Lecture Notes for February 3, 2015: Convergence of the core of a large economy

22.1 Replication; a large economy

We will treat a Q-fold replica economy, denoted Q-H. Q will be a positive integer; $Q = 1, 2, \ldots$. In a Q-fold replica economy we take an economy consisting of households $i \in H$, with endowments r^i and preferences $\succeq_i$, and create a similar larger economy with Q times as many agents in it, totaling $\#H \times Q$ agents. There will be Q agents with preferences $\succeq_1$ and endowment r^1, Q agents with preferences $\succeq_2$ and endowment $r^2, \ldots,$ and Q agents with preferences $\succeq_{\#H}$ and endowment $r^{\#H}$. Each household $i{\in}H$ now corresponds to a household type. There are Q individual households of type i in the replica economy Q-H. Note that the competitive equilibrium prices in the original H economy will be equilibrium prices of the Q-H economy. Household i's competitive equilibrium allocation x^i in the original H economy will be a competitive equilibrium allocation to all type i households in the Q-H replica economy. Agents in the Q-H replica economy will be denoted by their type and a serial number. Thus, the agent denoted i, q will be the qth agent of type i, for each $i \in H, q = 1, 2, \ldots, Q$.

22.2 Equal treatment

Theorem 22.1 (Equal treatment in the core) Assume C.IV*, C.V, and C.VI(SC). Let $\{x^{i,q}, i \in H, q = 1, \ldots, Q\}$ be in the core of Q-H, the Q-fold replica of economy H. Then for each $i, x^{i,q}$ is the same for all q. That is, $x^{i,q} = x^{i,q'}$ for each $i \in H, q \neq q'$.

Theorem 8.1, Bounding Hyperplane Theorem (Minkowski) Let K be convex, $K \subseteq \mathbf{R}^N$. There is a hyperplane H through z and bounding for K if z is not interior to K. That is, there is $p \in \mathbf{R}^N, p \neq 0$, so that for each $x \in K, p \cdot x \geq p \cdot z$.

Theorem 22.2 (Debreu-Scarf) Assume C.IV*, C.V, C.VI(SC). Let $X^i = \mathbf{R}^N_+$ and $r^i >> 0$ for all $i \in H$. Let $\{x^{\circ i}, i \in H\} \in \text{core}(Q\text{-}H)$ for all $Q = 1, 2, 3, 4, \ldots$. Then $\{x^{\circ i}, i \in H\}$ is a competitive equilibrium allocation for Q-H, for all Q.

Proof We must show that there is a price vector p so that for each household type $i, p \cdot x^{\circ i} \leq p \cdot r^i$ and that $x^{\circ i}$ optimizes preferences $\succeq_i$ subject to this budget. The strategy of proof is to create a set of net trades preferred to those that achieve $\{x^{\circ i}, i \in H\}$. We will show that it is a convex set with a supporting hyperplane through the origin. The normal to the supporting hyperplane will be designated p. We will then argue that p is a competitive equilibrium price vector supporting $\{x^{\circ i}, i \in H\}$.

For each $i \in H$, let $\Gamma^i = \{z \mid z \in \mathbf{R}^N, z + r^i \succ_i x^{oi}\}$. What is this set of vectors Γ^i? Γ^i is defined as the set of net trades from endowment r^i so that an agent of type i strictly prefers these net trades to the trade $x^{oi} - r^i$, the trade that gives him the core allocation. We now define the convex hull (set of convex combinations) of the family of sets $\Gamma^i, i \in H$. Let $\Gamma = \{\sum_{i \in H} a_i z^i \mid z^i \in \Gamma^i$, $a_i \geq 0, \sum a_i = 1\}$, the set of convex combinations of preferred net trades. The set Γ is the convex hull of the union of the sets Γ^i. (See Figure 22.1.) Note that $(x^{\circ i} - r^i) \in \text{boundary}(\Gamma^i), (x^{\circ i} - r^i) \in \overline{\Gamma}^i$, and $(x^{\circ i} - r^i) \in \overline{\Gamma}$ for all i.

The strategy of proof now is to show that Γ and the constituent sets Γ^i are arrayed strictly above a hyperplane through the origin. The normal to the hyperplane will be the proposed equilibrium price vector.

