Your Name:

SUGGESTED ANSWERS

Please answer all questions. Each of the six questions marked with a big number counts equally. Designate your answers clearly.

Short correct answers are sufficient and get full credit. Including irrelevant (though correct) information in an answer will not increase the score.

All notation not otherwise defined is taken from Starr's *General Equilibrium Theory: An Introduction (2nd edition).*

If you need additional space to answer a question, write "(over)" at the end of text on the first page and continue on the back of the page. If you still need additional space, use an additional sheet of paper, making sure that it has your name on it and is attached to your exam when submitted.

This examination is open book, open notes (hard copy only). Calculators, cell phones, computers, iPads, etc., advice of classmates, are not allowed.

1 Where's the Competitive Equilibrium?

Consider a two-household, two-commodity pure exchange economy — an Edgeworth Box. Let there be two goods, x and y, so that a household's consumption may be represented by (x, y). Both households have the same preferences, $\succ_i$ of the following form.

$\succ_i$ is read "is strictly preferred to;" $\sim_i$ is read " is indifferent to."

$$(x, y) \succ_i (x', y') \text{ if } 3x + y > 3x' + y'; \text{ or}$$

$$(x, y) \succ_i (x', y') \text{ if } 3x + y = 3x' + y' \text{ and } x > x'.$$

$$(x, y) \sim_i (x', y') \text{ if } (x, y) = (x', y').$$

That is, a bundle (x, y) is evaluated by the value of the expression $3x + y$ except when two bundles have the same value of $3x + y$. Then the tie breaker is which one has more x.

	Household 1	Household 2
Preferences	$\succ_\mathbf{i}$, $\sim_\mathbf{i}$	$\succ_\mathbf{i}$, $\sim_\mathbf{i}$
Endowment	$r^1 = (50, 50)$	$r^2 = (50, 50)$

Is there a competitive equilibrium in this Edgeworth Box? Find the equilibrium allocation.

Circle the letter(s) of the correct answer(s) and explain your reasoning. Draw an illustration if that will help.

A. There is apparently no equilibrium price vector, despite fulfillment of all of the usual assumptions C.I - C.V, C.VI(C), C.VII and (trivially) P.I - P.IV. Those assumptions are not quite strong enough to assure existence of general equilibrium price vector.

B. The equilibrium price vector on the unit simplex is $(p_x, p_y) = (\frac{3}{4}, \frac{1}{4})$. Everyone likes x three times as much as y. The initial endowment is the equilibrium allocation.

C. Since x is slightly preferred to y, it carries a small premium price in equilibrium. The equilibrium price vector on the unit simplex is $(p_x, p_y) = (\frac{3}{4} + \varepsilon, \frac{1}{4} - \varepsilon)$ for $\varepsilon \to 0$.

D. There is no equilibrium price vector. Assumption C.V, Continuity of Preferences, is not fulfilled.

E. None of the above answers is correct.

Enter your explanation for Question 1 on the next page.

Explanation for Question 1

The correct answer is option **D**. The preferences do not fulfill the continutiy assumption, C.V. Formally this shows up as the upper and lower contour sets not being closed. The elementary demonstration is to consider a sequence of possible consumption plans $(x^{\nu}, y^{\nu}) \to (x^o, y^o)$ so that $x^{\nu} + y^{\nu} > x^o + y^o = x' + y'$ but $x^o < x'$. Then the sequence (x^{ν}, y^{ν}) goes from strictly preferred to (x', y') to inferior to (x', y') without going through indifference.

The obvious candidate equilibrium price vector is $(\frac{3}{4}, \frac{1}{4})$. But at this price vector, prevailing demand is $(\frac{50\times 4}{3}, 0)$ for both households, generating an excess demand for good x. There is no market clearing price vector.

2 Fundamental Theorems of Welfare Economics

Recall **Theorem 19.1: First Fundamental Theorem of Welfare Economics** For each $i \in H$, assume C.II (unbounded possible consumption set), C.IV (non-satiation), and C.VI(C) (convexity)

or assume C.IV* (weak montonicity) and $X^i = \mathbf{R}^N_+$.

