Lecture Notes for January 20, 2015 - Part 1

24.4 The market economy

We now bring the two sides, households and firms, of the set-valued economic model together. The demand correspondence of the unrestricted model is defined as

$$D(p) = \sum_{i \in H} D^i(p).$$

For the artificially restricted model, the demand side is characterized as

$$\tilde{D}(p) = \sum_{i \in H} \tilde{D}^i(p).$$

The economy's resource endowment is

$$r = \sum_{i \in H} r^i.$$

The supply side of the unrestricted economy is characterized as

$$S(p) = \sum_{j \in F} S^j(p),$$

and for the artificially restricted economy we have

$$\tilde{S}(p) = \sum_{j \in F} \tilde{S}^j(p).$$

We can now summarize supply, demand, and endowment as an excess demand correspondence.

Definition The excess demand correspondence at prices $p \in P$ is $Z(p) \equiv D(p) - S(p) - \{r\}$.

The excess demand correspondence of the artificially restricted model is $\tilde{Z}(p) = \tilde{D}(p) - \tilde{S}(p) - \{r\}$.

Having defined excess demand, we can now state and prove the Walras' Law, first for the unrestricted economy and then for the artificially restricted economy.

Theorem 24.5 (Walras' Law) Assume C.IV, C.V, and C.VI(C). Suppose $Z(p)$ is well defined and let $z \in Z(p)$. Then $p \cdot z = 0$.

Proof Let $z \in Z(p)$. Substituting into the definition of $Z(p)$, we have

$$p \cdot z = p \cdot \sum_{i \in H} x^i - p \cdot \sum_{j \in F} y^j - p \cdot \sum_{i \in H} r^i$$
$$\text{for some } x^i \in D^i(p), y^j \in S^j(p).$$

For each $i \in H$, by Lemma 24.4,

$$p \cdot x^i = M^i(p) = p \cdot r^i + \sum_{j \in F} \alpha^{ij} \pi^j(p)$$
$$= p \cdot r^i + \sum_{j \in F} \alpha^{ij} p \cdot y^j.$$

Now summing over $i \in H$, we get

$$\sum_{i \in H} p \cdot x^i = \sum_{i \in H} p \cdot r^i + \sum_{i \in H} \sum_{j \in F} \alpha^{ij} (p \cdot y^j).$$

Taking the vector p outside the sums and reversing the order of summation in the last term yields

$$p \cdot \sum_{i \in H} x^i = p \cdot \sum_{i \in H} r^i + p \cdot \sum_{j \in F} \sum_{i \in H} \alpha^{ij} y^j.$$

Recall that $\sum_{i \in H} \alpha^{ij} = 1$ for each j, and that $r = \sum_{i \in H} r^i$. We have then

$$p \cdot \sum_{i \in H} x^i = p \cdot r + p \cdot \sum_{j \in F} y^j.$$

That is, the value at market prices p of aggregate demand equals the value of endowment plus aggregate supply. Transposing the right-hand side to the left and recalling that $z = \sum_{i \in H} x^i - \sum_{j \in F} y^j - r$, we obtain

$$p \cdot \left[\sum_{i \in H} x^i - \sum_{j \in F} y^j - r \right] = p \cdot z = 0.$$

QED

The Walras' Law tells us that at prices where supply, demand, profits, and income are well defined, planned aggregate expenditure equals planned income from profits and sales of endowment. Hence, the value of planned purchases equals the value of planned sales and the net value at market prices of excess demand is nil. Unfortunately, $Z(p)$ is not always well defined. This arises because Y^j and $B^i(p)$ may be unbounded and hence may not include well-defined maxima of $\pi^j(\cdot)$ or $u^i(\cdot)$, respectively. This shifts our focus to $\tilde{Z}(p)$, which we know to be well defined for all $p \in P$. We now establish the counterpart of the Walras' Law for $\tilde{Z}(p)$.

