Lecture Notes for January 20 - Part 2
U-Shaped Cost Curves and Concentrated Preferences

25.1 U-Shaped Cost Curves and Concentrated Preferences

Using the Shapley-Folkman theorem we'll establish the existence of approximate equilibrium in cases of non-convex preferences (a preference for concentrated consumption) and U-shaped cost curves (small scale economies) in production. The approximation will depend on the dimension of the commodity space, N. Holding N fixed while the number of firms #F and households #H becomes large (as in a fully competitive model), will allow the approximate equilibrium to be arbitrarily close to a full equilibrium as a proportion of the size of the economy.

The strategy of proof is to consider a fictional mathematical construct of an economy where we replace the (possibly nonconvex) typical firm's production technology Y^j with it's convex hull, $con(Y^j)$. We replace the households', $i \in H$, nonconvex preference contour sets, $A^i(x)$, by their convex hulls, $con(A^i(x))$. This fictional construct will fulfill the model of Chapter 24. It will have a market-clearing general equilibrium price vector p*. The artificial convex-valued supply and demand correspondences are formed from the convex hulls of the true underlying non-convex-valued supply and demand correspondences. Then the Shapley-Folkman Theorem implies that the market-clearing plans of the fictional convex-valued supply and demand correspondences are within a small bounded distance of the the true economy's underlying nonconvex-valued supply and demand correspondences. That is, the non-convex-valued demand and supply correspondences at p^* are nearly market-clearing. Further, the bound depends on the size of non-convexities in the original economy's sets, L, and on the dimension of the space, N, not on the number of firms or households in the economy. Thus, in a large economy, where the number of households in H becomes large, the average disequilibrium per household becomes small. Thus, in the limit as the economy becomes large (the setting where we expect the economy to behave competitively), the approximation to market clearing can be as close as you wish.

25.1.1 U -shaped Cost Curves versus Natural Monopoly

Our economic intuition tells us that U-shaped cost curves — a small bounded scale economy — for the firms in an economy should be consistent with the existence of a competitive equilibrium. But unbounded scale economies — a natural monopoly — are inconsistent with competitive equilibrium. The intuition is correct. It shows up in the mathematics of the problem in the following way: $con(Y^j)$ will typically be closed for Y^j representing a firm with a U-shaped cost curve. For $Y^{j'}$ representing a natural monopoly $con(Y^{j'})$ will not be closed. Closedness of $con(Y^j)$ will be one of the assumptions of the convexified model, ruling out natural monopoly in the underlying non-convex economy.

8.2 The Shapley-Folkman Theorem

8.2.1 Nonconvex sets and their convex hulls

A typical nonconvex set contains a hole or indentation.

The convex hull of a set S will be the smallest convex set containing S. The convex hull of S will be denoted con(S). We can define con(S) , for $S \subset R^N$ as follows

$$\text{con(S)} \equiv \{x \mid x = \sum_{i=0}^{N} \alpha^i x^i, \text{where } x^i \in \text{S}, \alpha^i \geq 0 \text{ all i, and } \sum_{i=0}^{N} \alpha^i = 1\}.$$

or equivalently as

$$\text{con(S)} \equiv \bigcap_{S\subset\ T;T\ convex} T.$$

That is con(S) is the smallest convex set in R^N containing S.

Example Consider V^1 $=\{x \in R^2 | 3 \leq |x| \leq 10\}$. V^1 is a disk in R^2 with a hole in the center. The hole makes it nonconvex. Let V^2 $=\{x \in R^2| \ |x| \leq 10; x_1 \geq 0$ or $x_2 \geq 0\}$. V^2 is the disk of radius 10 centered at the origin with the lower left quadrant omitted. The indentation at the lower left makes V^2 nonconvex. con$(V^1) = \{x \in R^2 || x| \leq 10\}$, and con$(V^2) = \{x \in R^2 ||x| \leq 10$ for $x_1 \geq 0$ or $x_2 \geq 0$; for x_1, $x_2 \leq 0$, $x_1 + x_2 \geq -10\}$. Taking the convex hull of a set means filling in the holes just enough to make the amended set convex.

