K_{jk}$ ) and  $\alpha_{ij} \approx 0.5$  for all  $i \neq j \in \{a, b, c\}$ , agent  $X_{ij}$  prefers indirect trade to direct trade if  $(1 - 0.5F_{ik}/I_{ik})(1 - 0.5F_{jk}/K_{jk}) > 1 - 0.5F_{ij}/I_{ij}$ . If the fixed costs are equal, indirect trade is appealing if  $I_{ik}$  and  $K_{jk}$  are large compared to  $I_{ij}$ . Notice that  $I_{ik}$ ,  $K_{jk}$  and  $I_{ij}$  represent the traffic at each trading post due to the market balance.

## 2.4. Equilibrium

We study the Nash equilibria of the game described above. An equilibrium is a configuration of trading strategies for those agents such that no agent prefers a different trading strategy, given the prices determined by all agents' current strategies. Given the assumptions about preferences introduced above, each agent chooses the action that allows him to consume the maximum amount of his consumption good that he can purchase given his endowment and everyone else's actions.

Equilibria are characterized by three conditions: (i) zero profit for the firms, (ii) arbitrage free pricing and (iii) market clearing. These properties are the result of optimal household behavior and the mechanical pricing rule used by the firms. Arbitrage free pricing means that no agent can come up with a trading plan that will result in acquiring more of his endowed good than he started with in his endowment. Such a trading plan would allow the agent eventually to trade for an unlimited amount of any of the goods. Market clearing occurs at two levels. First, each trading post must buy the same amount of any good that it either sells or burns up to cover its fixed cost. This condition is just material balance for trading posts, and holds in or out of equilibrium. Second, the total amount of each good brought to the market by households less the amount needed to cover the post's operating costs must equal the amount consumed by households. This condition represents material balance for the households given that all trading posts earn zero profit.

The equilibrium patterns of trade are distinguished by which trading posts are active. A monetary equilibrium is characterized by each active post trading a different good for a distinguished good that is common to all active trading posts, which serves as the common medium of exchange. In the case of three goods a monetary equilibrium consists of two active trading posts and one inactive trading post. Agents that are endowed with or consume the commodity money trade directly and the other agents trade indirectly. A barter equilibrium is characterized by an active trading post for every pair of consumption goods, which means that all trading posts remain active. All agents trade directly for their desired consumption goods. In the case of three goods there can be as many as four possible equilibria, depending on the number of each type of agent and the

fixed cost of each trading post. The possible equilibria are barter and three monetary equilibria.

In the absence of ties for the high-cost post there is a unique Pareto efficient equilibrium, in which the trading post with the highest fixed cost is inactive and the unique good that is traded at both of the other two trading posts serves as the medium of exchange. Due to the cost sharing rules, a move from an inefficient to an efficient equilibrium need not be Pareto improving. However if each trading post can use any of the three goods to cover its fixed cost, a central planner could choose those amounts to cover the fixed costs such that no agents are worse off and some agents are strictly better off.

### 3. Long Run Equilibrium Selection

This paper performs a long run equilibrium analysis in the style of Kandori, Mailath and Rob (1993). KMR present a dynamic adjustment process for adaptive agents that consists of two parts, a deterministic component and a stochastic component. Under the deterministic adjustment, each agent best responds to the distribution of actions of the population. The deterministic adjustment is characterized by the basins of attraction as explained below. Under the stochastic component, there is a low probability that any agent will deviate from the best response and play a different action. The stochastic component leads to a steady-state probability of transitioning between equilibria that is independent of the initial state of the system. KMR also show that as the probability of deviation goes to zero that the limiting distribution can be determined by counting the minimum number of deviations it takes to leave each equilibrium or state. This limiting distribution can be interpreted as the proportion of time the system spends at each equilibrium in the long run.

The analysis in this paper proceeds in three steps. First we characterize the basin of attraction of each equilibrium. Then we use the probabilities of switching between equilibria to construct the transition matrix. Finally we calculate the limiting probability distribution of the states as the probability of error approaches zero.



### 3.1. Basins of attraction

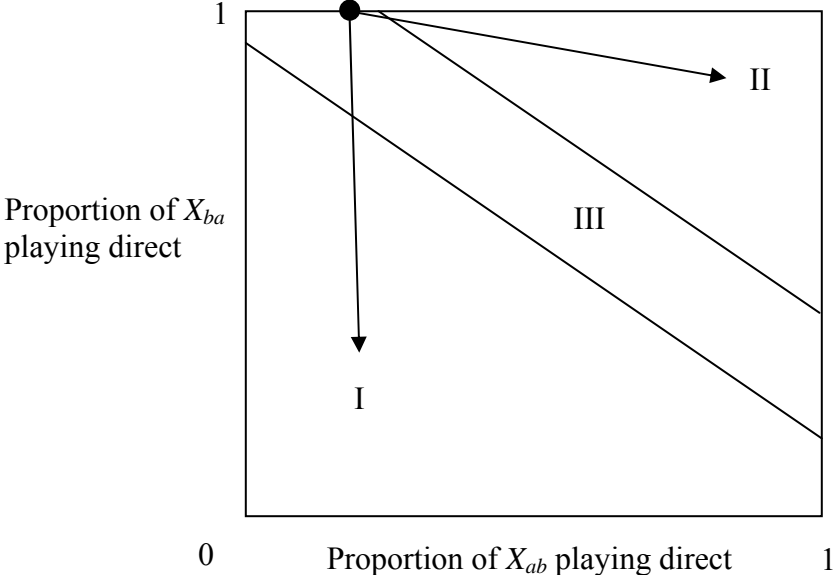
In standard analysis the basin of attraction for an equilibrium is the region of the action space where best response dynamics lead to that equilibrium. The standard best response dynamics have each agent in the population play his best response to the given action distribution. However in this paper if the entire population best responds, there is a large portion of the action space that will lead to cyclic behavior. To avoid this cyclic behavior a damped best response dynamic is used where exactly one agent is chosen at random each period to respond to the population's action distribution.<sup>3</sup> This dynamic ensures that an equilibrium will eventually be reached. However a given point in the action space may lead to different equilibria depending on the specific types of agents that are chosen to respond; it is path dependent. For this paper we define the basin of attraction of an equilibrium as the region of the action space that puts positive probability on reaching that equilibrium given the damped best-response dynamic.