Theorem 24.6 (Weak Walras' Law) Assume C.I - C.V, C.VI(C). Let $z \in \tilde{Z}(p)$. Then $p \cdot z \leq 0$. Further, if $p \cdot z < 0$ then there is $k = 1, 2, 3, \ldots, N$ so that $z_k > 0$.

Proof $p \cdot z = p \cdot \sum_{i \in H} x^i - p \cdot \sum_{j \in F} y^j - p \cdot \sum_{i \in H} r^i$, where $x^i \in \tilde{D}^i(p)$, $y^j \in \tilde{S}^j(p)$. For each $i \in H$,

$$p \cdot x^i \leq \tilde{M}^i(p) = p \cdot r^i + \sum_{j \in F} \alpha^{ij} \tilde{\pi}^j(p)$$
$$= p \cdot r^i + \sum_{j \in F} \alpha^{ij} (p \cdot y^j),$$

and

$$\sum_{i \in H} p \cdot x^i \leq \sum_{i \in H} p \cdot r^i + \sum_{i \in H} \sum_{j \in F} \alpha^{ij} (p \cdot y^j)$$
$$p \cdot \sum_{i \in H} x^i \leq p \cdot \sum_{i \in H} r^i + p \cdot \sum_{j \in F} \sum_{i \in H} \alpha^{ij} y^j.$$

Note the changed order of summation in the last term. Recall that $\sum_{i \in H} \alpha^{ij} = 1$ for each j and that $r = \sum_{i \in H} r^i$. We have then

$$p \cdot \sum_{i \in H} x^i \leq p \cdot r + p \cdot \sum_{j \in F} y^j.$$

Transposing the right-hand side to the left and recalling that $z = \sum_{i \in H} x^i - \sum_{j \in F} y^j - r$, we get

$$p \cdot \left[\sum_{i \in H} x^i - \sum_{j \in F} y^j - r \right] = p \cdot z \leq 0.$$

The left-hand side in this expression is

$$\sum_{i \in H} [p \cdot x^i] - \sum_{i \in H} [\tilde{M}^i(p)].$$

If $p \cdot z < 0$ then for some $i \in H, p \cdot x^i < \tilde{M}^i(p)$. In that case, by Lemma 24.5, $|x^i| = c$ and hence x^i is not attainable. Unattainability implies $z_k > 0$ for some $k = 1, 2, \ldots, N$. QED

Lemma 24.7 Assume C.I - C.V, C.VI(C), C.VII, and P.I–P.IV. The range of $\tilde{Z}(p)$ is bounded. $\tilde{Z}(p)$ is upper hemicontinuous and convex-valued.

Proof $\tilde{Z}(p) = \sum_{i \in H} \tilde{D}^i(p) - \sum_{j \in F} \tilde{S}^j(p) - \{\sum_{i \in H} r^i\}$ is the finite sum of bounded sets and is therefore bounded. It is a finite sum of upper hemicontinuous convex correspondences and is hence convex and upper hemicontinuous. QED

As an artificial construct to allow us to prove the existence of equilibrium in the market economy, we introduce an artificially restricted economy.

24.5 The artificially restricted economy

We will describe the artificially restricted economy by taking the production technology of each firm j to be $\tilde{Y}^j$ rather than Y^j, thus making the supply correspondence $\tilde{S}^j(p)$ rather than $S^j(p)$, and by taking the demand correspondence of each household i to be $\tilde{D}^i(p)$ rather than $D^i(p)$. In this special restricted case we will refer to the excess demand correspondence of the economy as $\tilde{Z}(p)$. By Theorems 24.1 and 24.3, the artificially restricted excess demand correspondence is well defined for all $p \in P$:

$$\tilde{Z}: P \to \mathbf{R}^N.$$

We use the artificially restricted economy above as a mathematical construct, which is convenient because supply, demand, and excess demand are everywhere well defined. The unrestricted economy is defined by Y^j, D^i, and Z. As demonstrated in Theorem 24.1 and Lemma 24.6, $Z(p)$ and $\tilde{Z}(p)$ will coincide for elements of $Z(p)$ corresponding to attainable points in $\tilde{S}^j(p)$ and $\tilde{D}^i(p)$. The set $\tilde{Z}(p)$ is nonempty for all $p \in P$, whereas $Z(p)$ may not be well defined (nonempty) for some elements of $p \in P$.