We wish to show that $0 \notin \Gamma$. We will show that the possibility that $0 \in \Gamma$ corresponds to the possibility of forming a blocking coalition against the core allocation x^{oi}, a contradiction. The typical element of Γ can be represented as $\sum a_i z^i$, where $z^i \in \Gamma^i$. Suppose that $0 \in \Gamma$. Then there are $0 \leq a_i \leq 1, \sum_{i \in H} a_i = 1$ and $z^i \in \Gamma^i$ so that $\sum_{i \in H} a_i z^i = 0$. We'll focus on these values of a_i, z^i, and consider the k-fold replication of H, eventually letting k become arbitrarily large. Let the notation $[\cdot]$ represent the smallest integer greater than or equal to the argument $\cdot$. Consider the hypothetical net trade for a household of type i, $\frac{ka_i}{[ka_i]} z^i$. We have $\frac{ka_i}{[ka_i]} z^i \to z^i$ as $k \to \infty$. Therefore, by (C.V, continuity) for k sufficiently large,

$$[r^i + \frac{ka_i}{[ka_i]} z^i] \succ_i x^{oi} \qquad (\dagger)$$

Further,

$$\sum_{i \in H} [ka_i] \frac{ka_i}{[ka_i]} z^i = k \sum_{i \in H} a_i z^i = 0 \qquad (\ddagger).$$

It is now time to form a blocking coalition. We confine attention to those $i \in H$ so that $a_i > 0$. The blocking coalition is formed by $[\hat{k} a_i]$ households of type i where $\hat{k}$ is the smallest integer so that (†) is fulfilled for all $i \in H$ for $a_i > 0$. That is, let $\hat{k} \equiv \inf\{k \in \mathcal{N} | (\dagger) \text{ is fulfilled for all } i \in H \text{ such that } a_i > 0\}$ where $\mathcal{N}$ is the set of positive integers. Consider Q larger than $\hat{k}$. Form the coalition S consisting of $[\hat{k} a_i]$ households of type i for all i so that $a_i > 0$. The blocking allocation to each household of type i

is $r^i + \frac{ka_i}{[ka_i]} z^i$. This allocation is attainable to the coalition by (‡) and it is preferable to the coalition by (†). This is how replication with large Q overcomes the indivisibility of the individual agents. Thus S blocks x^{oi}, which is a contradiction. Hence, as claimed, $0 \notin \Gamma$.

Having established that 0 is not an element of Γ, we should recognize that 0 is nevertheless very close to Γ. Indeed $0 \in$ boundary of Γ. This occurs inasmuch as $0 = (1/\#H) \sum_{i\in H}(x^{\circ i} - r^i)$, and the right-hand side of this expression is an element of $\overline{\Gamma}$, the closure of Γ. Thus 0 represents just the sort of boundary point through which a supporting hyperplane may go in the Bounding Hyperplane Theorem. The set Γ is trivially convex. Hence we can invoke the Bounding Hyperplane Theorem. There is $p \in \mathbf{R}^N, p \neq 0$, so that for all $v \in \Gamma$, $p \cdot v \geq p \cdot 0 = 0$. Noting $X^i = \mathbf{R}^N_+$, C.IV* and C.VI(SC) , we know that $p \geq 0$. Now $(x^{\circ i} - r^i) \in \overline{\Gamma}$ for each i, so $p \cdot (x^{\circ i} - r^i) \geq 0$. But $\sum_{i\in H}(x^{\circ i} - r^i) = 0$, so $p \cdot \sum_{i\in H}(x^{\circ i} - r^i) = 0$. Hence $p \cdot (x^{\circ i} - r^i) = 0$ each i. Equivalently, $p \cdot x^{\circ i} = p \cdot r^i$. This gives us

$$0 = p \cdot \sum_{i\in H} \frac{1}{\#H}(x^{\circ i} - r^i) = \inf_{x\in\Gamma} p \cdot x = \sum_{i\in H} \frac{1}{\#H}\Big[\inf_{z^i\in\Gamma^i} p \cdot z^i\Big],$$

so

$$p \cdot (x^{\circ i} - r^i) = \inf_{z^i\in\Gamma^i} p \cdot z^i.$$

We have then for each i, that $p \cdot (x^{\circ i} - r^i) = \inf p \cdot y$ for $y \in \Gamma^i$. Equivalently, $x^{\circ i}$ minimizes $p \cdot (x - r^i)$ subject to $x \succeq_i x^{\circ i}$. In addition, $p \cdot x^{\circ i} = p \cdot r^i$. Further, by the specification of X^i and r^i, there is an ε-neighborhood of $x^{\circ i}$ contained in X^i. By C.IV*, C.V, and C.VI(SC) , and strict positivity of r^i, expenditure minimization subject to a utility constraint is equivalent to utility maximization subject to budget constraint. Hence $x^{\circ i}, i \in H$, is a competitive equilibrium allocation. QED