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EXISTENCE OF COMPETITIVE EQUILIBRIA IN MARKETS WITH A CONTINUUM OF TRADERS[1]

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An appropriate model for a market with many individually insignificant traders is one with a continuum of traders. Here it is proved that competitive equilibria exist in such markets, even though individual preferences may not be convex. Such a result is not true for markets with finitely many traders.

1. INTRODUCTION

THE PROBLEM of rigorously establishing the existence of a competitive equilibrium in a market was first brought to the attention of economists by Wald [11]. Since the appearance of his pioneering paper, other authors[2] have established the existence of competitive equilibria under various sets of assumptions. In all this work, it was invariably assumed that the traders have convex preferences.[3] Indeed, if this assumption is abandoned it is easy to give examples of markets that do not possess any competitive equilibria.

Attention has recently been called[4] to the possibility of dispensing with the convexity assumption if the market in question has a large number of traders, no individual one of whom can significantly affect the outcome of trading. In a heuristic, imprecise way it was argued that the preferences of a large number of individually insignificant traders would have a convex effect in the aggregate, even if none of the individual preferences were convex. A rigorous treatment of this theme was given very recently by Shapley and Shubik [10], though not directly in connection with the competitive equilibrium. Their work will be discussed in Section 8.

In a previous paper [2], we suggested that the most appropriate model for a market with many individually insignificant traders is one with a continuum of traders. Analogous models are used in physics, for example, when the large number of particles in a fluid are replaced for mathematical convenience by a con-

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[2] Such as Arrow-Debreu [1] ,Gale [7], and McKenzie [9].

[3] I.e., that the set of commodity bundles preferred or indifferent to a given bundle is convex.

[4] See the articles by Bator, Farell, Koopmans, and Rothenberg in the *Journal of Political Economy*: (Vol. 67, 1959, pp. 377–391; Vol. 68, 1960, pp. 435–468; Vol. 69, 1961, pp. 478–493).

tinuum of particles. This raises the question of whether it would be possible to establish the existence of competitive equilibria in markets with a continuum of traders, even when the preferences need not be convex. The purpose of this paper is to give an affirmative answer to that question, and thus to underscore the power and scope of the continuum-of-traders approach to market theory.

We remark that the concept of competitive equilibrium is generally agreed to be significant only in a market with "perfect competition," i.e., one with a large number of individually insignificant traders. The concept makes no sense for a small number of traders. Thus, we show here that when competitive equilibria are at all relevant, convex preferences are not needed to establish their existence.

The proof is based on McKenzie's beautiful existence proof [**9**] for competitive equilibria in finite markets. Major modifications are required, however, because of the presence of a continuum of traders (which necessitates the use of Banach-space methods) and the nonavailability of convex preferences.

In Section 2 we give a precise statement of the model and the main theorem. Section 3 is devoted to the statement of an auxiliary theorem. In Section 4 the proof of the auxiliary theorem is outlined, and in Section 5 it is completed. In Section 6 the main theorem is deduced from the auxiliary theorem.

Section 7 is devoted to a detailed comparison of our proof with McKenzie's, and Section 8 to a discussion of the relation of our current result to that of our previous paper [**2**] and to the Shapley-Shubik results [**10**].

Our result concerns true markets only, i.e., pure exchange economies. Presumably it can be extended to economies with production (at least if one assumes constant returns to scale), but we have not done this.

2. MATHEMATICAL MODEL AND STATEMENT OF MAIN THEOREM

We shall be working in a Euclidean space E^n; the dimensionality n of the space represents the number of different commodities being traded in the market. Superscripts will be used exclusively to denote coordinates. Following standard practice, for x and y in E^n we take $x>y$ to mean $x^i>y^i$ for all i; $x\geqq y$ to mean $x^i\geqq y^i$ for all i; and $x\geq y$ to mean $x\geqq y$ but not $x=y$. The integral of a vector function is to be taken as the vector of integrals of the components. Superscripts will be used exclusively to denote coordinates. The scalar product $\Sigma_{i=1}^n x^i y^i$ of two members x and y of E^n is denoted $x\cdot y$. The symbol 0 denotes the origin in E^n as well as the real number zero; no confusion will result. The symbol $\setminus$ will be used for set-theoretic subtraction, whereas $-$ will be reserved for ordinary algebraic subtraction.

A *commodity bundle* x is a point in the nonnegative orthant Ω of E^n. The set of *traders* is the closed unit interval [0, 1]; it will be denoted T. The words "measure," "measurable," "integral," and "integrable" are to be understood in the sense of Lebesgue. All integrals are with respect to the variable t (which stands for trader), and in most cases the range of integration is all of T. In an integral we will therefore omit the symbol dt and the indication of dependence of the integrand on t, and will specifically indicate the range of integration only when it differs from all of T. Thus $\int \boldsymbol{x}$ means $\int_T \boldsymbol{x}(t)\,dt$. A *null set* is a set of measure 0. *Null sets of traders are systematically ignored throughout the paper.* Thus a statement asserted for "all" traders, or "each" trader, or "each" trader in a certain set, is to be understood to hold for all such traders except possibly for a null set of traders.

An *assignment* (of commodity bundles to traders) is an integrable function on T to Ω. There is a fixed *initial assignment* $\boldsymbol{i}$; *intuitively*, $\boldsymbol{i}(t)$ is the bundle with which trader t comes to market. We assume

(2.1) $\int \boldsymbol{i} > 0$.

Intuitively, this asserts that no commodity is totally absent from the market.

For each trader t there is defined on Ω a relation $\succsim_t$ called *preference-or-indifference*. This relation is assumed to be a *quasi-order*, i.e., transitive, reflexive, and complete.[5] From $\succsim_t$ we define relations $\succ_t$ and $\sim_t$ called *preference* and *indifference*, respectively, as follows:

$$x \succ_t y \quad \text{if} \quad x \succsim_t y \quad \text{but not} \quad y \succsim_t x\,;$$

$$x \sim_t y \quad \text{if} \quad x \succsim_t y \quad \text{and} \quad y \succsim_t x\,.$$

The following assumptions are made:

(2.2) *Desirability* (*of the commodities*): $x \geq y$ implies $x \succ_t y$.

(2.3) *Continuity* (*in the commodities*): For each $y \in \Omega$, the sets $\{x : x \succ_t y\}$ and $\{x : y \succ_t x\}$ are open (relative to Ω).

(2.4) *Measurability*: If $\boldsymbol{x}$ and $\boldsymbol{y}$ are assignments, then the set $\{t : \boldsymbol{x}(t) \succ_t \boldsymbol{y}(t)\}$ is measurable.

The intuitive content of these assumptions should be fairly clear from their names. Note that together with the assumption that $\succsim_t$ is a quasi-order, the continuity assumption (2.3) yields the existence of a continuous utility function $v_t(x)$ on Ω for each fixed trader t **[4]**. Then the measurability assumption[6] (2.4) says that the v_t can be chosen so that $v_t(x)$ is simultaneously measurable in t and x.

An *allocation* is an assignment $\boldsymbol{x}$ such that $\int \boldsymbol{x} = \int \boldsymbol{i}$. A *price vector* is a member p of R^n such that $p \geq 0$; though it is in Ω, it should not be thought of as a commodity bundle. A *competitive equilibrium* is a pair consisting of a price vector p and an allocation $\boldsymbol{x}$, such that for all traders t, $\boldsymbol{x}(t)$ is maximal with respect to $\succ_t$ in the "budget set" $\boldsymbol{B}_p(t) = \{x \in \Omega : p \cdot x \leqq p \cdot \boldsymbol{i}(t)\}$.