For the case of 3 goods with 6 types of agents, the action space can be described as a unit hyper-cube in the positive orthant with one corner at the origin. The coordinates of any point inside the cube represent the proportions of each type that play direct trade. The point (1, 1, 1, 1, 1, 1) represents the (possible) barter equilibrium where all pairs trade directly. In order to assign each point to the correct basin(s), prices are calculated according to the posts' mechanical pricing rules and the agents' specified actions. The basin of attraction for an equilibrium is the region of points that place a positive probability on reaching that equilibrium given the best response dynamic (including the probability that each agent is given the opportunity to adjust, but not including the error probabilities). A two-dimensional representation of the basins of attraction is illustrated in figure 1 below. If

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<sup>3</sup> KMR use a damped best response dynamic for the case of games with no symmetric pure strategy equilibrium. They demonstrate that an undamped best response dynamic may result in a limiting distribution that does not correspond to the mixed strategy equilibrium for the system. Although the only stable equilibria for the model in this paper are pure strategy equilibria, a damped best response dynamic is appropriate because market imbalances result in the same type of cyclic behavior as considered by KMR. Cyclic behavior results from market imbalances because if one side of a market sees a favorable price the other side will see an unfavorable price. The following period agents on both sides of this market will change actions and the situation will reverse.

all agents of types  $X_{ac}$ ,  $X_{ca}$ ,  $X_{bc}$  and  $X_{cb}$  play direct trade, this representation gives one face of the hyper-cube that represents the action space for all types. The point (1, 1) on this graph represents the barter equilibrium and the point (0, 0) represents the equilibrium where good C is used as the medium of exchange.



**Figure 1: Basins of attraction**

The union of regions I and III is the basin of attraction for the monetary equilibrium with good C used as the common medium of exchange. The union of regions II and III is the basin of attraction for the barter equilibrium. Given that one agent is chosen to best respond each period, points in region III put positive probability on reaching either equilibrium. Appendix B provides an example of assigning a point in the action space to a basin of attraction. It continues the example used in Appendix A.

### 3.2. Transitions

The best response dynamic leads to movement within a basin of attraction towards an equilibrium. The probability of error leads to periodic large jumps from one equilibrium to another. These transitions occur when enough individual errors accumulate to move the system to a new basin of attraction. Given a low probability of an error the probability of a transition from one equilibria to another is approximately equal to the probability of

moving from the first equilibrium to any point in the second equilibrium's basin of attraction. The probability of transitioning from equilibrium  $i$  to equilibrium  $j$  is calculated by summing the probabilities of error necessary to move the system from equilibrium  $i$  to each point in equilibrium  $j$ . Calculating the transition probabilities is computationally intensive, but relatively straightforward. One step of this calculation is presented in Appendix C.

### 3.3. Limiting steady-state distribution

Define  $[T]$  as the transition probability matrix where  $[t]_{ij}$  is equal to the probability of reaching the basin of attraction for equilibrium  $i$  at time  $t + 1$  given that the system is at equilibrium  $j$  at time  $t$ . Once the transition probabilities are determined, the system is characterized as a Markov process where each equilibrium corresponds to a state. Define  $\mathbf{P}_t$  as the vector of probabilities of being at the different equilibria at time  $t$ . The steady state distribution is calculated from the equation  $[T] \mathbf{P}_t = \mathbf{P}_{t+1}$  when  $\mathbf{P}_t = \mathbf{P}_{t+1}$ . Finally the limit of the steady state distribution will be taken as the error rate goes to 0. This limit is independent of the initial distribution and can be interpreted as the percentage of time the system spends at each equilibrium in the long run.

