Supplement to Lecture Notes for February 3, 2015, pages 3 and 4 (updated February 1, 2015)

Notation: Let b be a positive scalar, and let the notation $[b]$ represent the smallest integer greater than or equal to the argument b .

We wish to show that $0 \notin \Gamma$ where 0 is the zero vector in R^N. We will show that the possibility that $0 \in \Gamma$ corresponds to the possibility of forming a blocking coalition against the core allocation $\{x^{oi} | i \in H\}$, a contradiction. The typical element of Γ can be represented as $\sum_{i \in H} a_i z^i$, where $z^i \in \Gamma^i$, and where $r^i + z^i \succ_i x^{\circ i}$, $0 \leq a_i \leq 1, \sum_{i \in H} a_i = 1$. We use a proof by contradiction, showing that the possibility $0 \in \Gamma$ leads to the recognition that there is a coalition in Q-H, for Q sufficiently large, that blocks $\{x^{oi} | i \in H\}$. If that were to occur, $\{x^{oi} | i \in H\}$ would not be a core allocation. Suppose, on the contrary, that that $0 \in \Gamma$. Then there are $0 \leq a_i \leq 1, \sum_{i \in H} a_i = 1$ and $z^i \in \Gamma^i$ so that

$$\sum_{i \in H} a_i z^i = 0.$$

That is, for those $i \in H$ so that $a_i > 0$ there are $x'^i \in X^i$ so that $z^i = x'^i - r^i$ where $x'^i = (r^i + z^i) \succ x^{\circ i}$ and $0 = \sum a_i z^i = \sum a_i (x'^i - r^i)$. We'll focus on the households of type i where $a_i > 0$, and on their preferred net trades z^i. Consider the k-fold replication of H and the hypothetical net trade for a household of type i, $\frac{k a_i}{[k a_i]} z^i$. We have $\frac{k a_i}{[k a_i]} z^i \to z^i$ as $k \to \infty$. Therefore, by (C.V, continuity) for k sufficiently large (and all larger values of k),

$$[r^i + \frac{k a_i}{[k a_i]} z^i] \succ_i x^{oi} \qquad (\dagger)$$

Further,

$$\sum_{i\in H}[ka_i]\frac{ka_i}{[ka_i]}z^i = k\sum_{i\in H}a_i z^i = 0 \qquad (\ddagger).$$

It is now time to form a blocking coalition. We confine attention to those $i \in H$ so that $a_i > 0$. The blocking coalition is formed by $[\hat{k}a_i]$ households of type i where $\hat{k}$ is the smallest integer so that (†) is fulfilled for all $i \in H$ where $a_i > 0$. Consider Q larger or equal to $\hat{k}$. Form the coalition S consisting of $[\hat{k}a_i]$ households of type i for all i so that $a_i > 0$. The intial resource endowment of this coalition (summing over S) is $\sum[\hat{k}a_i]r^i$. We need to find an allocation to S, attainable for S, that blocks the core allocation $\{x^{oi}|i \in H\}$. The blocking allocation to each household of type i is $r^i + \frac{\hat{k}a_i}{[\hat{k}a_i]}z^i$. Summing this allocation over S, we have

$$\sum[\hat{k}a_i]\{r^i + \frac{\hat{k}a_i}{[\hat{k}a_i]}z^i\} = \sum[\hat{k}a_i]r^i + \sum[\hat{k}a_i]\frac{\hat{k}a_i}{[\hat{k}a_i]}z^i = \sum[\hat{k}a_i]r^i.$$

The last equality follows from expression (‡), and it demonstrates that the allocation is attainable to S. The allocation $r^i + \frac{\hat{k}a_i}{[\hat{k}a_i]}z^i$ is attainable to the coalition S and it is preferable to all members of the coalition by (†). Thus S blocks x^{oi} if $0 \in \Gamma$. The contradiction implies $0 \notin \Gamma$.