5 A relation $\mathscr{R}$ is called *transitive* if $x\mathscr{R}y$ and $y\mathscr{R}z$ imply $x\mathscr{R}z$; *reflexive* if $x\mathscr{R}x$ for all x; and *complete* if for all x and y, either $x\mathscr{R}y$ or $y\mathscr{R}x$.

6 In this context (but not in [2]), the measurability assumption is equivalent to the assumption that $\{t : x \succ_t y\}$ is measurable for all x and y in Ω.

MAIN THEOREM: *Under the conditions of this section, there is a competitive equilibrium.*

3. STATEMENT OF AUXILIARY THEOREM

To prove the main theorem, we first establish an auxiliary theorem, which has some interest in its own right. Let us define a *market* $\mathscr{M}$ to consist of a positive integer n (the number of commodities), an initial assignment $\boldsymbol{i}$, and preference-or-indifference relations $\succsim_t$ on Ω for each of the traders t. The markets that we consider heıe differ from those described in the previous section in a number of ways. First, condition (2.1) on the initial assignments is strengthened to read

(3.1) $\boldsymbol{i}(t) > 0$ for all t.

This means that a positive amount of each commodity is initially held by each trader.

Second, a bundle x is said to *saturate*, or more explicitly, to *saturate trader t's desire*, if $x \succsim_t y$ for all $y \in \Omega$. Assumption 2.2 is weakened to read as follows:

(3.2) *Weak Desirability*: Unless y saturates, $x > y$ implies $x \succ_t y$.

Notice that this is a double weakening of (2.2); the hypothesis $x \geq y$ is replaced by $x > y$, and allowance is made for saturation (saturation is impossible under (2.2)).

Third, under the auxiliary theorem we do not only permit saturation, we specifically require it. Let $\boldsymbol{v}$ be an assignment. We say that trader t's desire is *commodity-wise-saturated* at $\boldsymbol{v}(t)$ if for all bundles x and commodities i such that $x^i \geqq \boldsymbol{v}^i(t)$, we have

$$x \sim_t (x^1, \ldots, x^{i-1}, \boldsymbol{v}^i(t), x^{i+1}, \ldots, x^n) .$$

In other words, changing the value of the ith coordinate above $\boldsymbol{v}^i(t)$ does not change the indifference level. Intuitively, this means that desire for the ith commodity is saturated when the quantity of that commodity is $\boldsymbol{v}^i(t)$, although trader t may still

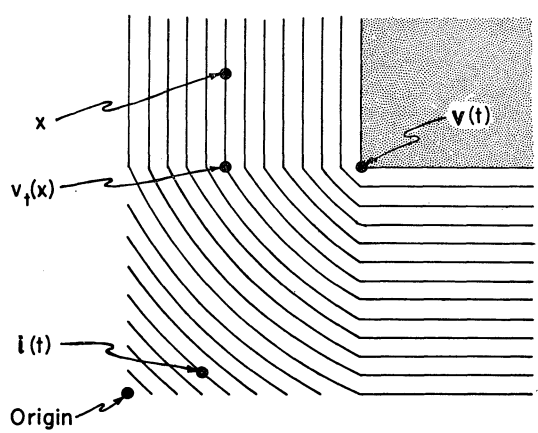

FIGURE 1

want moıe of other commodities j, of which he holds less than $\boldsymbol{v}^j(t)$. To rephrase the condition, let $V(t) = \{x \in \Omega \colon x \leqq \boldsymbol{v}(t)\}$ be the "hyper-rectangle" of bundles that are $\leqq \boldsymbol{v}(t)$, and define a mapping v_t from Ω into $V(t)$ as follows: $v_t(x)$ is the bundle formed from x by replacing by $\boldsymbol{v}^i(t)$ all coordinates x^i of x that exceed $\boldsymbol{v}^i(t)$. Then commodity-wise saturation at $\boldsymbol{v}(t)$ asserts that $v_t(x) \sim_t x$. It follows that the entire preference order is determined by its behavior in the hypercube $V(t,)$ since $x \succsim_t y$ if and only if $v_t(x) \succsim_t v_t(y)$. A preference order with commodity-wise saturation is illustrated in Figure 1.

The existence of a $\boldsymbol{v}(t)$ that commodity-wise saturates desire is intuitively very acceptable; it simply means that there is an upper bound on the amount of a commodity that can be profitably used by an individual, no matter what other commodities are or are not available. The demand that $\boldsymbol{v}$ be an assignment, i.e., integrable, means that "the market as a whole can be commodity-wise saturated"; more precisely, it means that there is a bundle (namely $\int \boldsymbol{v}$) that can be distributed among the traders in such a way as to commodity-wise saturate each trader's desire. We now assume

(3.3) There is an assignment $\boldsymbol{v}$ such that each trader t's desire is commodity-wise saturated at $\boldsymbol{v}(t)$.

Finally, we need the following assumption:

(3.4) *Saturation restriction*: x cannot saturate unless $x > \boldsymbol{i}(t)$.

AUXILIARY THEOREM: *Let $\mathscr{M}$ be a market satisfying the assumptions of this section as well as* (2.3) *and* (2.4). *Then $\mathscr{M}$ has a competitive equilibrium.*

4. OUTLINE OF THE PROOF OF THE AUXILIARY THEOREM

The starting point of the proof is the *preferred set* $\boldsymbol{C}_p(t)$, defined for each trader t and each price vector p to be the set of commodity bundles preferred or indifferent to all elements of the budget set $\boldsymbol{B}_p(t)$; formally,

$$\boldsymbol{C}_p(t) = \{x \in \Omega \colon \quad \text{for all} \quad y \in \boldsymbol{B}_p(t),\ x \succsim_t y\}$$

(see Figure 2). Next, define

$$\int \boldsymbol{C}_p = \{\int \boldsymbol{x} \colon\ \boldsymbol{x} \text{ is an assignment such that } \boldsymbol{x}(t) \in \boldsymbol{C}_p(t) \text{ for all } t\}\ ;$$

this is called the *aggregate preferred set*. $\int \boldsymbol{C}_p$ is the set of all aggregate bundles that can be distributed among the traders in such a way that each trader is at least as satisfied as he is when he sells his initial bundle and buys the best (by his standards) that he can with the proceeds, at prices p.

Since we have made no convexity assumption on the preferences, the individual preferred sets $\boldsymbol{C}_p(t)$ need not be convex. The *aggregate* preferred set $\int \boldsymbol{C}_p$, on the other hand, *is* convex; as we shall see, that fact holds only because there is a continuum of traders, and it constitutes the nub of the proof. By using the convexity of the

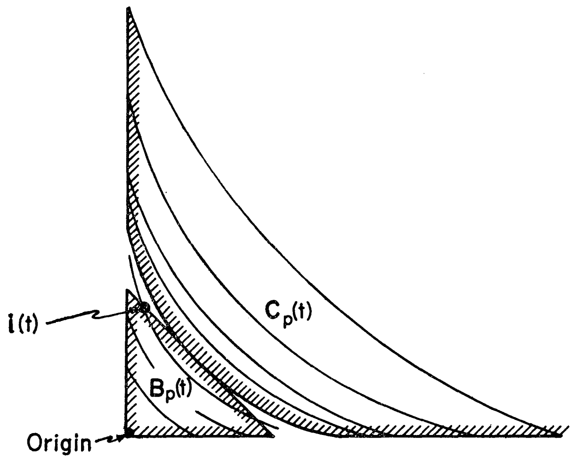

FIGURE 2

aggregate preferred set $\int C_p$, we shall be able to show that there is a unique point $c(p)$ in $\int C_p$ that is nearest to $\int \boldsymbol{i}$; set $h(p) = c(p) - \int \boldsymbol{i}$.