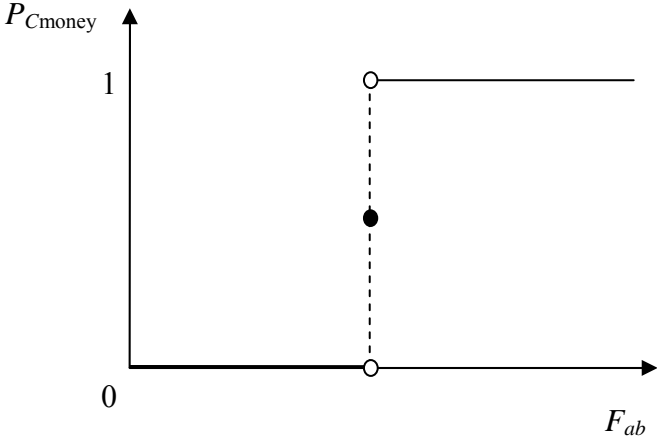
## 4. General Results

This section gives theorems that show how the long run equilibria distribution changes as the parameters change. Two factors drive the equilibrium selection, the number of each type of agent and the fixed costs of the trading posts. For instance, if the number of types  $X_{ab}$ ,  $X_{ba}$ ,  $X_{ac}$  and  $X_{ca}$  all increase then the basin of attraction grows for the equilibrium where good A is used as the medium of exchange. Intuitively this proposition holds because the higher traffic on these trading routes drives down the fixed cost per agent and makes the option of indirect trade look relatively more appealing to types  $X_{bc}$  and  $X_{cb}$ . If the fixed cost of trading post  $Y_{bc}$  increases then the basin of attraction grows for the equilibrium where good A is used as the medium of exchange. This proposition holds because the cost of trading at post  $Y_{bc}$  increases and makes indirect trade look more appealing for types  $X_{bc}$  and  $X_{cb}$ .

There are three possible regions of parameter values, one with no barter equilibrium, one with a small basin of attraction for the barter equilibrium, and one with a large basin of attraction for the barter equilibrium. A large barter basin is defined when the transition from one money to barter to another money requires no more deviations than the transition from the first money directly to the second. The theorems in this section only apply to the case of a large barter basin whereas the techniques presented in section 5 apply to all the regions. These techniques allow the computation of the limiting steady state distribution for any specific economy but they do not allow for analytic comparative statics.

Assume there are equal numbers of corresponding types of agents ( $n_{ab} = n_{ba}$ ,  $n_{ac} = n_{ca}$ , and  $n_{bc} = n_{cb}$ ). Let  $P_i$  represent the probability mass assigned by the limiting steady state distribution to the equilibrium where good  $i$  is used as money.

Note that these probabilities are not smooth functions of the parameters. The limiting steady-state distribution is generally flat with respect to the parameters in the model and makes discrete jumps when these parameters cross threshold values. A sample graph of the probability of the good  $C$  monetary equilibrium is given in figure 2 below.



**Figure 2: Sample probability of the  $C$  money equilibrium as a function of  $F_{ab}$**

Note that when the number of types  $X_{jk}$  and  $X_{kj}$  decreases the relative number the other four types increases (who are all endowed with or consume good  $i$ ). Theorem 1 states that as the number of types  $X_{jk}$  and  $X_{kj}$  decreases, then the limiting steady-state distribution (weakly) places more probability mass on the equilibrium where good  $i$  is used as the medium of exchange.

**Theorem 1:**  $\frac{\Delta P_i}{\Delta n_{jk}} \leq 0$

Theorem 2 states that as the fixed cost of a trading post rises, then the limiting steady-state distribution (weakly) places more probability mass on the equilibrium where the good that is not exchanged at that post is used as the medium of exchange.

**Theorem 2:**  $\frac{\Delta P_i}{\Delta F_{jk}} \geq 0$

The intuition behind both these proofs is that as the average cost of trading at a post increases its likelihood of closing increases. These theorems are proven in Appendix D.

## 5. Examples

Theorems 1 and 2 give comparative static results for cases with a large barter equilibrium. We can use the techniques described in section 3 to compute the actual long run equilibria for any specific economy (regardless of the size of the basins of attraction). We now examine two examples, one symmetric and one asymmetric. In the symmetric case, there are equal numbers of each type of agent and all firms have identical fixed costs. In the asymmetric case the model's parameters vary across goods and across agents as explained below.

### 5.1. Case I – Symmetric

First consider the symmetric case where there are 10 of each type of agent and all fixed costs are equal to 2. Each firm that operates divides its cost evenly between the 2 goods that are traded at its post ( $\alpha_{ij} = 0.5$  for  $i, j \in \{A, B, C\}$ ). The perfectly symmetric case seems like the most difficult case for a money to emerge because the advantage that the best potential money enjoys over barter is smaller than in any asymmetric case. Given these parameters, this example has all four equilibria, barter and the three monetary. A monetary equilibrium is an equilibrium where one good is used as the common medium of exchange.

The transition probabilities are calculated as explained in section 3 and the limiting distribution is calculated as  $\varepsilon$  approaches 0. The steady state distribution puts probability 1/3 on each of the monetary equilibria, meaning that in the long run, in any given period,

the probability that the system is in a given monetary equilibrium is  $1/3$ . This steady state distribution may be counterintuitive at first, but it is explained by the symmetry in the model. In the long run, the model will remain in one of the monetary states for a long time. However the model will eventually move back to the barter equilibrium. The model will spend a relatively short time in the barter state and will then switch to a (possibly different) monetary equilibrium. There is an equal probability of switching to any of the monetary equilibria due to the perfect symmetry in this example. The final result is that the probability of economy being in any given monetary equilibrium is  $1/3$  and the probability of being in the barter equilibrium is 0.