Let $p^\circ \in \mathbf{R}^N_+$ be a competitive equilibrium price vector of the economy. Let $w^{\circ i} \in X^i$, $i \in H$, be the associated individual consumption bundles, and let $y^{\circ j}$, $j \in F$, be the associated firm supply vectors. Then $w^{\circ i}$ is Pareto efficient.

Consider a two-person pure exchange economy (Edgeworth Box) made up of the following two households. The notation "min$[xy, 16]$" means the minimum of xy and 16. Superscripts denote the household name — nothing in this problem is raised to a power.

	Household 1	Household 2
Endowment	$r^1 = (1, 9)$	$r^2 = (9, 1)$
Utility Function	$u^1(x, y) = xy$	$u^2(x, y) = \min[xy, 16]$

Adopt the notation: (x^1, y^1) is household 1's consumption plan of x and y; (x^2, y^2) is household 2's consumption plan of x and y.

Household 2's maximum possible utility is 16; whenever household 2's holdings of x and y fulfill $xy > 16$, household 2's utility remains at $u^2(x, y) = 16$. Set $p = (\frac{1}{2}, \frac{1}{2})$. At this price vector $(x^1, y^1) = (5, 5)$, $(x^2, y^2) = (5, 5)$ are household utility maximizing demand vectors subject to budget constraint.

Evaluate this example in terms of the First Fundamental Theorem of Welfare Economics (1FTWE).

Circle the letter(s) of the correct answer(s) and explain your reasoning.

A. The proposed prices and allocation do not represent a competitive equilibrium. 1FTWE does not apply. It is not possible to determine whether the allocation is Pareto efficient.

B. The allocation, $(x^1, y^1) = (5, 5)$, $(x^2, y^2) = (5, 5)$ is a competitive equilibrium. Markets clear. 1FTWE applies and the allocation is Pareto efficient.

C. The allocation, $(x^1, y^1) = (5, 5)$, $(x^2, y^2) = (5, 5)$, with $p = (\frac{1}{2}, \frac{1}{2})$ is a competitive equilibrium but it is not Pareto efficient. 1FTWE requires non-satiation, so it cannot be correctly applied in this example.

D. Nonconvexity of preferences means that 1FTWE is not applicable.

E. None of the above proposed answers is sound.

Explain your answer on the next page

Explanation for answer to question 2:

The correct answer is **C**. Assumptions of 1FTWE are not fulfilled. Household 2 is satiated at $xy = 16$, but its equilibrium consumption (not unique, but among the equilbria) is $(5, 5)$ where it would be equally satisfied with $(4, 4)$. The result is a waste of $(1, 1)$ which could be allocatd to Household 1, generating a Pareto improvement.

3 Existence of General Equilibrium

Theorem 5.2 is a statement and proof of existence of general equilibrium prices, p^* so that $Z(p^*) \leq 0$ in an economy with a point-valued continuous excess demand function. The theorem was proved in class and appears in *General Equilibrium Theory: An Introduction* chapter 5. Theorem 5.2 makes two principal assumptions on the excess demand function $Z : P \to R^N$:

Walras's Law: For all $p \in P$,

$$p \cdot Z(p) = \sum_{n=1}^{N} p_n \cdot Z_n(p) = \sum_{i \in H} p \cdot D^i(p) - \sum_{j \in F} p \cdot S^j(p) - p \cdot r = 0.$$

Continuity:

$$Z : P \to \mathbf{R}^N, Z(p) \text{ is a continuous function for all } p \in P.$$

Recall that the proof is based on the price adjustment function $T : P \to P$ where the i^{th} co-ordinate of T, is defined as

$$T_i(p) \equiv \frac{\max[0, p_i + \gamma^i Z_i(p)]}{\sum_{n=1}^{N} \max[0, p_n + \gamma^n Z_n(p)]}. \tag{1}$$

Suppose for some $p^\circ \in P$, $Z_i(p^\circ) = \frac{p_i^\circ}{\gamma^i} \geq 0$ for all i = 1, ..., N, and of course $Z_i(p^\circ) = \frac{p_i^\circ}{\gamma^i} > 0$ for some i. Then p° is clearly not market clearing but

$$T_i(p^\circ) = \frac{\max[0, p_i^\circ + \gamma^i \frac{p_i^\circ}{\gamma^i}]}{\sum_{n=1}^{N} \max[0, p_n^\circ + \gamma^n \frac{p_n^\circ}{\gamma^n}]} = \frac{\max[0, 2p_i^\circ]}{\sum_{n=1}^{N} \max[0, 2p_n^\circ]} = \frac{2p_i^\circ}{2} = p_i^\circ.$$

Thus p° is a fixed point of T but markets do not clear, apparently violating Theorem 5.2.