Let P be the simplex of price vectors normalized so that their sum is 1, i.e., $P = \{p \in \Omega : \Sigma_{i=1}^n p^i = 1\}$. The central idea of the proof is to use h to construct a continuous function f from P to itself, and then to apply Brouwer's fixed point theorem;[7] the resulting fixed point—denoted q—turns out to be an equilibrium price vector. The function f is defined by

$$f(p) = \frac{p + h(p)}{1 + \sum_{i=1}^{n} h^i(p)} .$$

We shall show later that $h(p) \geqq 0$. Therefore, the denominator in the definition of f does not vanish, and so $f(p) \in P$ for all $p \in P$. Suppose q is a fixed point of f. Then

$$q\left(1 + \sum_{i=1}^{n} h^i(q)\right) = q + h(q) ,$$

i.e.,

(4.1) $$h(q) = \alpha q ,$$

where, because $h(p) \geqq 0$,

$$\alpha = \sum_{i=1}^{n} h^i(q) \geqq 0 .$$

[7] Brouwer's theorem asserts that every continuous single-valued function f from P to itself has a fixed point, i.e., a point p such that $f(p) = p$. For a proof, see Dunford and Schwartz [5, Sec. V. 12, p. 468].

We wish to show that

(4.2) $\quad h(q)=0\,.$

Indeed, suppose (4.2) is false. From the definition of h and the convexity of $\int \boldsymbol{C}_p$ it follows that for all p, the hyperplane through $h(p)+\int \boldsymbol{i}$ perpendicular to $h(p)$ supports[8] $\int \boldsymbol{C}_p$. Applying this for $p=q$, we obtain

$$(y-\textstyle\int \boldsymbol{i})\cdot h(q) \geqq h(q)\cdot h(q)$$

for all $y \in \int \boldsymbol{C}_q$. Because (4.2) is false, $\alpha>0$; so by (4.1), we obtain

$$(y-\textstyle\int \boldsymbol{i})\cdot \alpha q \geqq \alpha^2 q\cdot q\,,$$

and hence

(4.3) $\quad (y-\int \boldsymbol{i})\cdot q \geqq \alpha(q\cdot q)>0 \quad \text{for all} \quad y\in \int \boldsymbol{C}_q\,.$

Now if for each t we let $\boldsymbol{x}(t)$ be a point in the budget set $\boldsymbol{B}_q(t)$ that is maximal with respect to t's preference order, then on the one hand we have $(\boldsymbol{x}(t)-\boldsymbol{i}(t))\cdot q \leqq 0$, and on the other hand $\boldsymbol{x}(t)\in \boldsymbol{C}_q(t)$. Hence, by integrating we obtain $(\int \boldsymbol{x}-\int \boldsymbol{i})\cdot q\leqq 0$, and $\int \boldsymbol{x}\in\int \boldsymbol{C}_q$; this contradicts (4.3), and establishes (4.2).

Equation (4.2) says that $\int \boldsymbol{i}\in\int \boldsymbol{C}_q$, i.e., there is an assignment $\boldsymbol{x}$ such that $\int \boldsymbol{x}=\int \boldsymbol{i}$ and $\boldsymbol{x}(t)\in \boldsymbol{C}_q(t)$ for all t. Thus, $\boldsymbol{x}$ is an allocation, and $\boldsymbol{x}(t)$ is preferred or indifferent to all elements of $\boldsymbol{B}_q(t)$. To complete the proof that $(q, \boldsymbol{x})$ is a competitive equilibrium, it is only necessary to show that $\boldsymbol{x}(t)$ is in $\boldsymbol{B}_q(t)$ for all t. Suppose now that $q\cdot \boldsymbol{x}(t)<q\cdot \boldsymbol{i}(t)$ for some t. Then $\boldsymbol{x}$ does not saturate (because of the saturation restriction (3.4)), and so from the desirability assumption (3.2), it follows that $\boldsymbol{x}(t)+(\delta,\ldots,\delta)\succ_t \boldsymbol{x}(t)$ for $\delta>0$. But for δ sufficiently small, we shall still have

$$q\cdot(\boldsymbol{x}(t)+(\delta,\ldots,\delta))=q\cdot \boldsymbol{x}(t)+\delta<q\cdot \boldsymbol{i}(t)\,,$$

so $\boldsymbol{x}(t)+(\delta,\ldots,\delta)\in \boldsymbol{B}_q(t)$; this contradicts $\boldsymbol{x}(t)\in \boldsymbol{C}_q(t)$. So $q\cdot \boldsymbol{x}(t)<q\cdot \boldsymbol{i}(t)$ is impossible, and we conclude that $q\cdot \boldsymbol{x}(t)\geqq q\cdot \boldsymbol{i}(t)$ for all t. If the $>$ sign would hold for some t, we could deduce $\int q\cdot \boldsymbol{x}>\int q\cdot \boldsymbol{i}$, contradicting $\int \boldsymbol{x}=\int \boldsymbol{i}$. So $q\cdot \boldsymbol{x}(t)=q\cdot \boldsymbol{i}(t)$ for all t, and it follows that $\boldsymbol{x}(t)\in \boldsymbol{B}_q(t)$ for all t. So $(q, \boldsymbol{x})$ is a competitive equilibrium.

The foregoing proof, which follows McKenzie's ideas [9] rather closely, is incomplete in two respects: The required properties of $h(p)$—existence, uniqueness, continuity, and nonnegativity—have not been established; and it has not been shown that the $\boldsymbol{x}$ whose integral $\int \boldsymbol{x}$ contradicts (4.3) may be chosen to be measurable. These items will be taken up in the next section.

5. COMPLETION OF THE PROOF OF THE AUXILIARY THEOREM

In this section we make considerable use of the theory of integrals of set-valued functions, as developed in [3]. Before stating the results from [3] that are used in the sequel, we recall the necessary definitions.

[8] This is a standard method of constructing a supporting hyperplane. An explicit proof is given by McKenzie [9, Lemma 7 (1), p. 61].

Let $\boldsymbol{F}$ be a function defined on T whose values are subsets of Ω. Define

$$\int \boldsymbol{F} = \{\int \boldsymbol{f} : \boldsymbol{f} \text{ is integrable and } \boldsymbol{f}(t) \in \boldsymbol{F}(t) \text{ for all } t\} .$$

$\boldsymbol{F}$ is called *Borel-measurable* if its *graph* $\{(x, t): x \in E^n, x \in \boldsymbol{F}(t)\}$ is a Borel subset of $\Omega \times T$. $\boldsymbol{F}$ is called *integrably bounded* if there is an integrable point-valued function $\boldsymbol{b}$ such that for all t, $x \in \boldsymbol{F}(t)$ implies $x \leqq \boldsymbol{b}(t)$. For each t, $\boldsymbol{F}^*(t)$ denotes the convex hull of $\boldsymbol{F}(t)$.