## 5.2. Case II – Asymmetric

In the next example we will see that a single money will be predicted with probability one in the limit in a case where the symmetry is disturbed. In this example there are 10 of each type,  $X_{ab}$ ,  $X_{ba}$ ,  $X_{ac}$  and  $X_{ca}$ , and there are 7 of type  $X_{bc}$  and  $X_{cb}$ . This assumption tends to favor the use of good  $A$  as money because agents will pay a lower share of the fixed costs at trading posts  $Y_{ab}$  and  $Y_{ca}$ . The fixed cost of post  $Y_{ca}$  is equal to 1.25 and that the fixed costs of the other two posts are equal to 1. This assumption tends to favor the use of good  $B$  as money so that agents can avoid paying the higher fixed cost at trading post  $Y_{ca}$ . The economy still has four possible equilibria and the unique efficient equilibrium is the one where good  $B$  is used as the commodity money. Again, each firm that operates divides its cost evenly between the 2 goods that are traded at its post.

Markov chain-transition probabilities  
(calculated from the asymmetric example)

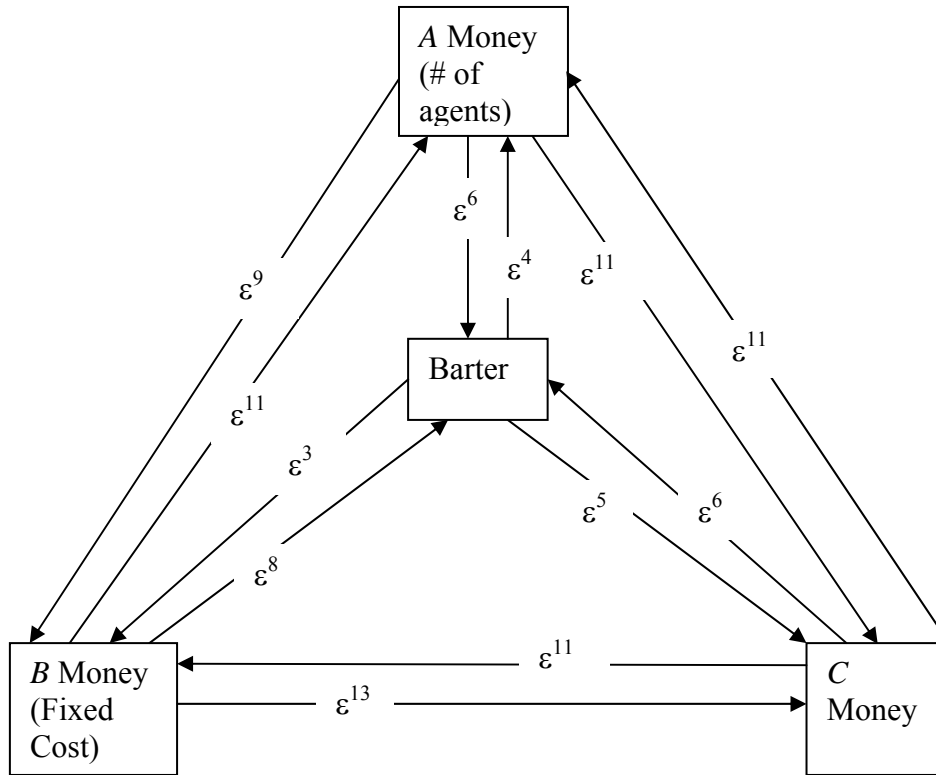


Figure 3: Markov chain transition probabilities

These transition probabilities (shown in figure 3) are the probabilities of switching from one equilibrium to another given that the probability of making an error is  $\epsilon$ . They are calculated using the techniques described in section 3. These probabilities are used to construct the transition probability matrix which is used to solve the following equation for the steady state equilibrium probabilities.

$$\begin{vmatrix} 1-\epsilon^4-\epsilon^3-\epsilon^5 & \epsilon^6 & \epsilon^8 & \epsilon^6 \\ \epsilon^4 & 1-\epsilon^6-\epsilon^9-\epsilon^{11} & \epsilon^{11} & \epsilon^{11} \\ \epsilon^3 & \epsilon^9 & 1-\epsilon^8-\epsilon^{11}-\epsilon^{13} & \epsilon^{11} \\ \epsilon^5 & \epsilon^{11} & \epsilon^{13} & 1-\epsilon^6-\epsilon^{11}-\epsilon^{11} \end{vmatrix} \begin{vmatrix} P_{\text{barter}} \\ P_{\text{Amoney}} \\ P_{\text{Bmoney}} \\ P_{\text{Cmoney}} \end{vmatrix} = \begin{vmatrix} P_{\text{barter}} \\ P_{\text{Amoney}} \\ P_{\text{Bmoney}} \\ P_{\text{Cmoney}} \end{vmatrix}$$

As  $\varepsilon$  goes to 0, the limiting steady state distribution becomes  $(0, 0, 1, 0)$  which implies that in the long run, the system will spend almost all of the time with good  $B$  used as the commodity money.