8.2.2 The Shapley-Folkman Lemma

Lemma 8.2 (Shapley-Folkman): Let S^1, S^2, S^3, ...,S^m, be nonempty compact subsets of R^N. Let $x \in con(S^1 + S^2 + S^3 + \ldots + S^m)$. Then for each i=1,2,...,m, there is $y^i \in con(S^i)$ so that $\sum_{i=1}^m y^i = x$ and with at most N exceptions, $y^i \in S^i$. Equivalently: Let F be a finite family of nonempty compact sets in R^N and let $y \in con(\sum_{S \in F} S)$. Then there is a partition of F into two disjoint subfamilies F' and F'' with the number of elements in $F' \leq N$ so that $y \in \Sigma_{S \in F'} con(S) + \Sigma_{S \in F''} S$.

8.2.3 Measuring Non-Convexity, The Shapley-Folkman Theorem

We now introduce a scalar measure of the size of a non-convexity.

Definition: The radius of a compact set S is defined as $rad(S) \equiv \inf_{x \in R^N} \sup_{y \in S} |x - y|$.

That is, $rad(S)$ is the radius of the smallest closed ball containing S.

Theorem 8.3 (Shapley - Folkman): Let F be a finite family of compact subsets $S \subset R^N$ and $L > 0$ so that $rad(S) \leq L$ for all $S \in F$. Then for any $x \in con(\Sigma_{S \in F} S)$ there is $y \in \Sigma_{S \in F} S$ so that $|x - y| \leq L\sqrt{N}$.

8.2.4 Corollary: A tighter bound

Definition: We define the inner radius of $S \subset R^N$ as
$r(S) \equiv \sup_{x \in con(S)} \inf_{T \subset S; x \in con(T)} rad(T)$

Corollary 8.1 Corollary to the Shapley-Folkman Theorem: Let F be a finite family of compact subsets $S \subset R^N$ and $L > 0$ so that $r(S) \leq L$ for all $S \in F$. Then for any $x \in con(\Sigma_{S \in F} S)$ there is $y \in \Sigma_{S \in F} S$ so that $|x - y| \leq L\sqrt{N}$.

25.2 The Non-Convex Economy

We start with a model of the economy with the same notation and same assumptions as in Chapter 24 with the omission of two assumptions: P.I and C.VI(C). Neither technology nor preferences are assumed to be convex.

25.2.1 Non-Convex Technology and Supply

Supply behavior of firms, $S^j(p)$, when it is well defined, may no longer be convex-valued. Since Y^j admits scale economies $S^j(p)$ may include many distinct points and not the line segments connecting them. A supply curve might look like figure 24.2. Alternatively, $S^j(p)$ might include 0 and a high level of output, but none of the values in between. This is, of course, the U-shaped cost curve case.

25.2.2 Non-Convex Preferences and Demand

Demand behavior of households, $D^i(p)$, when it is well defined, may no longer be convex-valued. Thus it is possible that $x, y \in D^i(p)$ but that $\alpha x + (1-\alpha)y \notin D^i(p)$ for $0 < \alpha < 1$.

25.2.3 Nonexistence of Market Equilibrium

The proof of Theorem 24.7, relying on the Kakutani Fixed-Point Theorem, requires convexity of $S^j(p)$ for all $j \in F$ and of $D^i(p)$ for all $i \in H$. Theorem 24.7 cannot be applied to the non-convex economy. We cannot rely on the existence of general competitive equilibrium. What can go wrong? Roughly, a demand curve (or correspondence) can run through the holes in a supply curve (or correspondence), resulting in no nonull intersection and no equilibrium prices.