For each p in P, let F_p be a subset of Ω. F is said to be *upper-semicontinuous in p* if for each convergent sequence $p_1, p_2, \ldots$ in P and each convergent sequence $x_1, x_2, \ldots$ in Ω such that $x_1 \in F_{p_1}$, $x_2 \in F_{p_2}, \ldots$, we have $\lim x_k \in F_{\lim p_k}$. It is *lower-semicontinuous in p* if for each convergent sequence $p_1, p_2, \ldots$ in P, every point in $F_{\lim p_k}$ is the limit of a sequence $x_1, x_2, \ldots$ in Ω such that $x_1 \in F_{p_1}$, $x_2 \in F_{p_2}, \ldots$. It is *continuous* if it is both upper- and lower-semicontinuous.

If $F_1, F_2, \ldots$ are subsets of E^n, then $\limsup F_k$ is defined to be the set of all x in E^n such that every neighborhood of x intersects infinitely many F_k.

The following lemmas are proved in [**3**]:

LEMMA 5.1: *$\int \boldsymbol{F}$ is convex.*

LEMMA 5.2: *If $\boldsymbol{F}$ is Borel-measurable, and $\boldsymbol{F}(t)$ is non-empty for each t, then there is a measurable function $\boldsymbol{f}$ such that $\boldsymbol{f}(t) \in \boldsymbol{F}(t)$ for all t.*

LEMMA 5.3: *If $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots$ is a sequence of set-valued functions that are all bounded by the same integrable point-valued function, then $\int \limsup \boldsymbol{F}_k \supset \limsup \int \boldsymbol{F}_k$.*

LEMMA 5.4: *If $\boldsymbol{F}_p(t)$ is continuous in p for each fixed t and Borel-measurable in t for each fixed p in P, and if all the $\boldsymbol{F}_p$ are bounded by the same integrable point-valued function, then $\int \boldsymbol{F}_p$ is continuous in p.*

We now wish to establish the existence, uniqueness, continuity, and nonnegativity of the function h. In principle, the first three of these properties will follow from the closedness and nonemptiness, convexity, and continuity (in p) of $\int \boldsymbol{C}_p$ respectively; nonnegativity will follow from weak desirability. In carrying out the proofs, however, the unboundedness of the $\boldsymbol{C}_p(t)$ causes difficulties. To circumvent these difficulties, we shall find a bounded "substitute" for $\boldsymbol{C}_p$.

Let $\boldsymbol{v}$ be a commodity-wise saturating assignment (i.e., an assignment satisfying (3.3)), and recall the notation $V(t) = \{x : x \leqq \boldsymbol{v}(t)\}$. We shall work with the sets $V(t) \cap \boldsymbol{C}_p(t)$, which we shall denote $\boldsymbol{D}_p(t)$, passing back to the consideration of $\boldsymbol{C}_p$ itself only at the very end of the section. Note that the $\boldsymbol{D}_p(t)$ are integrably bounded, uniformly in p, by the function $\boldsymbol{v}$.

LEMMA 5.5: *For each t, $\boldsymbol{D}_p(t)$ is continuous in p.*

PROOF: A similar lemma was proved by McKenzie [**9**, Lemma 4, pp. 57, 68]; we repeat the proof for the sake of completeness. Let $p_1, p_2, \ldots \in P$ have limit p. Suppose first that $x_k \in \boldsymbol{D}_{p_k}(t)$ are such that $\lim x_k = x$. Certainly $x \in V(t)$; so if $x \notin \boldsymbol{D}_p(t)$, then there is $y \in \boldsymbol{B}_p(t)$ such that $y \succ_t x$. Then $p \cdot y \leqq p \cdot \boldsymbol{i}(t)$, and since assumption (3.1) asserts that $\boldsymbol{i}(t) > 0$, it follows that $p \cdot \boldsymbol{i}(t) > 0$. So we can find a z that is sufficiently close to y so that we still have $z \succ_t x$ (by continuity (2.3)), but $p \cdot z < p \cdot \boldsymbol{i}(t)$. Then for k sufficiently large, we shall still have $p_k \cdot z < p_k \cdot \boldsymbol{i}(t)$. Again applying continuity (2.3), we deduce from $z \succ_t x$ that for k sufficiently large $z \succ_t x_k$; but this contradicts $x_k \in \boldsymbol{D}_{p_k}(t)$. Hence $x \in \boldsymbol{D}_p(t)$, and upper-semicontinuity is proved.

Next, let $x \in \boldsymbol{D}_p(t)$. If x saturates, then it is a member of all $\boldsymbol{D}_{p_k}(t)$, so we can set $x_k = x$ in the definition of lower-semicontinuity. Assume therefore that x does not saturate. Let x_k be a point in $\boldsymbol{D}_{p_k}(t)$ closest to x; the existence of x_k follows from the closedness of $\boldsymbol{D}_{p_k}(t)$, which in turn follows from upper-semicontinuity. For arbitrary $\delta > 0$, set $y_\delta = v_t(x + (\delta, \ldots, \delta))$; then $y_\delta \sim_t x + (\delta, \ldots, \delta) \succ_t x$, by (3.2). Either for all δ, $y_\delta \in \boldsymbol{D}_{p_k}(t)$ for all sufficiently large k, or else for some δ, there are infinitely many k such that $y_\delta \notin \boldsymbol{D}_{p_k}(t)$. In the first case we have for all δ, by the definition of x_k, that $\|x_k - x\| \leqq \|y_\delta - x\| \leqq \delta\sqrt{n}$, where $\| \;\, \|$ represents the euclidean norm (i.e., the distance from the origin). Since δ can be chosen arbitrarily small, this shows that $x_k \to x$, and establishes lower-semicontinuity. In the second case, we can assume without loss of generality that $y_\delta \notin \boldsymbol{D}_{p_k}(t)$ for all k. Then for each k there is a $z_k \in \boldsymbol{B}_{p_k}(t) \cap V(t)$ such that $z_k \succ_t y_\delta$. Since the z_k are all in $V(t)$, they have a limit point z; again without loss of generality, we can let it be the limit. Since $p_k \to p$, and $p_k \cdot z_k \leqq p_k \cdot \boldsymbol{i}(t)$, it follows that $p \cdot z \leqq p \cdot \boldsymbol{i}(t)$, i.e., $z \in \boldsymbol{B}_p(t)$. On the other hand, from continuity (2.3) it follows that $z \succsim_t y$; since $y_\delta \succ_t x$, it follows that $z \succ_t x$. But this contradicts $x \in \boldsymbol{D}_p(t) \subset \boldsymbol{C}_p(t)$, and completes the proof of the lemma.

The proof of upper-semicontinuity in this lemma is the only place where use is made of $\boldsymbol{i}(t) > 0$ (3.1), rather than the far weaker $\int \boldsymbol{i} > 0$ (2.1).