## 6. Conclusion

Theorems 1 and 2 provide conditions that favor the use of a particular commodity as the common medium of exchange in a trading post economy. Theorem 1 demonstrates that as a good becomes more common (in terms of consumption and endowment) it is more likely to be used as the medium of exchange. Theorem 2 shows that as the fixed cost of operating a trading post increases the likelihood of that post shutting down increases. Theorem 2 shows that there is a tendency for equilibrium selection to favor efficiency, but it does not always suffice for full efficiency. Theorems 1 and 2 show that equilibrium selection via evolutionary dynamics has a strong but not perfect tendency to favor more efficient equilibria. Theorem 2 favors efficiency but theorem 1 does not.

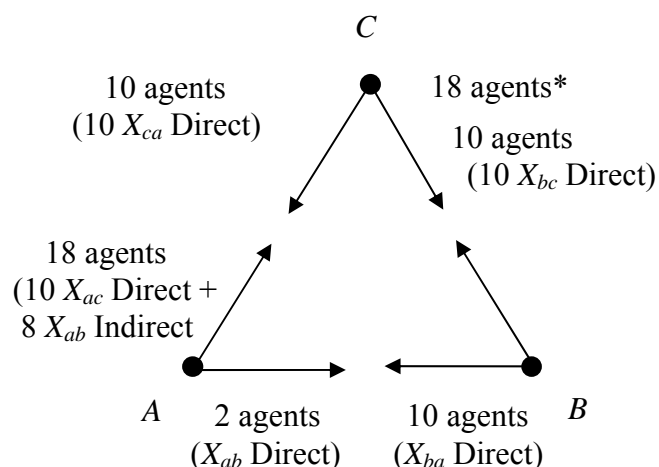
The examples demonstrate that in the long run, as the probability of error approaches 0, the proportion of time spent in the monetary equilibria approaches 1. In the symmetric case, all possible commodity monies have positive prior probabilities, and the barter equilibrium has zero prior probability. In the asymmetric example, by contrast, a unique money will emerge with probability one. In the examples considered in this paper, the costs are such that all equilibria that have positive probability in the long run are efficient. In the symmetric example, exactly one post will be shut down at a time, and in the asymmetric example, the high cost post will be shut down.

There are several questions open for further investigation elsewhere. An interesting issue to consider is whether increasing the number of goods in the model to four will significantly change the analysis. Adding a good will have two effects. The first is that monetized equilibria will have a greater efficiency advantage over barter equilibria because more trading posts will be shut down. The second is that it will be harder for agents to coordinate on a particular money since there will be an increase in both the number of types and in the number of actions available to each type. With four goods, there are more monies to choose from and there are additional equilibrium patterns of trade that are neither strictly barter nor monetary.



## Appendix A: Sample Calculation of Equilibrium Prices

Assume that there are ten of each type of agent, that the fixed cost of each post is 2, and that each firm pays the fixed cost using 1 unit of each good that it trades ( $\alpha_{ij} = 0.5$  for  $i, j \in \{a, b, c\}$ ). Consider the case where two agents of type  $X_{ab}$  trade directly and eight agents of type  $X_{ab}$  trade indirectly. Furthermore, assume all ten agents of each type  $X_{ba}$ ,  $X_{ac}$ ,  $X_{ca}$ ,  $X_{bc}$  and  $X_{cb}$  trade directly. This point is (2, 10, 10, 10, 10, 10) in the action space. The pattern of trade is shown in figure 4 below.



**Figure 4: Pattern of trade (agents)**

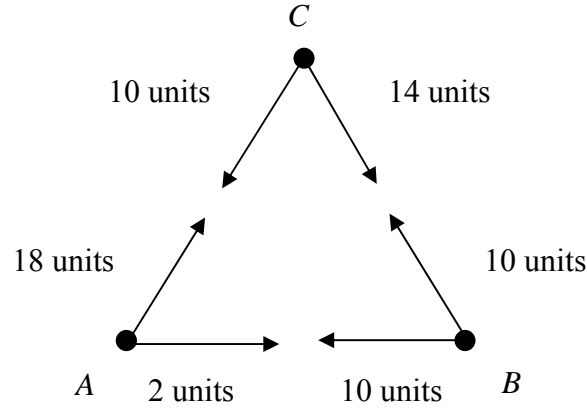
This graph shows the number of agents that trade (or attempt to trade) at each post and in which direction. For instance, eighteen agents (marked with a \* above) are bringing good C to the  $Y_{bc}$  trading post, all ten agents of type  $X_{cb}$  that play direct trade and the eight agents of type  $X_{ab}$  that play indirect trade. The number of agents bringing each good to each post is different than the actual amount of each good that is brought because the agents that trade in 2 steps do not generally trade at a 1:1 ratio at the first post that they visit. The exact quantity of each good that is supplied at each trading post is found by solving simultaneously a system of six equations, two for each post.

