

## Lecture Notes, January 29, 2015

Bargaining and equilibrium: The core of a market economy

Set  $X^i = \mathbf{R}_+^N$ , all  $i$ .

Each  $i \in H$  has an endowment  $r^i \in \mathbf{R}_+^N$  and a preference quasi-ordering  $\succeq_i$  defined on  $\mathbf{R}_+^N$ .

An allocation is an assignment of  $x^i \in \mathbf{R}_+^N$  for each  $i \in H$ . A typical allocation,  $x^i \in \mathbf{R}_+^N$  for each  $i \in H$ , will be denoted  $\{x^i, i \in H\}$ . An allocation,  $\{x^i, i \in H\}$ , is feasible if  $\sum_{i \in H} x^i \leq \sum_{i \in H} r^i$ , where the inequality holds coordinatewise.

We assume preferences fulfill weak monotonicity (C.IV\*), continuity (C.V), and strict convexity (C.VI(SC)).

The core of a pure exchange economy

**Definition** A *coalition* is any subset  $S \subseteq H$ . Note that every individual comprises a (singleton) coalition.

**Definition** An allocation  $\{x^i, i \in H\}$  is *blocked* by  $S \subseteq H$  if there is a coalition  $S \subseteq H$  and an assignment  $\{y^i, i \in S\}$  so that:

- (i)  $\sum_{i \in S} y^i \leq \sum_{i \in S} r^i$  (where the inequality holds coordinatewise),
- (ii)  $y^i \succeq_i x^i$ , for all  $i \in S$ , and
- (iii)  $y^h \succ_h x^h$ , for some  $h \in S$

**Definition** The *core* of the economy is the set of feasible allocations that are not blocked by any coalition  $S \subseteq H$ .

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- Any allocation in the core must be individually rational. That is, if  $\{x^i, i \in H\}$  is a core allocation then we must have  $x^i \succeq_h r^i$ , for all  $i \in H$ .
  - Any allocation in the core must be Pareto efficient.
- (i) The competitive equilibrium is always in the core (Theorem 21.1).

Theorems 22.2 and 22.3 say that

- (ii) For a large economy, the set of competitive equilibria and the core are virtually identical. All core allocations are (nearly) competitive equilibria.

The competitive equilibrium allocation is in the core

Definition  $p \in \mathbf{R}_+^N$ ,  $p \neq 0$ ,  $x^i \in \mathbf{R}_+^N$ , for each  $i \in H$ , constitutes a competitive equilibrium if

- (i)  $p \cdot x^i \leq p \cdot r^i$ , for each  $i \in H$ ,
- (ii)  $x^i \succeq_i y$ , for all  $y \in R_+^N$ , such that  $p \cdot y \leq p \cdot r^i$ , and
- (iii)  $\sum_{i \in H} x^i \leq \sum_{i \in H} r^i$  (the inequality holds coordinatewise) with  $p_k = 0$  for any  $k = 1, 2, \dots, N$  so that the strict inequality holds.

Theorem 21.1 Let the economy fulfill C.II, C.IV\*, C.VI(SC) and let  $X^i = \mathbf{R}_+^N$ . Let  $p, x^i, i \in H$ , be a competitive equilibrium. Then  $\{x^i, i \in H\}$  is in the core of the economy.

Proof We will present a proof by contradiction. Suppose the theorem were false. Then there would be a blocking coalition

$S \subseteq H$  and a blocking assignment  $y^i, i \in S$ . We have

$$\begin{aligned} \sum_{i \in S} y^i &\leq \sum_{i \in S} r^i \text{ (attainability, the inequality holds coordinatewise)} \\ y^i &\succeq_i x^i, && \text{for all } i \in S, \text{ and} \\ y^h &\succ_h x^h, && \text{some } h \in S. \end{aligned}$$

But  $x^i$  is a competitive equilibrium allocation. That is, for all  $i \in H$ ,  $p \cdot x^i = p \cdot r^i$  (recalling Lemma 17.1), and  $x^i \succeq_i y$ , for all  $y \in R_+^N$  such that  $p \cdot y \leq p \cdot r^i$ .

Note that  $\sum_{i \in S} p \cdot x^i = \sum_{i \in S} p \cdot r^i$ . Then for all  $i \in S$ ,  $p \cdot y^i \geq p \cdot r^i$ . That is,  $x^i$  represents  $i$ 's most desirable consumption subject to budget constraint.  $y^i$  is at least as good under preferences  $\succeq_i$  fulfilling C.II, C.IV\*, C.VI(SC), (local non-satiation). Therefore,  $y^i$  must be at least as expensive. Furthermore, for  $h$ , we must have  $p \cdot y^h > p \cdot r^h$ . Therefore, we have

$$\sum_{i \in S} p \cdot y^i > \sum_{i \in S} p \cdot r^i.$$

Note that this is a strict inequality. However, for coalitional feasibility we must have

$$\sum_{i \in S} y^i \leq \sum_{i \in S} r^i.$$

But since  $p \geq 0$ ,  $p \neq 0$ , we have  $\sum_{i \in S} p \cdot y^i \leq \sum_{i \in S} p \cdot r^i$ . This is a contradiction. The allocation  $\{y^i, i \in S\}$  cannot simultaneously be smaller or equal to the sum of endowments  $r^i$  coordinatewise and be more expensive at prices  $p, p \geq 0$ . The contradiction proves the theorem. QED

### Convergence of the core of a large economy

#### Replication; a large economy

In replication, the economy keeps cloning itself.

duplicate to triplicate,  $\dots$ , to  $Q$ -tuplicate, and so on, the set of core allocations keeps getting smaller, although it always includes the set of competitive equilibria (per Theorem 21.1).