LEMMA 5.6: *$\boldsymbol{C}_p$ and $\boldsymbol{D}_p$ are Borel-measurable for each fixed p.*

PROOF: Since every measurable function differs on at most a null set from a Borel-measurable function, we may assume that $\boldsymbol{v}$ and $\boldsymbol{i}$ are Borel-measurable. The statement "$x \in \boldsymbol{D}_p(t)$" is equivalent to "$x \leqq \boldsymbol{v}(t)$ and $x \in \boldsymbol{C}_p(t)$." The statement "$x \in \boldsymbol{C}_p(t)$" is equivalent to "for all $y \in \boldsymbol{B}_p(t)$, $x \succsim_t y$"; because of continuity (2.3), this is equivalent to "for each rational point[9] $r \in \boldsymbol{B}_p(t)$, $x \succsim_t r$." For fixed r, "$x \succsim_t r$" is equivalent to "not $r \succ_t x$." Because of continuity, "$r \succ_t x$" is equivalent to "there is a rational point s in Ω such that $s \geqq x$ and $r \succ_t s$." Hence $\{(x, t) : x \succsim_t r\}$, which equals

$$\Omega \times T \setminus \bigcup_{\text{rational } s \text{ in } \Omega} [\{x : s \geqq x\} \times \{t : r \succ_t s\}],$$

is a Borel set. Hence $\{(x, t) : x \in \boldsymbol{C}_p(t)\}$, which equals $\bigcap_{\text{rational } r \text{ in } \Omega} [(\Omega \times \{t : p \cdot r > p \cdot \boldsymbol{i}(t)\}) \cup \{(x, t) : x \succsim_t r\}]$, is a Borel set, and this proves that $\boldsymbol{C}_p$ is Borel-measurable.

[9] I.e., point with rational coordinates.

Hence $\{(x, t): x \in \boldsymbol{D}_p(t)\}$, which equals

$$\{(x, t): x \leqq \boldsymbol{v}(t)\} \cap \{(x, t): x \in \boldsymbol{C}_p(t)\},$$

is a Borel set, and the proof is complete.

COROLLARY 5.7: *$\int \boldsymbol{D}_p$ is closed, non-empty, convex, and continuous in p.*

PROOF: $\int \boldsymbol{v} \in \int \boldsymbol{D}_p$, so non-emptiness is proved. Convexity follows from Lemma 5.1. Since $\boldsymbol{D}_p$ is uniformly integrably bounded by $\boldsymbol{v}$, continuity follows from Lemmas 5.4, 5.5, and 5.6. Since the values of a continuous set-valued function are always closed, the corrollary is proved.

For each p in P, let $d(p)$ be the point in $\int \boldsymbol{D}_p$ that is closest to $\int \boldsymbol{i}$. Such a point exists because $\int \boldsymbol{D}_p$ is non-empty and closed; it is unique because $\int \boldsymbol{D}_p$ is convex.

LEMMA 5.8: *$d(p)$ is a continuous (point-valued) function of p.*

PROOF: A similar lemma was proved by McKenzie [**9**, Lemma 10, p. 62]; we repeat the proof for the sake of completeness. Let $p_k \to p$, and let x be a limit point of $d(p_k)$. From the upper-semicontinuity of $\int \boldsymbol{D}_p$ it follows that $x \in \int \boldsymbol{D}_p$. Suppose that there is a point $y \in \int \boldsymbol{D}_p$ such that $\|y - \int \boldsymbol{i}\| < \|x - \int \boldsymbol{i}\|$. By the lower-semicontinuity of $\int \boldsymbol{D}_p$, there is a sequence of points $y_k \in \int \boldsymbol{D}_{p_k}$ converging to y. Let $\{d(p_{k_j})\}$ be a subsequence of $\{d(p_k)\}$ converging to x. Since the norm is continuous, it follows that for j sufficiently large,

$$\|y_{k_j} - \textstyle\int \boldsymbol{i}\| < \|d(p_{k_j}) - \textstyle\int \boldsymbol{i}\|,$$

contradicting the definition of $d(p_{k_j})$. Hence $x = d(p)$. So the only limit point of $\{d(p_k)\}$ is $d(p)$, and the lemma is proved.

LEMMA 5.9: *For each p in P, $d(p) \geqq \int \boldsymbol{i}$.*

PROOF: If not, then $d(p)$ has a coordinate—without loss of generality, we can let it be the first—such that $d^1(p) < \int \boldsymbol{i}^1$. Now $d(p) = \int \boldsymbol{x}$, where $\boldsymbol{x}(t) \in \boldsymbol{D}_p(t)$ for all t. Let $\boldsymbol{y}(t) = (\boldsymbol{v}^1(t), \boldsymbol{x}^2(t), \ldots, \boldsymbol{x}^n(t))$. Then $\boldsymbol{y}(t) \geqq \boldsymbol{x}(t)$ and $\boldsymbol{y}(t) \leqq \boldsymbol{v}(t)$; therefore $\boldsymbol{y}(t) \in \boldsymbol{D}_p(t)$ for all t. Therefore

$$\left(\textstyle\int \boldsymbol{v}^1, d^2(p), \ldots, d^n(p)\right) = \textstyle\int \boldsymbol{y} \in \textstyle\int \boldsymbol{D}_p.$$

Now $d^1(p) < \int \boldsymbol{i}^1$, and by the saturation restriction (3.4), $\int \boldsymbol{i}^1 < \int \boldsymbol{v}^1$; so there is an α with $0 < \alpha < 1$ such that $\alpha \int \boldsymbol{v}^1 + (1-\alpha) d^1(p) = \int \boldsymbol{i}^1$. Setting $\boldsymbol{z} = \alpha \boldsymbol{y} + (1-\alpha)\boldsymbol{x}$ and $z = \int \boldsymbol{z}$, we obtain $z \in \int \boldsymbol{D}_p$ (by the convexity of $\int \boldsymbol{D}_p$), and $z = (\int \boldsymbol{i}^1, d^2(p), \ldots, d^n(p))$. Then from $d^1(p) < \int \boldsymbol{i}^1$, we deduce $\|z - \int \boldsymbol{i}\|^2 = \Sigma_{i=2}^n (d^i(p) - \int \boldsymbol{i}^i)^2$

$$< \sum_{i=1}^{n} \left(d^i(p) - \textstyle\int \boldsymbol{i}^i\right)^2 = \|d(p) - \textstyle\int \boldsymbol{i}\|^2.$$

Thus z is closer than $d(p)$ to $\int \boldsymbol{i}$, a contradiction. This proves the lemma.

Let $g(p) = d(p) - \int \boldsymbol{i}$. We have established for $g(p)$ all the properties that we set out to establish for $h(p)$: existence, uniqueness, continuity, and nonnegativity (the last by Lemma 5.9). So with the following lemma we achieve our aim:

LEMMA 5.10: $g(p) = h(p)$.

PROOF: Fix p, and write $g = g(p)$, $h = h(p)$, $c = c(p)$, $d = d(p)$. If $g = 0$ there is nothing to prove. Otherwise, by the definition of g, the hyperplane through d perpendicular to g supports $\int \boldsymbol{D}_p$ (see footnote 8). This means that

(i) $\quad x \cdot g \geqq \|g\|^2$ for all $x \in \int \boldsymbol{D}_p - \int \boldsymbol{i}$.

Suppose there is a point in $\int \boldsymbol{C}_p$ that is nearer to $\int \boldsymbol{i}$ than d is. This means that there is a point y in $\int \boldsymbol{C}_p - \int \boldsymbol{i}$ that is nearer to 0 than g is. Then

(ii) $\quad \|y\|^2 < \|g\|^2$.

Furthermore $\|y\|^2 - 2y \cdot g + \|g\|^2 = \|y - g\|^2 > 0$. Hence $\|y\|^2 > y \cdot g + [y \cdot g - \|g\|^2]$. If $y \cdot g - \|g\|^2 \geqq 0$, then it follows that $\|y\|^2 > y \cdot g \geqq \|g\|^2$, contradicting (ii). Hence

(iii) $\quad y \cdot g < \|g\|^2$.

Formula (iii) expresses the geometrically obvious fact that any point nearer than d to $\int \boldsymbol{i}$ must be on the near side of the hyperplane through d perpendicular to g.