25.3 Artificial Convex Counterpart to the Non-Convex Economy

We now form a convex counterpart to the non-convex economy. This artificial convex economy will be designed to fulfill the conditions of Chapter 24 and sustain competitive general equilibrium prices. We will then show, using the Shapley-Folkman Theorem, that the equilibrium price vector of the artificial convex economy supports an approximate equilibrium allocation of the original non-convex economy. The remaining disequilibrium (unsatisfied demand and supply at these prices) is independent of the size of the economy, as measured by the number of households, total output, or number of firms. Hence as a proportion of a large economy the remaining disequilibrium can be arbitrarily small.

25.3.1 Convexified Technology and Supply

Starting from the non-convex technology set Y^j, we merely substitute its convex hull, $con(Y^j)$, for each $j \in F$. Then substitute the convex hull of the aggregate technology set for the aggregate set Y, $con(Y) = con(\sum_{j\in F} Y^j) = \sum_{j\in F} con(Y^j)$. Then we assume the convexified counterpart to P.III (the notation K is intended as a nmemonic for "convex")

PK.III $con(Y^j)$ is closed for all $j \in F$.

The economic implication of PK.III is that scale economies are bounded — as in the U-shaped cost curve case; average costs are not indefinitely diminishing. Assumption PK.III rules out natural monopoly in the underlying non-convex economy. Thus, for example,

$Y^j = \{(x,y)|y \leq (-x)^2, x \leq 0\}$ would not fulfill PK.III but

$Y'^j = \{(x,y)|y \leq (-x)^2,$ for $-1 \leq x \leq 0, y \leq \sqrt{-x}$ for $x \leq -1\}$ would fulfill PK.III.

Now we introduce a counterpart to P.IV for the convexified economy.

PK.IV (a) if $y \in con(Y)$ and $y \neq 0,$ then $y_k < 0$ for some k.

(b) if $y \in con(Y)$ and $y \neq 0,$ then $-y \notin con(Y)$.

Then we consider a production sector characterized by firms with technologies $con(Y^j)$ for all $j \in F$. We assume P.II, PK.III, PK.IV. Since the technology of each firm j is $con(Y^j)$, P.I is trivially fulfilled. Then the production sector fulfills all of the assumptions of Theorem 24.7.

The artificially convex supply behavior of firm k then is

$S^{kj}(p) \equiv \{y^o \in con(Y^j)|p \cdot y^o \geq p \cdot y$ for all $y \in con(Y^j)\}$.

The artificially convex profit function of firm j is

$\pi^{kj}(p) \equiv p \cdot y^o$, where $y^o \in S^{kj}(p)$.

Under PK.III, a typical point of $S^{kj}(p)$ will be a point of $S^j(p)$ or a convex combination of points of $S^j(p)$.

Lemma 25.1 Assume P.II, PK.III, PK.IV and suppose $S^{kj}(p)$ is nonempty (exists and is well defined). Then $y^j \in S^{kj}(p)$ implies $y^j \in con(S^j(p))$ and $\pi^{kj}(p) = \pi^j(p)$.

Proof $y^j \in S^{kj}(p)$ implies $y^j \in con(Y^j), y^j = \sum \alpha^\eta y^\eta$ where $y^\eta \in Y^j, 0 \leq \alpha^\eta \leq 1$, and $\sum \alpha^\eta = 1$. We claim for each η such that $\alpha^\eta > 0$, that $p \cdot y^\eta = p \cdot y^j$. $y^\eta \in con(Y^j)$

so if $p \cdot y^\eta > p \cdot y^j$ then $y^j \notin S^{kj}(p)$ contrary to assumption. So $p \cdot y^\eta \leq p \cdot y^j$ for each η. But if for any η so that $\alpha^\eta > 0$, $p \cdot y^\eta < p \cdot y^j$ then there is another η' with $\alpha^{\eta'} > 0$ so that $p \cdot y^{\eta'} > p \cdot y^j$, a contradiction. So $p \cdot y^\eta = p \cdot y^j$ for all η so that $\alpha^\eta > 0$ and $y^\eta \in S^j(p)$. But $y^j = \sum \alpha^\eta y^\eta$ so $y^j \in con(S^j(p))$, $p \cdot y^\eta = \pi^j(p)$, but $p \cdot y^j = p \cdot y^\eta = \pi^j(p)$, so $\pi^{kj}(p) = \pi^j(p)$. QED

25.3.2 Artificial Convex Preferences and Demand

Household i's budget set $B^i(p)$ is described in Chapter 24, and as in Chapter 24, there may be price vectors where $B^i(p)$ is not well defined.