Recall the following properties of $\tilde{Z}(p)$:

(1) Weak Walras' Law (Theorem 24.6): Assuming P.I - P.IV, C.IV and C.VI(C), we have $z \in \tilde{Z}(p)$ implies $p \cdot z \leq 0$. Further, if $p \cdot z < 0$ then there is $k = 1, 2, 3, \ldots, N$, so that $z_k > 0$.

(2) $\tilde{Z}(p)$ is well defined for all $p \in P$ and is everywhere upper hemicontinuous and convex valued, assuming C.I - C.V, C.VI(C), C.VII and P.I–P.IV. This is Theorems 24.1 and 24.3 and lemma 24.7.

We will use these properties to prove the existence of market clearing prices in the artificially restricted economy. We will then use Theorems 24.1 and 24.6 and C.VI(C) to show that the equilibrium of the artificially restricted economy is also an equilibrium of the unrestricted economy. To start the process of establishing the existence of an equilibrium for the artificially restricted economy, we need a price adjustment function. We plan to use the Kakutani Fixed-Point Theorem, and thus we hope to construct an upper hemicontinuous, convex-valued price adjustment correspondence.

Let $\rho(z) \equiv \{p^* | p^* \in P, p^* \cdot z \text{ maximizes } p \cdot z \text{ for all } p \in P\}$. $\rho(z)$ is the price adjustment correspondence. For each excess demand vector z, ρ chooses a price vector based on increasing the prices of goods in excess demand while reducing the prices of goods in excess supply. Choose positive real C so that $|\tilde{Z}(p)| < C$ for all $p \in P$. We know that C exists (by lemma 24.7) since $\#F$ and $\#H$ are finite and each of the $\tilde{D}^i(p)$, $\tilde{S}^j(p)$ is chosen from a bounded set (the set of attainable allocations is bounded by Theorem 15.2). Then let $\Delta = \{x | x \in \mathbf{R}^N, |x| \leq C\}$. Note that Δ is compact and convex:

$$\rho : \Delta \to P$$
$$\tilde{Z} : P \to \Delta.$$

Lemma 24.8 $\rho(z)$ is upper hemicontinuous for all $z \in \Delta$; $\rho(z)$ is convex and nonnull for all $z \in \Delta$.

Proof Exercise 24.6.

24.6 Existence of competitive equilibrium

We are now ready to establish existence of competitive general equilibrium. We focus first on the artificially restricted economy and then extend our results to the unrestricted economy.

Definition $p^\circ \in P$ is said to be a competitive equilibrium price vector (of the unrestricted market economy) if there is $z^\circ \in Z(p^\circ)$ so that $z^\circ \leq 0$ (coordinatewise) and $p^\circ_k = 0$ for k so that $z^\circ_k < 0$.

Theorem 24.7 Let the economy fulfill C.I - C.V, C.VI(C), C.VII, and P.I–P.IV. Then there is a competitive equilibrium p° for the economy.

The strategy of proof is to create a grand upper hemicontinuous convex-valued mapping, $\Phi(\cdot)$, from $\Delta \times P$, the Cartesian product of (artificially restricted) excess demand space, Δ, with price space, P, into itself. The mapping takes prices and maps them into the corresponding excess demands and takes excess demands and maps them into corresponding prices. The mapping Φ will have a fixed point by (the corollary to) the Kakutani Fixed-Point Theorem. The fixed point of the price adjustment correspondence, $\rho(\cdot)$, will take place at a market equilibrium of the artificially restricted economy. We will then use Theorems 24.1 and 24.6 and Lemma 24.6 to show that the equilibrium of the artificially restricted economy is also an equilibrium of the original (unrestricted) economy. This follows because the equilibrium of the artificially restricted economy is attainable. Hence, at the artificially restricted economy's equilibrium prices, artificially restricted and unrestricted demands and supplies coincide.