Now $y = \int \boldsymbol{x} - \int \boldsymbol{i}$, where $\boldsymbol{x}(t) \in \boldsymbol{C}_p(t)$ for all t. Then by commodity-wise saturation, $v_t(\boldsymbol{x}(t)) \in \boldsymbol{D}_p(t)$ for all t. Furthermore $v_t(\boldsymbol{x}(t) \leqq \boldsymbol{x}(t)$, and $v_t(\boldsymbol{x}(t)) \leqq \boldsymbol{v}(t)$. Setting $\boldsymbol{z}(t) = v_t(\boldsymbol{x}(t))$, we obtain $\int \boldsymbol{z} \in \int \boldsymbol{D}_p$ and $\int \boldsymbol{z} - \int \boldsymbol{i} \leqq y$. Since $g \geqq 0$ (Lemma 5.9), it follows that $(\int \boldsymbol{z} - \int \boldsymbol{i}) \cdot g \leqq y \cdot g$. Hence by (iii), $(\int \boldsymbol{z} - \int \boldsymbol{i}) \cdot g < \|g\|^2$. But since $\int \boldsymbol{z} - \int \boldsymbol{i} \in \int \boldsymbol{D}_p - \int \boldsymbol{i}$, it follows from (i) that $(\int \boldsymbol{z} - \int \boldsymbol{i}) \cdot g \geqq \|g\|^2$, and this is the contradiction that proves our lemma.

It remains to show that a measurable $\boldsymbol{x}$ may be chosen whose integral will contradict (4.3). According to Section 4, it is sufficient to show that there is a measurable $\boldsymbol{x}$ such that for all t, $\boldsymbol{x}(t)$ is maximal in $\boldsymbol{B}_q(t)$ with respect to t's preference order. Let $\boldsymbol{X}(t)$ be the set of all maximal points in $\boldsymbol{B}_q(t)$. As in the proof of Lemma 5.6, we may assume that $\boldsymbol{i}$ is Borel-measurable. Then

$$\begin{aligned}\{(x, t) \colon x \in \boldsymbol{B}_q(t)\} &= \{(x, t) \colon q \cdot x \leqq q \cdot \boldsymbol{i}(t)\} \\ &= \Omega \times T \setminus \textstyle\bigcup_\theta [\{x \colon q \cdot x > \theta\} \times \{t \colon \theta > q \cdot \boldsymbol{i}(t)\}],\end{aligned}$$

where θ runs over the rational numbers. Hence the left side is a Borel set. Applying Lemma 5.6 we deduce that $\{(x, t) \colon x \in \boldsymbol{X}(t)\}$, which equals

$$\{(x, t) \colon x \in \boldsymbol{B}_q(t)\} \cap \{x, t) \colon x \in \boldsymbol{C}_q(t)\},$$

is a Borel set. Hence $\boldsymbol{X}$ is Borel-measurable.

Next, we show that $\boldsymbol{X}(t)$ is non-empty for each t. From the compactness of $\boldsymbol{V}(t) \cap \boldsymbol{B}_q(t)$ and the continuity condition (2.3) for preferences, it follows that $\boldsymbol{V}(t) \cap \boldsymbol{B}_q(t)$

has a maximal element y. Then because of commodity-wise saturation, y is also maximal in $\boldsymbol{B}_q(t)$. Indeed, suppose $z \in \boldsymbol{B}_q(t)$ is such that $z \succ_t y$. Now $z \in \boldsymbol{B}_q(t)$ means $q \cdot z \leqq q \cdot \boldsymbol{i}(t)$; therefore $q \cdot v_t(z) \leqq q \cdot z \leqq q \cdot \boldsymbol{i}(t)$, and therefore also $v_t(z) \in \boldsymbol{B}_q(t)$. But by definition, $v_t(z) \in V(t) \cap \boldsymbol{B}_q(t)$. Finally, $v_t(z) \sim_t z \succ_t y$. Thus $v_t(z)$ contradicts the maximality of z in $V(t) \cap \boldsymbol{B}_q(t)$, proving the existence of a maximal element in $\boldsymbol{B}_q(t)$.

From Lemma 5.2 we may now deduce the existence of an appropriate $\boldsymbol{x}$. This completes the proof of the auxiliary theorem.

6. PROOF OF THE MAIN THEOREM

The general idea is to approximate a given market $\mathscr{M}$ satisfying the conditions of the main theorem by a sequence of markets $\mathscr{M}_k$ satisfying the conditions of the auxiliary theorem. Then by the auxiliary theorem, the $\mathscr{M}_k$ have competitive equilibria $(q_k, \boldsymbol{y}_k)$; from these competitive equilibria we shall construct a pair $(q, \boldsymbol{y})$ that is a competitive equilibrium in the original market $\mathscr{M}$.

To define the markets $\mathscr{M}_k$, we must specify their initial assignments $\boldsymbol{i}_k$ and their preference orders $\precsim_t^k$; the number of commodities is taken to be n in all the $\mathscr{M}_k$. Let δ_k be a monotone sequence of numbers tending to 0, and define

$$\boldsymbol{i}_k(t) = \boldsymbol{i}(t) + (\delta_k, \ldots, \delta_k) .$$

To define the preference orders, let γ_k be a monotone sequence of numbers tending to ∞ such that $\gamma_1 > \delta_1$, let

$$\boldsymbol{v}_k(t) = \boldsymbol{i}(t) + (\gamma_k, \ldots, \gamma_k) ,$$

and let "hyper-rectangles" $V_k(t)$ and functions $v_{k,t}$ from Ω onto $V_k(t)$ be defined as in Section 3, with $\boldsymbol{v}_k$ in place of $\boldsymbol{v}$. Now define the preference orders by

$$x \succsim_t^k y \quad \text{if and only if} \quad v_{k,t}(x) \succsim_t v_{k,t}(y) .$$

It may be verified that the $\mathscr{M}_k$ satisfy the conditions of the auxiliary theorem, with $\boldsymbol{v}_k$ as the commodity-wise saturating assignment. Furthermore, note that the preference orders in $\mathscr{M}_k$ coincide with those in $\mathscr{M}$ for all x and y such that x and y are $\leqq \boldsymbol{i}(t) + (\gamma_k, \ldots, \gamma_k)$.

Let $(q_k, \boldsymbol{y}_k)$ be a competitive equilibrium of $\mathscr{M}_k$. Because of the compactness of P, the sequence $\{q_k\}$ has a convergent subsequence, and we may suppose without loss of generality that this subsequence is the original sequence. Let $q = \lim_k q_k$. The following is the crucial lemma of this section:

LEMMA 6.1: $q > 0$.

PROOF: Suppose, on the contrary, that some coordinate of q vanishes, say $q^1 = 0$. First we establish

(i) *if* $q \cdot \boldsymbol{i}(t) > 0$, *then* $\{\boldsymbol{y}_k(t)\}$ *has no limit point as* $k \to \infty$.