The formal definition of i's demand behavior $D^i(p)$ is precisely the same as in Chapter 24. However, without the convexity assumption, C.VI(C), on $\succeq_i$ the demand correspondence $D^i(p)$ may look rather different. $D^i(p)$ will be upper hemicontinuous in neighborhoods where it is well defined, but it may include gaps that look like jumps in demand behavior. That's because $D^i(p)$ may not be convex-valued.

In order to pursue the plan of the proof we need to formalize the notion of artificially convex preferences.

Definition: Let $x, y \in X^i$. We say $x \succeq_{ki} y$ if for every $w \in X^i$, $y \in con(A^i(w))$ implies $x \in con(A^i(w))$.

This definition creates a convex preference ordering $\succeq_{ki}$ for household i, by substituting the family of convex hulls of i's upper contour sets $con(A^i(w))$ for i's original upper contour sets $A^i(w)$. Without going more deeply into the geometry of these new upper contour sets, it is sufficient to assume

(CK.0) $\succeq_{ki}$ is a complete quasi-order on X^i.
(CK.IV) For each $i \in H$, $\succeq_{ki}$ fulfills C.IV.
(CK.V) For each $i \in H$, $\succeq_{ki}$ fulfills C.V.
(CK.VI) For each $i \in H$, $\succeq_{ki}$ fulfills C.VI(C).

We need to develop the notion of a convex-valued counterpart to $D^i(p)$. Define $D^{ki}(p) \equiv \{x^o | x^o \in B^i(p), x^o \succeq_{ki} x \text{ for all } x \in B^i(p)\}$. Under assumptions C.I - C.III, CK.0, CK.IV, CK.V, CK.VI,C.VII, $D^{ki}(p)$ is very well behaved in neighborhoods where it is well defined: upper hemicontinuous, convex-valued. Using $\succeq_{ki}$ as the preference ordering, rather than the nonconvex ordering $\succeq_i$, fills in the gaps left in $D^i(p)$ by the nonconvex ordering.

Lemma 25.2 Assuming C.I - C.III, CK.0, CK.IV, CK.V, CK.VI,C.VII, for each $i \in H, x^i \in X^i$, there is $\xi^i \in X^i$ so that $A^{ki}(x^i) = con(A^i(\xi^i))$. Further, if $M^i(p) > \inf_{x \in X^i} p \cdot x$ (consistent with C.VII), and if $D^{ki}(p)$ is non-empty, then $D^{ki}(p) = con(D^i(p))$.

Proof The presence of ξ^i as specified, follows directly from definition of $\succeq_{ki}$ under completeness and continuity, CK.0 and CK.V.

Let $x^i \in D^{ki}(p)$. x^i minimizes $p \cdot x$ in X^i subject to $x \succeq_{ki} x^i$. $x^i \in con(A^i(\xi^i)) \supseteq A^i(\xi^i)$. Then there is a finite set $\{w^\nu\} \subset A^i(\xi^i)$ so that $x^i = \sum_\nu \alpha^\nu w^\nu; 0 < \alpha^\nu \leq 1; \sum_\nu \alpha^\nu = 1$. Note that we disregard any w^ν with $\alpha^\nu = 0$. Then $p \cdot x^i = p \cdot \sum_\nu \alpha^\nu w^\nu = \sum_\nu \alpha^\nu p \cdot w^\nu$. We claim that for each ν, $p \cdot w^\nu = p \cdot x^i$. If not then for some $\nu', \nu", p \cdot w^{\nu'} > p \cdot x^i > p \cdot w^{\nu"}$. But this is a contradiction: x^i is then no longer the minimizer of $p \cdot x$ in $A^{ki}(x^i)$. Note then that even though x^i

may not be an element of $A^i(\xi^i)$, $p \cdot x^i = \inf_{x \in A^i(\xi^i)} p \cdot x$. Thus, $D^{ki}(p) = con(D^i(p))$.