The following equation says that the amount of good A brought to trading post  $Y_{ab}$  is equal to the number of  $X_{ab}$  agents that trade directly plus the number of  $X_{ac}$  agents that trade

indirectly plus the total amount of good  $A$  received by agents of type  $X_{cb}$  that trade indirectly after they trade good  $C$  for good  $A$ .  $A_{ab}$  refers to the amount of good  $A$  brought to trading post  $Y_{ab}$  and  $d_{ij}$  refers to the proportion of type  $X_{ij}$  that chooses direct trade.

$$A_{ab} = n_{ab}d_{ab} + n_{ac}(1 - d_{ac}) + n_{cb}(1 - d_{cb})\text{Max}\left[\left(\frac{A_{ac} - \alpha_{ac}^a F_{ac}}{C_{ac}}\right), 0\right]$$

The other five equations are similar. There is a corresponding equation for the amount of good  $B$  brought to trading post  $Y_{ab}$  as well as two equations for each of the other two posts. The quantities are determined by solving these equations for  $A_{ab}$ ,  $B_{ab}$ ,  $A_{ac}$ ,  $C_{ac}$ ,  $B_{bc}$  and  $C_{bc}$ . The closed-form solution to these equations does not exist but can be approximated recursively. This solution is illustrated in figure 5 below.



**Figure 5: Pattern of trade (goods)**

In the above example, prices are as follows:

- $Y_{ab}$  sells 4.5 units of  $B$  for 1 unit of  $A$  and sells 0.1 units of  $A$  for 1 unit of  $B$ .
- $Y_{ac}$  sells 0.5 units of  $C$  for 1 unit of  $A$  and sells 1.7 units of  $A$  for 1 unit of  $C$ .
- $Y_{bc}$  sells 1.3 units of  $C$  for 1 unit of  $B$  and sells 0.64 units of  $B$  for 1 unit of  $C$ .

Given the trading volumes and prices it's straight forward to check for consistency. For instance, the amount of  $C$  that flows into the  $BC$  trading post,  $C_{bc}$  is equal to 14 (10 from the  $X_{cb}$  agents that trade directly plus the 4 units received by the  $X_{ab}$  agents that trade indirectly). Post  $Y_{ac}$  will give 0.5 units of  $C$  to each of the eight agents of type  $X_{ab}$  trading indirectly. We'll refer back to this example in Appendix B to illustrate the basins of attraction.

## Appendix B: Sample Calculation of Equilibrium Prices

Consider 10 agents of each type, with all agents playing direct trade except 8 agents of type  $X_{ab}$ , which is represented by the point  $(0.2, 1, 1, 1, 1, 1)$  in the action space. This is the example we considered in section Appendix A. Recall that prices are as follows:

- $Y_{ab}$  sells 4.5 units of  $B$  for 1 unit of  $A$  and sells 0.1 units of  $A$  for 1 unit of  $B$ .
- $Y_{ac}$  sells 0.5 units of  $C$  for 1 unit of  $A$  and sells 1.7 units of  $A$  for 1 unit of  $C$ .
- $Y_{bc}$  sells 1.3 units of  $C$  for 1 unit of  $B$  and sells 0.64 units of  $B$  for 1 unit of  $C$ .

An agent of type  $X_{ab}$  will receive 4.5 units of  $B$  if he chooses direct trade and will receive 0.32 units of  $B$  if he chooses indirect trade. First he'll receive 0.5 units of  $C$  for one unit of  $A$ ; then he'll receive 0.32 units of  $B$  for the 0.5 units of  $C$ . All agents of type  $X_{ab}$  will choose direct trade given the opportunity to change actions (provided they do not make errors). There is a positive probability that the 8 agents of type  $X_{ab}$  that were initially playing indirect trade will be chosen successively over the following 8 periods to be allowed to choose new strategies. Each agent will choose to play direct trade, leading to a positive probability of reaching the barter equilibrium. (The prices will change as each agent changes actions but direct trade will still beat indirect trade for these agents.) A single point may be in the basin for more than one equilibrium because the best response can vary based on which agent is chosen each period to be allowed to change strategies. In the example above, if type  $X_{ba}$  agents are chosen first, they will change to indirect trade. Eventually every agent will desire to play the actions that correspond with the equilibrium that uses good  $C$  as money. The point  $(0.2, 1, 1, 1, 1, 1)$  is in the basin of attraction for both the barter equilibrium and the equilibrium with good  $C$  used as money.

## Appendix C: Partial Calculation of Transition Probabilities

Continuing from the example given in Appendix B, the point being examined is  $(2, 10, 10, 10, 10, 10)$ . This point has been found to be in the basin of attraction for both the good  $C$  monetary equilibrium and the barter equilibrium. The probability of switching from any equilibrium to either of these equilibria by way of this point can be calculated. To

calculate the probability of the system switching from the good  $B$  monetary equilibrium to the barter equilibrium by way of this point, the number of deviations from the good  $B$  monetary equilibrium to this point is summed. The good  $B$  monetary equilibrium corresponds to the point  $(10, 10, 10, 10, 0, 0)$  in the action space. In this example there are 28 deviations. All twenty agents of types  $X_{AC}$  and  $X_{CA}$  must switch from indirect to direct trade. Additionally 8 agents of type  $X_{AB}$  must switch from direct to indirect trade. The probability of this event occurring is  $\varepsilon^{28}$ . The overall probability of switching from the good  $B$  monetary equilibrium to the barter equilibrium is determined by summing the probabilities of switching equilibria over all of the points in the basin of attraction for the barter equilibrium.