Indeed, suppose y were such a limit point; without loss of generality, assume that it is actually the limit. Now because $(q_k, \boldsymbol{y}_k)$ is a competitive equilibrium in $\mathscr{M}_k$, we have $q_k \cdot \boldsymbol{y}_k(t) \leqq q_k \cdot \boldsymbol{i}_k(t)$. Using this and the saturation restriction (3.4) in $\mathscr{M}_k$, we deduce that $\boldsymbol{y}_k(t)$ does not saturate. Hence if $q_k \cdot \boldsymbol{y}_k(t) < q_k \cdot \boldsymbol{i}_k(t)$, then by weak desirability (3.2) in $\mathscr{M}_k$, it would be possible to find a member of $\boldsymbol{B}_{q_k}(t)$ preferred to $\boldsymbol{y}_k(t)$, contradicting the definition of competitive equilibrium. Thus $q_k \cdot \boldsymbol{y}_k(t) = q_k \cdot \boldsymbol{i}(t)$, and so from the hypothesis of (i) we obtain

(ii) $$q \cdot y = \lim_k q_k \cdot \boldsymbol{y}_k(t) = \lim_k q_k \cdot \boldsymbol{i}_k(t) = q \cdot \boldsymbol{i}(t) > 0.$$

Hence there is a coordinate j such that $y^j > 0$ and $q^j > 0$; assume without loss of generality that $j = 2$. Now by desirability (2.2), $y + \{1, 0, \ldots, 0\} \succ_t y$. If for sufficiently small $\delta > 0$ we define $z = y + \{1, -\delta, 0, \ldots, 0\}$, then $z \in \Omega$, and by continuity we deduce $z \succ_t y$. Again using continuity, we obtain $z \succ_t \boldsymbol{y}_k(t)$ for k sufficiently large. Since $(q_k, \boldsymbol{y}_k)$ is a competitive equilibrium in $\mathscr{M}_k$, we obtain $q_k \cdot z > q_k \cdot \boldsymbol{i}_k(t)$. Letting $k \to \infty$ and applying (ii), we get

(iii) $$q \cdot z = \lim_k q_k \cdot z \geqq \lim q_k \cdot \boldsymbol{i}_k(t) = q \cdot y\,.$$

But since $q^1 = 0$ and $q^2 > 0$, we have

$$q \cdot z = q \cdot y + q^1 - \delta q^2 = q \cdot y - \delta q^2 < q \cdot y\,,$$

contradicting (iii). This proves (i).

Since $q \in P$ and $\int \boldsymbol{i} > 0$ (2.1), it follows that $\int q \cdot \boldsymbol{i} = q \cdot \int \boldsymbol{i} > 0$. Let $S = \{t \colon q \cdot \boldsymbol{i}(t) > 0\}$; then S is non-null, and we denote its measure by $\mu(S)$. Define

$$A = \left\{ x \in \Omega \colon \sum_{i=1}^{n} x^i \leqq 2 \int \sum_{j=1}^{n} \boldsymbol{i}^j / \mu(S) \right\}.$$

For $t \in S$, it follows from (i) and the compactness of A that $\boldsymbol{y}_k(t) \in A$ for at most finitely many k; that is, for each $t \in S$ there is an integer $\boldsymbol{k}(t)$ such that $\Sigma_i \boldsymbol{y}_k^i(t) > 2 \int \Sigma_j \boldsymbol{i}^j / \mu(S)$ for $k \geqq \boldsymbol{k}(t)$. Hence for $t \in S$,

(iv) $$\liminf_k \sum_i \boldsymbol{y}_k^i(t) \geqq 2 \int \sum_j \boldsymbol{i}^j / \mu(S)\,.$$

Because $\boldsymbol{y}_k$ is an allocation in $\mathscr{M}_k$, we have

(v) $$\lim_k \int \sum_i \boldsymbol{y}_k^i = \lim_k \int \sum_i \boldsymbol{i}_k^i = \lim_k \left[\int \sum_i \boldsymbol{i}^i + n\delta_k \right] = \int \sum_i \boldsymbol{i}^i\,.$$

But by Fatou's Lemma[10] and (iv),

$$\lim_k \int \sum_i \boldsymbol{y}_k^i \geqq \int \liminf_k \sum_i \boldsymbol{y}_k^i \geqq \int_S \liminf_k \sum_i \boldsymbol{y}_k^i$$
$$\geqq \int_S \left[2 \int \sum_j \boldsymbol{i}^j / \mu(S) \right] = \left[2 \int \sum_j \boldsymbol{i}^j \right] \int_S 1/\mu(S) = 2 \int \sum_j \boldsymbol{i}^j > \int \sum_j \boldsymbol{i}^j\,,$$

[10] Fatou's Lemma states that if φ_k are nonnegative measurable real functions, then $\liminf_k \int \varphi_k \geqq \int \liminf_k \varphi_k$. See [**5**, III. 6.19, p. 152].

where the last inequality follows from $\int \boldsymbol{i} > 0$ (2.1). This contradicts (v) and proves Lemma 6.1.

Since $q_k \to q > 0$, there is a $\delta > 0$ such that $q_k^i \geqq \delta$ for k sufficiently large and all i. Without loss of generality, we may assume that $q_k^i \geqq \delta$ for all i and k, and that $\boldsymbol{i}_k^i(t) \leqq \boldsymbol{i}^i(t) + \delta$ for all i, k, and t. Hence for all i, k, and t,

$$\delta \boldsymbol{y}_k^i(t) \leqq q_k \cdot \boldsymbol{y}_k(t) \leqq q_k \cdot \boldsymbol{i}_k(t) \leqq q_k \cdot \boldsymbol{i}(t) + \delta \leqq \sum_{j=1}^{n} \boldsymbol{i}^j(t) + \delta .$$

Thus we obtain

$$\boldsymbol{y}_k^i(t) \leqq 1 + \sum_{j=1}^{n} \frac{\boldsymbol{i}^j(t)}{\delta} . \tag{6.2}$$

For each t, let $\boldsymbol{Y}(t)$ be the set of limit points of $\boldsymbol{y}_k(t)$ as $k \to \infty$. Let $\boldsymbol{Y}_k(t)$ be the set consisting of the single point $\boldsymbol{y}_k(t)$; then $\boldsymbol{Y}(t) = \limsup \boldsymbol{Y}_k(t)$. By (6.2), all the $\boldsymbol{Y}_k$ are bounded by the same integrable function. Hence by Lemma 5.3,

$$\int \boldsymbol{i} = \lim \int \boldsymbol{i}_k = \lim \int \boldsymbol{y}_k \in \limsup \int \boldsymbol{Y}_k \subset \int \limsup \boldsymbol{Y}_k = \int \boldsymbol{Y} .$$

Let $\boldsymbol{y}$ be such that $\boldsymbol{y}(t) \in \boldsymbol{Y}(t)$ for all t, and

$$\int \boldsymbol{y} = \int \boldsymbol{i} . \tag{6.3}$$

We shall show that $(q, \boldsymbol{y})$ is a competitive equilibrium in $\mathscr{M}$.