QED

25.3.3 Competitive Equilibrium in the Artificial Convex Economy

One of the great powers of mathematics is that you only have to solve a problem once: when it reappears, you already know the answer. Even when it reappears under a new wrapping, if it's the same underneath you can say "reduced to the previous case." That's what we've been working on in sections 25.3.1 and 25.3.2: taking the non-convex economy of section 25.2 and restating it in a fashion where we can reduce consideration of its general equilibrium to a "previous case," the model of Chapter 24.

Consider a convex economy characterized in the following way:

Firms: $j \in F$, technologies are $con(Y^j)$, fulfilling P.I, P.II, PK.III, PK.IV.

Households: $i \in H$, tastes $\succeq_{ki}$, fulfilling C.I, C.II, C.III,CK.IV, CK.V, CK.VI, C.VII; endowments r^i, firm shares α^{ij}.

Then this economy fulfills all of the assumptions of Theorem 24.7. Applying that theorem, we know the convex economy has a general competitive equilibrium. That is,

Lemma 25.3 Assume P.II, PK.III, PK.IV, CK.0, C.I, C.II, C.III,CK.IV, CK.V, CK.VI, C.VII. Then there are prices $p^o \in P$, production plans $y^{oj} \in S^{kj}(p^o)$, consumption plans $x^{oi} \in D^{ki}(p^o)$ so that markets clear

$$\sum_{i\in H} r^i + \sum_{j\in F} y^{oj} \geq \sum_{i\in H} x^{oi}$$

where the inequality applies co-ordinatewise, and $p^o_n = 0$ for n so that the strict inequality holds.

Of course, the result of this lemma, in itself, should be of no interest at all. After all, the convex economy, is a figment of our imagination. The real economy is non-convex. But now we can apply the power of mathematics. The Shapley-Folkman Theorem (Theorem 8.3, Corollary 8.1) tells us that the actual economy is very near the artificial convex economy described above. This leads us to the result in the next section: the equilibrium of the constructed convex economy above is very nearly an equilibrium of the original non-convex economy.

25.4 Approximate Equilibrium

Recall the following definition and the corollary to the Shapley-Folkman Theorem:

Definition: We define the inner radius of $S \subset R^N$ as

$$r(S) \equiv \sup_{x\in con(S)} \inf_{T\subset S; x\in con(T)} rad(T)$$

The essence of this definition is to find the radius of the smallest subset $T \subset S$ that can be sure of spanning (including in its convex hull) an arbitrary point of $con(S)$.

Corollary 8.1 to the Shapley-Folkman Theorem: Let F be a finite family of compact subsets $S \subset R^N$ and $L > 0$ so that

$r(S) \leq L$ for all $S \in F$. Then for any $x \in con(\Sigma_{S\in F} S)$ there is $y \in \Sigma_{S\in F} S$ so that $|x - y| \leq L\sqrt{N}$.

Now we can apply this corollary to establish the existence of an approximate equilibrium.

Theorem 25.1 Let the economy fulfill P.II, PK.III, PK.IV, and CK.0, C.I, C.II, C.III, CK.IV, CK.V, CK.VI, C.VII. Let there be $L > 0$ so that for all $i \in H$, $x \in X^i$, $j \in F$,

$$r(A^i(x)) \leq L, \text{ and } r(Y^j) \leq L.$$

Then there are prices $p^* \in P$, production plans $y^{\dagger j} \in Y^j$, $y^{*j} \in con(Y^j)$, consumption plans $x^{*i} \in X^i$, and $x^{\dagger i} \in X^i$ so that