Evaluate this example.
Circle the response(s) below that best replies(y), then explain.

A. $Z(p)$ is set-valued so the point-valued price adjustment function $T(p)$ cannot successfully deal with $Z(p)$. This problem is resolved in Theorem 24.7 using a set-valued price adjustment correspondence, $\rho(z)$.

B. $Z(p)$ of the form described will not occur because it violates Walras's Law.

C. $Z(p)$ of the form described reflects external effects — non-market interactions —- between firms, inconsistent with the underlying model.

D. $Z(p)$ of the form described is discontinuous in p, so it is inconsistent with the assumptions of the theorem.

E. None of the above answers is sound.

Enter your explanation for Question 3 on the next page.

Explanation for Question 3:

The correct answer is **B**. $p^{\circ} \cdot Z(p^{\circ}) = \sum_{i=1,2,\ldots N} p_i^{\circ} \frac{p_i^{\circ}}{\gamma^i} = \sum_{i=1,2,\ldots N} \frac{1}{\gamma^i}[p_i^{\circ 2}] > 0$. The value at prices p° of excess demand $Z(p^{\circ})$ is strictly positive. This is a violation of Walras's Law. It means that all goods are in excess demand at once, so that there is an underlying violation of budget constraints.

4 Futures Markets

Consider an Arrow-Debreu general equilibrium model over time **without** uncertainty. There is a full set of futures markets covering allocation over time at dates t, $t = 1, 2, \ldots, T$. Prior to date 1 market equilibrium prices are established at the market date for goods deliverable at all dates. Firm j' plans to purchase inputs at dates $4, 5, 6$, to produce outputs at dates $7, 8, 9, 10$.

Describe firm j''s financial plan in the Arrow-Debreu model.
Circle the letter(s) of the correct answer(s) and explain your reasoning.

A. At the market date, prior to date 1, firm j' sells stock to households and uses the value of the stock sales to buy the inputs for delivery at $4, 5, 6$.

B. At the market date, prior to date 1, firm j' sells its planned output deliverable at dates $7, 8, 9, 10$ and purchases its planned inputs for delivery at $4, 5, 6$. Profits are paid to shareholders at the market date.

C. At dates $1, 2, 3$, firm j' sells shares and arranges loans. It uses the value of loans and share sales to pay for the purchase of inputs it purchases at dates $4, 5, 6$. Once it earns profits at dates $7, 8, 9, 10$, it repays the loans and distributes profits to shareholders.

D. Firm j' takes no action until date 4. It then evaluates the profitability of production based on its price expectations for periods $7, 8, 9, 10$. If production appears to be profitable, it purchases inputs on credit, promising to repay the sellers when the output is sold in periods $7, 8, 9, 10$.

E. None of the above responses is sound.

Explain: The correct answer is **B**. In the Arrow-Debreu model of futures contracts there is only one meeting of the market. All transactions take place then. The rest of economic activity consists in fulfilling the contracted plans.

5 Core Convergence

Consider a Debreu-Scarf pure exchange core model where the economy becomes large through replication. The original population of the economy is denoted H. Q (a positive integer) denotes the number of replications. Assume the equal treatment property so that core allocations can be described by household type ($i \in H$). Let $Core(Q - H)$ denote the set of allocations, by household type, in the core of the the Q-fold replica economy. Consider the core allocations for the original economy H (equivalently, $1 - H$), and the double replica $2 - H$.

We claim $Core(2 - H) \subseteq Core(H)$. Why is this inclusion valid?

Circle the letter(s) of the correct answer(s) and explain your reasoning.

A. The observation $Core(2 - H) \subseteq Core(H)$ is trivially true, since it includes the possibility $Core(2 - H) = Core(H)$. Replication means that the economy $2 - H$ is virtually identical — except for size — to the economy H. Competitive equilibria and the cores of the two economies are identical.

