# Lecture Notes for January 15, 2015

## 12.1 The structure of household consumption sets and preferences

Households are elements of the finite set $H$ numbered $1, 2, \ldots, \#H$. A household $i \in H$ will be characterized by its possible consumption set $X^i \subseteq \mathbf{R}^N_+$, its preferences $\succeq_i$, $\alpha^{ij} = i$'s share of firm $j$'s profits, where $0 \leq \alpha^{ij} \leq 1$, and its endowment $r^i \in \mathbf{R}^N_+$.

## 12.2 Consumption sets

(C.I) $X^i$ is closed and nonempty.

(C.II) $X^i \subseteq \mathbf{R}^N_+$. $X^i$ is unbounded above, that is, for any $x \in X^i$ there is $y \in X^i$ so that $y > x$, that is, for $n = 1, 2, ..., N, y_n \geq x_n$ and $y \neq x$.

(C.III) $X^i$ is convex.

It is usually simplest to take $X^i$ to be the nonnegative orthant (quadrant) of $\mathbf{R}^N$, denoted $\mathbf{R}^N_+$.

The possible aggregate (for the economy's household sector) consumption set is $X = \sum_{i \in H} X^i$.

### 12.2.1 Preferences

Each household $i \in H$ has a preference quasi-ordering on $X^i$, denoted $\succeq_i$. For typical $x, y \in X^i$, "$x \succeq_i y$" is read "$x$ is preferred or indifferent to $y$ (according to $i$)." We introduce the following terminology:

If $x \succeq_i y$ and $y \succeq_i x$ then $x \sim_i y$ ("$x$ is indifferent to $y$"),
If $x \succeq_i y$ but not $y \succeq_i x$ then $x \succ_i y$ ("$x$ is strictly preferred to $y$").

We will assume $\succeq_i$ to be complete on $X^i$, that is, any two elements of $X^i$ are comparable under $\succeq_i$. For all $x, y \in X^i$, $x \succeq_i y$, or $y \succeq_i x$ (or both). Since we take $\succeq_i$ to be a quasi-ordering, $\succeq_i$ is assumed to be transitive and reflexive.

The conventional alternative to describing the quasi-ordering $\succeq_i$ is to assume the presence of a utility function $u^i(x)$ so that $x \succeq_i y$ if and only if $u^i(x) \geq u^i(y)$. Under sufficient conditions, the utility function can be derived from the quasi-ordering. If you prefer the utility function formulation, use it at will. Just read $u^i(x) \geq u^i(y)$ wherever you see $x \succeq_i y$.

The assumption that household preferences can be characterized by a transitive, reflexive, complete relation, $\succeq_i$, is powerful. It says that the household knows what it wants and (transitivity) that its preferences are well defined and consistent (they do not cycle but rather represent a true ordering).

### 12.2.2 Non-Satiation

We will assume there is universal scarcity in the economy. For each household, and for any consumption plan $x \in X^i$ , there is always a preferable conceiveable alternative $y \in X^i$.

(C.IV) (Non-Satiation) Let $x \in X^i$. Then there is $y \in X^i$ so that $y \succ_i x$.

An occasionally useful special case is
(C.IV*) (Weak Monotonicity) Let $x, y \in X^i$ and $x >> y$. Then $x \succ_i y$.

### 12.2.3 Continuity

We now introduce the principal technical assumption on preferences, the assumption of continuity.

(C.V) (Continuity) For every $x^\circ \in X^i$, the sets
$A^i(x^\circ) = \{x \mid x \in X^i, x \succeq_i x^\circ\}$ and
$G^i(x^\circ) = \{x \mid x \in X^i, x^\circ \succeq_i x\}$ are closed.

Although C.V is more technical than economic, it proves to be extremely useful. The structure of the upper and lower contour sets of $\succeq_i$ assumed in C.V is precisely the behavior we'd expect if $\succeq_i$ were defined by a continuous utility function. This follows since the inverse image of a closed set under a continuous mapping is closed (Theorem 7.5). Thus, suppose household $i$'s preferences were represented by the utility function, $u^i(\cdot)$. Then the sets $A^i(x^\circ)$ and $G^i(x^\circ)$ are the inverse images of the closed intervals in R $[u^i(x^\circ), \infty)$ and $[\inf_{x \in X^i} u^i(x), u^i(x^\circ)]$.

The economic content of C.V is the following description of the structure of preferences: Begin with a typical point $x$ in $X^i$, consider a line segment in $X^i$ starting at one end with elements superior to $x$ according to $\succeq_i$ and progressing eventually to points inferior to $x$. Then the line segment must include points indifferent to $x$ as well. As we pass from superior to inferior according to $\succeq_i$, we must touch on indifference. This would seem trivially obvious. But there are —- otherwise well-behaved —- preference quasi-orderings that violate C.V that generate discontinuities in demand. The classic example is the lexicographic ordering.