$$\sum_{i\in H} x^{*i} \leq \sum_{j\in F} y^{*j} + r$$

$$p^*_k = 0 \text{ for } k \text{ so that } \sum_{i\in H} x^{*i}_k < \sum_{j\in F} y^{*j}_k + r_k$$

$$p \cdot y^{\dagger j} = \sup_{y\in Y^j} p \cdot y = \sup_{y\in con(Y^j)} p \cdot y = p \cdot y^{*j}$$

$$p^* \cdot x^{\dagger i} = p^* \cdot r^i + \sum_{j\in F} \alpha^{ij} p^* \cdot y^{\dagger j} = p^* \cdot r^i + \sum_{j\in F} \alpha^{ij} p^* \cdot y^{*j} = p \cdot x^{*i}$$

$$x^{\dagger i} \text{ maximizes } u^i(x) \text{ subject to } p^* \cdot x \leq p^* \cdot r^i + \sum_{j\in F} \alpha^{ij} p^* \cdot y^{\dagger j}, \text{ and}$$

$$|[\sum_{i\in H} x^{*i} - \sum_{j\in F} y^{*j}] - [\sum_{i\in H} x^{\dagger i} - \sum_{j\in F} y^{\dagger j}]| \leq L\sqrt{N}$$

$$|[\sum_{i\in H} x^{\dagger i} - \sum_{j\in F} y^{\dagger j}] - r| \leq L\sqrt{N}$$

Proof By Lemma 25.3, there is $p^* \in P$, $y^{*j} \in S^{kj}(p^*)$, $x^{*i} \in D^{ki}(p^*)$ so that

$$\sum_{i\in H} r^i + \sum_{j\in F} y^{*j} \geq \sum_{i\in H} x^{*i}$$, with $p^*_k = 0$ for k so that a strict inequality holds,

and $p^* \cdot x^{*i} = p^* \cdot r^i + \sum_{j\in F} \alpha^{ij} p^* \cdot y^{*j}$. Using lemmata 25.1, 25.2, $y^{*j} \in con(S^j(p^*))$ and $x^{*i} \in con(D^i(p))$. Applying Corollary 8.1 to the Shapley-Folkman Theorem, for each $j \in F$ there is $y^{\dagger j} \in S^j(p^*)$, and for each $i \in H$ there is $x^{\dagger i} \in D^i(p*)$ so that

$$|[\sum_{i\in H} x^{*i} - \sum_{j\in F} y^{*j}] - [\sum_{i\in H} x^{\dagger i} - \sum_{j\in F} y^{\dagger j}]| \leq L\sqrt{N}$$

.

$$|[\sum_{i\in H} x^{\dagger i} - \sum_{j\in F} y^{\dagger j}] - r| \leq L\sqrt{N}$$

The last inequality follows since $[\sum_{i\in H} x^{*i} - \sum_{j\in F} y^{*j} - r] \leq 0$.

QED

The theorem says that there are prices p^* so that households and firms can choose plans that are optimizing at p^* , fulfilling budget constraint, with the allocations nearly (but not perfectly) market clearing. The proof is a direct application of Corollary 8.1 to the Shapley-Folkman Theorem and Lemma 25.3. The Lemma establishes the existence of market clearing prices for an 'economy' characterized by the convex hulls of the actual economy. Then applying the Corollary 8.1 to the Shapley-Folkman Theorem there is a choice of approximating elements in the original economy that is within the bound $L\sqrt{N}$ of the equilibrium allocation of the artificial convex economy.

That's not the end of the story. Note that the bound in Theorem 25.1 depends on the underlying description of the firms and households in the economy, but is independent of the size of the economy, the number of households, $\#H$. The disequilibrium — gap between supply and demand — in Theorem 25.1 is $L\sqrt{N}$. Thus the disequilibrium per head is $\frac{L\sqrt{N}}{\#H}$. But $\frac{L\sqrt{N}}{\#H} \rightarrow 0$ as $\#H \rightarrow \infty$. In a large economy, the disequilibrium attributable to U-shaped cost curves or concentrated preferences is negligible.