B. The core is the set of unblocked allocations. As the size of the economy increases, the number and variety of blocking coalitions increases. The set of remaining unblocked allocations is (weakly) reduced.

C. The Debreu-Scarf model uses strict convexity of preferences, C.VI(SC). Convexity is equivalent to diminishing marginal rate of substitution. That means that in a growing economy there is a shrinking number and variety of core allocations.

D. Growth of the economy through replication uses non-satiation, C.IV, or weak monotonicity (C.IV*), along with convexity of preferences — concavity of the utility function. Therefore (weakly) diminishing marginal utility leads to a shrinking core.

E. None of the responses above is sound.

Explain: The correct answer is **B**. Every blocking coalition in $1-H$ is available in $2-H$. In $2 - H$ there are additional coalitions available that were not present in $1 - H$, in type proportions that were not available in $1 - H$. Hence the array of blocking coalitions expands — or does not contract. For example, for two household types, 1 and 2, with $Q = 1$ the coalition of one of each can form. At $Q = 2$, a coalition of two of type 1 and one of type 2 can form — a coalition that was not available at $Q = 1$. Thus the core shrinks, or does not expand, as Q increases.

6 The Uzawa Equivalence Theorem — No more multiple choice!

Let P be the unit simplex in R^N and let f be a function. Let $f : P \rightarrow P$, f continuous. The following **Walrasian Existence of Equilibrium Proposition (WEEP)** was proved in class and is proved in *General Equilibrium Theory: An Introduction*, chapter 5. For simplicity, the statement ignores free goods in excess supply.

WEEP: Let $Z{:}P \rightarrow \mathbf{R}^N$ so that $Z(p)$ is continuous for all $p \in P$ and $p \cdot Z(p) = 0$ (Walras's Law) for all $p \in P$. Then there is $p^* \in P$ so that $Z(p^*) = 0$ (the zero vector).

Define $Z(p) = f(p) - \left[\frac{p \cdot f(p)}{p \cdot p}\right] p$. The term in square brackets is just a scalar multiplying the vector p.

1. Show that Z fulfills Walras's Law, that $p \cdot Z(p) = 0$.

$$p \cdot Z(p) = p \cdot [f(p) - \left[\frac{p \cdot f(p)}{p \cdot p}\right] p] = p \cdot f(p) - \left[\frac{p \cdot f(p)}{p \cdot p}\right] p \cdot p = p \cdot f(p) - p \cdot f(p) = 0$$

.

2. Treat Z as an excess demand function. Apply WEEP. Then there is a competitive equilibrium price vector p^* so that $Z(p^*) = 0$ (the zero vector). This leads to $Z(p^*) = f(p^*) - \left[\frac{p^* \cdot f(p^*)}{p^* \cdot p^*}\right] p^* = 0$, and $f(p^*) = \left[\frac{p^* \cdot f(p^*)}{p^* \cdot p^*}\right] p^*$. But p^* and $f(p^*)$ both $\in P$, so $\left[\frac{p^* \cdot f(p^*)}{p^* \cdot p^*}\right] = 1$. **Does it follow that** $f(p^*) = p^*$? **Explain, briefly.**

Yes, it is obvious (takes a while). We have $f(p^*) = \left[\frac{p^* \cdot f(p^*)}{p^* \cdot p^*}\right] p^*$. and $\left[\frac{p^* \cdot f(p^*)}{p^* \cdot p^*}\right] = 1$. Then substituting in, $f(p^*) = 1 \times p^* = p^*$.

3. If you assume WEEP, does the the Brouwer Fixed Point Theorem follow as a consequence? **Explain.**

Yes, that's what we've just proved. Assume WEEP. Then formulate a cleverly chosen $Z(p)$. There's a bit of mathematician's cleverness here. This $Z(p)$ is not derived from underlying economic conditions, we just get to assume it. In this case, define $Z(p) \equiv f(p) - \left[\frac{p \cdot f(p)}{p \cdot p}\right] p$. Then in step 1 you showed that $Z(p)$ fulfills the Walras's Law, and of course it is trivially continuous. Then applying WEEP gets you $Z(p^*) = 0$. In step 2, you showed that p^* really is a fixed point. You've just proved the Brouwer fixed point theorem. Hence WEEP and BFPT are mathematically equivalent.