To this end we must demonstrate that $\boldsymbol{y}$ is an allocation, that $\boldsymbol{y}(t)$ belongs to $\boldsymbol{B}_q(t)$ for all t, and that $\boldsymbol{y}(t)$ is maximal in $\boldsymbol{B}_q(t)$ for all t, i.e., that no member of $\boldsymbol{B}_q(t)$ is preferred to $\boldsymbol{y}(t)$. We have already shown that $\boldsymbol{y}$ is an allocation (6.3). Next, since $\boldsymbol{y}(t) \in \boldsymbol{Y}(t)$, it follows that $\boldsymbol{y}(t)$ is a limit point of $\{\boldsymbol{y}_k(t)\}$, say $\boldsymbol{y}(t) = \lim_{m \to \infty} \boldsymbol{y}_{k_m}(t)$. Since

$$q_{k_m} \cdot \boldsymbol{y}_{k_m}(t) \leqq q_{k_m} \cdot \boldsymbol{i}_{k_m}(t) ,$$

we deduce by letting $m \to \infty$ that $q \cdot \boldsymbol{y}(t) \leqq q \cdot \boldsymbol{i}(t)$, and so for all t,

$$\boldsymbol{y}(t) \in \boldsymbol{B}_q(t) . \tag{6.4}$$

Finally, suppose that for t in a set of positive measure, there is a $z \in \boldsymbol{B}_q(t)$ such that $z \succ_t \boldsymbol{y}(t)$. Clearly $z \neq 0$; suppose without loss of generality that $z^1 > 0$. If for $\delta > 0$ sufficiently small we define $z_\delta = z - (\delta, 0, \ldots, 0)$, then we still have

$$z_\delta \succ \boldsymbol{y}(t) . \tag{6.5}$$

Moreover, since

$$\lim_k q_k \cdot z_\delta = q \cdot z - q^1 \delta < q \cdot z \leqq q \cdot \boldsymbol{i}(t) = \lim q_k \cdot \boldsymbol{i}_k(t) ,$$

it follows that

$$q_k \cdot z_\delta < q_k \cdot \boldsymbol{i}_k(t)$$

for all sufficiently large k, say for $k > k_0$. Now since $\boldsymbol{y}(t)$ is a limit point of $\{\boldsymbol{y}_k(t)\}$, there is a subsequence $\{\boldsymbol{y}_{k_m}(t)\}$ converging to $\boldsymbol{y}(t)$; hence for m sufficiently large,

$$z_\delta \succ_t \boldsymbol{y}_{k_m}(t) ,$$

by (6.5). If we also pick m so large so that $k_m \geqq k_0$, then z_δ contradicts the maximality of $\boldsymbol{y}_{k_m}(t)$ in the budget set $\{x: q_{k_m} \cdot x \leqq q_{k_m} \cdot \boldsymbol{i}_{k_m}(t)\}$. Thus the supposition $z \succ_t \boldsymbol{y}(t)$ has led to a contradiction, and we conclude that $\boldsymbol{y}(t)$ is maximal in $\boldsymbol{B}_q(t)$ for all t. Together with (6.3) and (6.4), this completes the proof that $(q, \boldsymbol{y})$ is a competitive equilibrium, and with it the proof of the main theorem.

7. COMPARISON WITH MCKENZIE'S PROOF

The differences between this proof and McKenzie's are caused by the different initial equipment: we have no convexity assumption to work with, and we have a continuum of traders rather than a finite number.

McKenzie needs the convexity assumption in only one place, to show that the aggregate preferred set (in his case the sum of the individual preferred sets) is convex. This is needed to define $h(p)$ uniquely, and follows from the convexity of the individual preferred sets. In a finite model there is no getting around this: no intuitive assumption other than individual convexity would lead to the convexity of the aggregate preferred set.

In a continuous model, however, this is superfluous, because of Lemma 5.1; this says that the integral of any set-valued function over a non-atomic measure space (in our case the unit interval) is convex, even if the individual values of the function are not convex. In particular, the aggregate preferred set, as the integral of the (possibly nonconvex) individual preferred sets, is convex.

Because of the presence of a continuum of traders, the space of assignments is no longer a subset of a finite-dimensional euclidean space, but of an infinite-dimensional function space. This necessitates the use of completely new methods to justify the passage from properties proved for individual traders to the corresponding properties for the aggregate of all traders. Consider, for example, the continuity of the aggregate preferred set as a function of the price vector. In the finite case, this follows trivially from the continuity of the individual preferred sets. Here, on the other hand, it involves Lemma 5.4, which is comparatively deep. In fact, Lemmas 5.1-5.4, which have been separately published, were originally proved for the purposes of this paper, and they embody the chief mathematical difficulties. The proofs of these lemmas involve Lyapunov's theorem on the range of a vector measure [**8**], and the methods of functional analysis (Banach spaces) and topology.

Another significant difference between this proof and McKenzie's is in the matter of boundedness. In the proof of the auxiliary theorem, the set of bundles under consideration must be in some sense bounded in order to establish the continuity —and indeed the existence—of the individual preferred sets. McKenzie does this by noting that no individual trader can have more goods than the whole market. This is not available here, because no matter how large an individual trader's bundle is, it is still infinitesimal compared with the whole market. We therefore used the notion of commodity-wise saturation, which does the job of bounding for us. In the passage from the auxiliary to the main theorem we do not have commodity-wise saturation, but need boundedness so that the sequence of competitive equilibria of the auxiliary markets $\mathscr{M}_k$ should converge. Here we first deduce from the desirability assumption (2.2) that all prices must be nonvanishing, and this bounds the bundles under consideration to a finite simplex.

8. THE CORE

Intimately connected with the concept of competitive equilibrium is that of *core*. This is the set of all allocations with the property that no "coalition" of traders can assure each of its members of a more desirable bundle by trading within itself only, without recourse to traders not in the coalition. Formally (in our model), an allocation $\boldsymbol{x}$ is in the core if there is no measurable nonnull set S of traders, for whom there is an allocation $\boldsymbol{y}$ such that $\boldsymbol{y}(t) \succ_t \boldsymbol{x}(t)$ for all $t \in S$ and $\int_S \boldsymbol{y} = \int_S \boldsymbol{i}$. In a finite market, the integral should be replaced by a sum.

In a finite market with convex preferences, the core is never empty; but when the preferences are not convex the core may be empty. As with the competitive equilibrium, it might be conjectured that this "pathology" would "tend to disappear" as the number of traders increases. Investigating this possibility, Shapley and Shubik **[10]** showed that though the core itself may remain empty for any (finite) number of traders, it is possible to define a kind of approximation to the core called an ε-core; and that for any positive ε, if the number n of traders is allowed to increase in a certain way, the ε-core will become non-empty for sufficiently large n. They concluded that, heuristically speaking, the true core lies "just below the surface" for sufficiently large n. The assumptions on which their theorem is based are comparatively strong: They assumed transferable utilities, that all traders have the same utility function, and that there is a fixed finite number of distinct types of traders (where two traders are of the same "type" if they have the same initial bundles).

We shall now describe how the concepts of core and competitive equilibrium are related. Let us define an *equilibrium allocation* to be an allocation that forms a competitive equilibrium when paired with an appropriate price vector. For finite markets, the core always contains the set of equilibrium allocations, but the two sets do not usually coincide. A long-standing conjecture states, however, that as the number of players in a market increases, the core of the market "tends," in some sense, to the set of equilibrium allocations. Recently this conjecture has been formalized and proved in a number of different ways.[11] In **[2]** we showed that for a market with a *continuum* of traders, the core actually *equals* the set of equilibrium allocations. This was shown under conditions that are even weaker than those of

[11] See **[2]** for a brief survey of these developments.

this paper.[12] A question that was left open was the *existence* of a competitive equilibrium, or equivalently, the non-emptiness of the core; though it had been shown that the two sets coincide, the possibility that both vanish was left open. From the theorem of this paper, it now follows that the core is non-empty as well. This agrees well with the Shapley-Shubik result (which was, however, obtained under considerably stronger assumptions): Since the ε-core is non-empty for large n, it is to be expected that the true core is non-empty for "infinite n."

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[12] The model of [2] differs from that of this paper in that there we started out directly with preference relations $\succ_t$ rather than deriving them from preference-or-indifference relations $\succsim_t$; furthermore, unlike here, we there made no assumptions of total or even partial order for the preference relations (for example, transitivity was not assumed). Otherwise, the two models are identical.