#### Appendix D: Proof of theorems 1 and 2

These proofs use a steady-state characterization given by Freidlin and Wentzel (1984) and discussed in Kandori, Mailath and Rob (1993). The characterization involves the construction of a vector,  $q$  that is proportional to the steady-state distribution. Each element in  $q$ ,  $q_z$  is defined as the sum of the product of all transition probabilities for all  $z$ -trees. A  $z$ -tree is a directed graph from all states except  $z$  to state  $z$  where each state except  $z$  has a unique successor. For instance, one ( $A$ -money)-tree is  $\{B \text{ money to Barter, } C \text{ money to Barter, Barter to } A \text{ money}\}$ . The transition probabilities used to construct  $q$  are the same transition probabilities used in the earlier analysis in this paper. Each  $q_z$  is a polynomial in terms of  $\varepsilon$ , the error rate. Let  $a^*$  denote the minimum power of  $\varepsilon$  that appears in  $q$  (with a nonzero coefficient). Define  $a_z$  as the coefficient of  $\varepsilon^{a^*}$  in  $q_z$ . As  $\varepsilon$  approaches 0, the limiting distribution of  $q$  is  $a_z/\sum_i a_i$  (from L'Hopital's rule). This distribution is the same as the limiting steady-state distribution.

Both proofs use the case with a large basin of attraction for the barter equilibrium. A large barter basin means that the probability of switching from one money to barter to another money is higher than the probability of switching directly from one money to the other. The only  $z$ -tree that needs to be considered for a monetary equilibrium is the one that has each of the other monetary equilibria leading to barter and has the barter equilibrium leading to that monetary equilibrium. The only Barter-tree that needs to be considered is

the one where each monetary equilibrium leads directly to the barter equilibrium. The relevant z-trees are as follows:

Barter-tree: **B-money to Barter**, C-money to Barter, **A-money to Barter**.

(A-money)-tree: **B-money to Barter**, C-money to Barter, Barter to A-money.

(B-money)-tree: **A-money to Barter**, C-money to Barter, Barter to B-money.

(C-money)-tree: **A-money to Barter**, **B-money to Barter**, **Barter to C-money**.

The two theorems are demonstrated by noting the effects that changing parameters in the model have upon these transition probabilities. These proofs both consider changes that increase the probability given to the equilibrium where good *C* is used as the common medium of exchange. These proofs are also appropriate for other changes due to the symmetry in the model. Both proofs demonstrate that the transition probabilities shown in bold face above become more likely and that the other transition probabilities become less likely. The sum of these probabilities for the (C-money)-tree must gain on the other trees because the all the segments that increase in the other trees (A-money to Barter and B-money to Barter) are included in the (C-money)-tree. The (C-money)-tree also has one segment that increases that is not included in the other relevant trees (Barter to C-money). Since the relevant (C-money)-trees increases relative to the other trees, the probability given to the equilibrium where good *A* is used as the common medium of exchange in the limiting steady-state equilibrium (weakly) increases because of the possibility that that tree now has a non-zero coefficient for the lowest power of  $\varepsilon$ .

Assume there are equal numbers of corresponding types of agents ( $n_{ab} = n_{ba}$ ,  $n_{ac} = n_{ca}$ , and  $n_{bc} = n_{cb}$ ). Let  $P_i$  represent the probability mass assigned by the limiting steady state distribution to the equilibrium where good *i* is used as money.

**Theorem 1:** 
$$\frac{\Delta P_i}{\Delta n_{jk}} \leq 0$$

Theorem 1 is shown by looking at the changes in the basins of attraction of the equilibria at the relevant regions, the points in action space where the minimum transitions in the relevant trees occur. We need to determine the minimum number of deviations needed to switch from the barter equilibrium to the basin of attraction for the equilibrium where good *C* is used as the common medium of exchange. (The proof is similar going

from the barter equilibrium to any monetary equilibrium or from any monetary equilibrium to the barter equilibrium.) Consider balanced trade where the number of agents of type  $X_{ij}$  playing direct trade is the same as the number of type  $X_{ji}$  playing direct trade for all  $i, j$ . Assume that all agents of types  $X_{ac}$ ,  $X_{ca}$ ,  $X_{bc}$  and  $X_{cb}$  are playing direct trade. If the following inequality is satisfied, the system is in the basin of attraction for the equilibrium where good  $C$  is used as the common medium of exchange.

$$\frac{F_{ab}}{n_{ab}z} > \frac{F_{ac}}{n_{ab}(1-z) + n_{ac}} + \frac{F_{bc}}{n_{ab}(1-z) + n_{bc}} \quad (\text{eq. 1})$$

The variable  $z$  represents the proportion of type  $X_{ab}$  (and type  $X_{ba}$ ) that play direct trade and  $z$  must be in the interval  $[0,1]$ .<sup>4</sup> The minimum number of deviations required to change from barter to the monetary equilibrium is equal to one minus the maximum value of  $n_{ab}z$  that satisfies the above inequality. The minimum number of deviations is achieved when all deviations occur from one side of the market (for instance, agents  $X_{ab}$ ). Then there is a positive probability that agents of type  $X_{ba}$  will be chosen to best respond. Once enough of these agents respond, prices will cause all agents of type  $X_{ab}$  and  $X_{ba}$  to choose indirect trade and the monetary equilibrium will be reached. If there are fewer deviations, agents of type  $X_{ab}$  and  $X_{ba}$  will eventually return to direct trade and the system will not reach the monetary equilibrium.