$Q$ -fold replica economy, denoted  $Q$ - $H$ .  $Q = 1, 2, \dots$

$\#H \times Q$  agents.

$Q$  agents with preferences  $\succeq_1$  and endowment  $r^1$ ,

$Q$  agents with preferences  $\succeq_2$  and endowment  $r^2, \dots$ , and  $Q$  agents with preferences  $\succeq_{\#H}$  and endowment  $r^{\#H}$ . Each household  $i \in H$  now corresponds to a household type. There are  $Q$  individual households of type  $i$  in the replica economy  $Q$ - $H$ .

Competitive equilibrium prices in the original  $H$  economy will be equilibrium prices of the  $Q$ - $H$  economy. Household  $i$ 's competitive equilibrium allocation  $x^i$  in the original  $H$  economy will be a competitive equilibrium allocation to all type  $i$  households in the  $Q$ - $H$  replica economy. Agents in the  $Q$ - $H$  replica economy will be denoted by their type and a serial number. Thus, the agent denoted  $i, q$  will be the  $q$ th agent of type  $i$ , for each  $i \in H, q = 1, 2, \dots, Q$ .

### Equal treatment

Theorem 22.1 (Equal treatment in the core) Assume C.IV, C.V, and C.VI(SC). Let  $\{x^{i,q}, i \in H, q = 1, \dots, Q\}$  be in the core of  $Q$ - $H$ , the  $Q$ -fold replica of economy  $H$ . Then for each  $i, x^{i,q}$  is the same for all  $q$ . That is,  $x^{i,q} = x^{i,q'}$  for each  $i \in H, q \neq q'$ .

Proof of Theorem 22.1 Recall that the core allocation must be feasible. That is,

$$\sum_{i \in H} \sum_{q=1}^Q x^{i,q} \leq \sum_{i \in H} \sum_{q=1}^Q r^i.$$

Equivalently,

$$\frac{1}{Q} \sum_{i \in H} \sum_{q=1}^Q x^{i,q} \leq \sum_{i \in H} r^i.$$

Suppose the theorem to be false. Consider a type  $i$  so that  $x^{i,q} \neq x^{i,q'}$ . For each type  $i$ , we can rank the consumptions attributed to type  $i$  according to  $\succeq_i$ .

For each  $i$ , let  $x^{i*}$  denote the least preferred of the core allocations to type  $i$ ,  $x^{i,q}$ ,  $q = 1, \dots, Q$ . For some types  $i$ , all individuals of the type will have the same consumption and  $x^{i*}$  will be this expression. For those in which the consumption differs,  $x^{i*}$  will be the least desirable of the consumptions of the type. We now form a coalition consisting of one member of each type: the individual from each type carrying the worst core allocation,  $x^{i*}$ .

Consider the average core allocation to type  $i$ , to be denoted  $\bar{x}^i$ .

$$\bar{x}^i = \frac{1}{Q} \sum_{q=1}^Q x^{i,q}.$$

We have, by strict convexity of preferences (C.VI(SC)),

$$\bar{x}^i = \frac{1}{Q} \sum_{q=1}^Q x^{i,q} \succ_i x^{i*} \text{ for those types } i \text{ so that } x^{i,q} \text{ are not identical,}$$

and

$$x^{i,q} = \bar{x}^i = \frac{1}{Q} \sum_{q=1}^Q x^{i,q} \sim_i x^{i*} \text{ for those types } i \text{ so that } x^{i,q} \text{ are identical.}$$

From feasibility, above, we have that

$$\sum_{i \in H} \bar{x}^i = \sum_{i \in H} \frac{1}{Q} \sum_{q=1}^Q x^{i,q} = \frac{1}{Q} \sum_{i \in H} \sum_{q=1}^Q x^{i,q} \leq \sum_{i \in H} r^i.$$

In other words, a coalition composed of one of each type (the worst off of each) can achieve the allocation  $\bar{x}^i$ . However, for each agent in the coalition,  $\bar{x}^i \succeq_i x^{i*}$  for all  $i$  and  $\bar{x}^i \succ_i x^{i*}$  for

some  $i$ . Therefore, the coalition of the worst off individual of each type blocks the allocation  $x^{i,q}$ . The contradiction proves the theorem. QED

$\text{Core}(Q) = \{x^i, i \in H\}$  where  $x^{i,q} = x^i, q = 1, 2, \dots, Q$ , and the allocation  $x^{i,q}$  is unblocked.

### Core convergence in a large economy

As  $Q$  grows there are more blocking coalitions, and they are more varied. Any coalition that blocks an allocation in  $Q$ - $H$  still blocks the allocation in  $(Q + 1)$ - $H$ , but there are new blocking coalitions and allocations newly blocked in  $(Q + 1)$ - $H$ .

Recall the Bounding Hyperplane Theorem:

Theorem 8.1, Bounding Hyperplane Theorem (Minkowski) Let  $K$  be convex,  $K \subseteq \mathbf{R}^N$ . There is a hyperplane  $H$  through  $z$  and bounding for  $K$  if  $z$  is not interior to  $K$ . That is, there is  $p \in \mathbf{R}^N, p \neq 0$ , so that for each  $x \in K, p \cdot x \geq p \cdot z$ .

Theorem 22.2 (Debreu-Scarf) Assume C.IV\*, C.V, C.VI(SC), and let  $X^i = \mathbf{R}_+^N$ . Let  $\{x^{oi}, i \in H\} \in \text{core}(Q)$  for all  $Q = 1, 2, 3, 4, \dots$ . Then  $\{x^{oi}, i \in H\}$  is a competitive equilibrium allocation for  $Q$ - $H$ , for all  $Q$ .

Proof TBA

QED