Proof Let $(p, z) \in P \times \Delta, \Phi(p, z) \equiv \{(\bar{p}, \bar{z}) | \bar{p} \in \rho(z), \bar{z} \in \tilde{Z}(p)\}$. Then $\Phi : P \times \Delta \to P \times \Delta$. Φ is nonnull, upper hemicontinuous, and convex valued. $P \times \Delta$ is compact and convex. Then by Corollary 23.1 to the Kakutani Fixed-Point Theorem there is $(p^\circ, z^\circ) \in P \times \Delta$ so that (p°, z°) is a fixed point of Φ:

$$(p^\circ, z^\circ) \in \Phi(p^\circ, z^\circ),$$
$$p^\circ \in \rho(z^\circ),$$
$$z^\circ \in \tilde{Z}(p^\circ).$$

We will now demonstrate that (p°, z°) represents an equilibrium of the artificially restricted economy. For each $i \in H$, and for each $j \in F$, there is $x^{\circ i} \in \tilde{D}^i(p^\circ)$, $y^{\circ j} \in \tilde{S}^j(p^\circ)$, so that $x^\circ = \sum_i x^{\circ i}$, $y^\circ = \sum_j y^{\circ j}$, with $z^\circ = x^\circ - y^\circ - r$, and by the Weak Walras' Law, $p^\circ \cdot z^\circ \leq 0$. But p° maximizes $p \cdot z^\circ$ for $p \in P$. This implies $z^\circ \leq 0$, since if there were any positive coordinate in z° then the maximum value of $p \cdot z^\circ$ would be positive. Moreover, we have either (Case 1) $p^\circ \cdot z^\circ = 0$ (in which case it follows that $z^\circ = 0$ or $z^\circ_k < 0$ implies $p^\circ_k = 0$) or (Case 2) $p^\circ \cdot z^\circ < 0$ (in which case the Weak Walras' Law implies $z^\circ_k > 0$ some k). But in Case 2, max $p \cdot z^\circ$ would then be positive, which is a contradiction. Hence Case 2 cannot arise and we have $p^\circ \cdot z^\circ = 0$, with either $z^\circ = 0$ or if for some $k, z^\circ_k < 0$, then $p^\circ_k = 0$. This establishes (p°, z°) as an equilibrium for the artificially restricted economy. Now we must demonstrate that it is an equilibrium for the unrestricted economy as well. We have

$$z^\circ = x^\circ - y^\circ - r$$

or

$$x^\circ - z^\circ = y^\circ + r.$$

Since $z^\circ \leq 0, x^\circ - z^\circ \geq x^\circ \geq 0$. Thus $y^\circ + r \geq 0$. Therefore, y° is attainable; this implies, by Theorem 24.1, that $y^{\circ j} \in S^j(p^\circ)$ for all $j \in F$. Furthermore, since $y^\circ + r \geq x^\circ, x^\circ$ is attainable. Hence, by Lemma 24.6, $x^{\circ i} \in D^i(p^\circ)$ for all $i \in H$. Thus we have $p^\circ \in P, y^{\circ j} \in S^j(p^\circ)$, and $x^{\circ i} \in D^i(p^\circ)$, so that $\sum_{i \in H} x^{\circ i} - \sum_{j \in F} y^{\circ j} - \sum_{i \in H} r^i \leq 0$, with $p_k = 0$ for all k such that $z^\circ_k < 0$. Hence (p°, z°) is an equilibrium for the unrestricted economy. QED