The derivative with respect to  $n_{ab}$  of the solution to the maximum  $z$  that satisfies equation 1 gives the change in the transition probability of switching from barter to good  $C$  used as money. The derivative can be found using the implicit function theorem.

$$\frac{\partial z}{\partial n_{ab}} = - \frac{-\frac{F_{ab}}{n_{ab}^2 z} + \frac{F_{ac}(1-z)}{[n_{ab}(1-z) + n_{ac}]^2} + \frac{F_{bc}(1-z)}{[n_{ab}(1-z) + n_{bc}]^2}}{-\frac{F_{ab}}{n_{ab}z^2} - \frac{F_{ac}n_{ab}}{[n_{ab}(1-z) + n_{ac}]^2} - \frac{F_{bc}n_{ab}}{[n_{ab}(1-z) + n_{bc}]^2}} \quad (\text{eq 2})$$

We're interested in the parameter values where equation 2 is negative. The denominator is always negative so the condition we're interested in cases where the following inequality holds.

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<sup>4</sup> The fixed costs are divided slightly differently here than in the rest of the paper. Each agent's share of the fixed costs at a post is divided proportionally among the number of users that use that post rather than based on the quantity traded there. This formulation is necessary for the theorem because it avoids the recursive price calculations. The prices are similar if trade is balanced and if the fixed costs are low relative to the number of agents.

$$F_{ab} > \frac{n_{ab}^2 z(1-z) F_{ac}}{[n_{ab}(1-z) + n_{ac}]^2} + \frac{n_{ab}^2 z(1-z) F_{bc}}{[n_{ab}(1-z) + n_{bc}]^2} \quad (\text{eq 3})$$

$z$  depends on the other parameters in the model so equation 3 is not of much use on its own. Note that the maximum value for  $z(1-z) = 1/4$  and the minimum value of  $(1-z) = 0$ . Therefore equation 3 will hold if the following holds.

$$F_{ab} > \frac{\frac{1}{4} n_{ab}^2 F_{ac}}{n_{ac}^2} + \frac{\frac{1}{4} n_{ab}^2 F_{bc}}{n_{bc}^2} \quad (\text{eq 4})$$

Equation 4 is a stronger condition than we need but it is more illustrative than the weaker condition. Furthermore a numerical search reveals that the derivative is negative given that all fixed costs are equal to 1, all  $\alpha$ 's are equal to 0.5,  $n_{ac} = n_{ca} = n_{bc} = n_{cb} = 10$  and  $n_{ab} = n_{ba} \in [1, 4000]$ .<sup>5</sup> Results for the other transition probabilities are computed similarly and they reveal that the transition probabilities shown in bold face on the relevant  $z$ -tree chart increase and that the other transition probabilities decrease. Therefore there is a (weak) increase in the probability mass given to the equilibrium where good  $C$  is used as the common medium of exchange in the limiting steady-state distribution.

**Theorem 2:**  $\frac{\Delta P_i}{\Delta F_{jk}} \geq 0$

Theorem 2 is more straightforward to prove. When the fixed cost of trading post  $Y_{ab}$  increases, the new basin of attraction for the equilibrium where good  $C$  is used as the medium of exchange contains the original basin (it's weakly larger everywhere). The new basins for the other monetary equilibria are contained inside their original basins. These results can be determined from the following equations.

$$u(\text{direct}) = \max\left[\frac{J_{ij} - \alpha_{ij}^j F_{ij}}{I_{ij}}, 0\right]$$

$$u(\text{indirect}) = \max\left[\frac{K_{ik} - \alpha_{ik}^k F_{ik}}{I_{ik}}, 0\right] \max\left[\frac{J_{jk} - \alpha_{jk}^j F_{jk}}{K_{jk}}, 0\right]$$

These utilities are for agents of type  $X_{ij}$ . The expression  $J_{ij}$  represents the amount of good  $j$  brought to the  $Y_{ij}$  trading post. The basin of attraction for the equilibrium with good  $C$  used as the medium of exchange consists of all points in the action space where  $u(\text{direct})$

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<sup>5</sup> Similar results hold when  $n_{ac}$ ,  $n_{ca}$ ,  $n_{bc}$ ,  $n_{cb}$ , and the fixed costs vary.

$\geq u(\text{indirect})$  for types  $X_{ac}$ ,  $X_{ca}$ ,  $X_{bc}$  and  $X_{cb}$  and where  $u(\text{indirect}) > u(\text{direct})$  for types  $X_{ab}$  and  $X_{ba}$ .

Given that the new basins either contain or are contained in the original basins, the possible change in the limiting distribution follows directly from the changes in the relevant transition probabilities. For instance, if  $F_{ab}$  increases, the transition probabilities decrease for Barter to A-money, Barter to B-money and C-money to Barter. The transition probabilities increase for A-money to Barter, B-money to Barter and Barter to C-money. Based on these changes, the (C-money)-tree increases. The other three trees have an indeterminate change, but any increase will be strictly less than the increase in the A-tree because each component that increases is contained in the changes in the A-tree. The minimum power of  $\varepsilon$  in the A-tree decreases relative to the other trees, so the limiting steady stated distribution assigns (weakly) more probability to the equilibrium where good A is used as the medium of exchange.

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