Example 12.1 (Lexicographic preferences) In this case it is not possible to represent the quasi-order $\succeq_L$ as a continuous real-valued utility function. The lexicographic (dictionary-like) ordering on $\mathbf{R}^N$ (let's denote it $\succeq_L$) is described in the following way. Let $x = (x_1, x_2, \ldots, x_N)$ and $y = (y_1, y_2, \ldots, y_N)$.

$$\begin{array}{ll} x \succ_L y & \text{if } x_1 > y_1, \text{ or} \\ & \text{if } x_1 = y_1 \text{ and } x_2 > y_2, \text{ or} \\ & \text{if } x_1 = y_1,\ x_2 = y_2, \text{ and } x_3 > y_3, \text{ and so forth } \ldots. \\ x \sim_L y & \text{if } x = y. \end{array}$$

$\succeq_L$ fulfills non-satiation, trivially fulfills strict convexity (C.VI(SC), introduced below), but does not fulfill continuity (C.V).

### 12.2.4 Attainable Consumption

Definition $x$ is an ***attainable*** consumption if $y + r \geq x \geq 0$, where $y \in Y$ and $r \in \mathbf{R}^N_+$ is the economy's initial resource endowment, so that $y$ is an attainable production plan.

Note that the set of attainable consumptions is bounded under P.I - P.IV.

### 12.2.5 Convexity of preferences

(C.VI)(C) (Convexity of Preferences) $x \succ_i y$ implies $((1-\alpha)x + \alpha y) \succ_i y$, for $0 < \alpha < 1$.

Of course, C.VI(C) includes as a special case

(C.VI)(SC) (Strict Convexity of Preferences): Let $x \succeq_i y$, (note that this includes $x \sim_i y$), $x \neq y$, and let $0 < \alpha < 1$. Then $\alpha x + (1-\alpha)y \succ_i y$.

An immediate consequence of C.VI(C) is that $A^i(x^\circ)$ is convex for every $x^\circ \in X^i$.

## 12.3 Representation of $\succeq_i$: Existence of a continuous utility function

Definition Let $u^i$: $X^i \to \mathbf{R}$. $u^i(\cdot)$ is a utility function that ***represents*** the preference ordering $\succeq_i$ if for all $x, y \in X^i$, $u^i(x) \geq u^i(y)$ if and only if $x \succeq_i y$. This implies that $u^i(x) > u^i(y)$ if and only if $x \succ_i y$.

The function $u^i(\cdot)$, $i$'s utility function, is merely a representation of $i$'s preference ordering $\succeq_i$; $u^i(\cdot)$ contains no additional information. In particular, it does not represent strength or intensity of preference. Utility functions like $u^i(\cdot)$ that represent an ordering $\succeq_i$, without embodying additional information or assumptions, are called ordinal (i.e., representing an ordering). In this sense, any monotone (order-preserving) transformation of $u^i(\cdot)$ is equally appropriate as a representation of $\succeq_i$.

### 12.3.1 Weak Conditions for Existence of a Continuous Utility Function

It is possible to prove the existence of a continuous utility function for $\succeq_i$ using C.I, C.II, C.III, and C.V only, without using any assumption on scarcity or desirability of commodities.

Theorem 12.1 Let $\succeq_i, X^i$, fulfill C.I, C.II, C.III, C.V. Then there is $u^i : X^i \to$ R, $u^i(\cdot)$ continuous on $X^i$, so that $u^i(\cdot)$ is a utility function representing $\succeq_i$.

Proof See Debreu (1959, Section 4.6) or Debreu (1954). QED

Lazy theorist's quick demonstration of existence of continuous utility function: Set $X^i = \mathbf{R}^{\mathbf{N}}_+$; assume "(C.IV*) Weak Monotonicity." Then set $u^i(x) =$ the length of the 45° line from 0 to the indifference curve through $x$.

### 12.4.1 Adequacy of income

The notations $M^i(p)$ and $\tilde{M}^i(p)$ will denote household $i$'s income. The notations $B^i(p)$ and $\tilde{B}^i(p)$ will denote his budget set, the set of affordable consumption bundles in $\mathbf{R}^{\mathbf{N}}$; the elements of $\tilde{B}^i(p)$ will be limited to Euclidean length less than or equal to $c > 0$.

To avoid possibly empty budget sets and discontinuities in demand behavior at the boundary of $X^i$ we will assume

(C.VII) For all $i \in H$,

$$\tilde{M}^i(p) > \inf_{x \in X^i \cap \{x \| x| \leq c\}} p \cdot x \qquad \text{for all } p \in P.$$

Assumption C.VII allows us to avoid discontinuities that may occur when the budget set coincides with the boundary of $X^i$, the Arrow corner[1].

Example 12.2 [The Arrow Corner] Consider household $i$ in a 2-commodity economy with sale of endowment as $i$'s only source of income ($i$ has no share

[1] A corner solution occurs when the solution is is up against a boundary constraint.

in firm profits). Let the household consumption set $X^i$ be the nonnegative quadrant, with $i$ endowed with one unit of good 1 and none of good 2. Consider consumption behavior in the neighborhood of a zero price of good 1. We have

$$X^i = \mathbf{R}^2_+,$$
$$r^i = (1, 0),$$
$$\tilde{M}^i(p) = p \cdot r^i.$$

Let $p^\circ = (0, 1)$. Then the household budget set $\tilde{B}^i(p^\circ)$ is $\{(x, y) \mid c \geq x \geq 0, y = 0\}$, the truncated nonnegative $x$ axis. Consider the sequence $p^\nu = (1/\nu, 1 - 1/\nu)$. $p^\nu \to p^\circ$. We have

$$\tilde{B}^i(p^\nu) \cap X^i = \left\{(x, y) \mid p^\nu \cdot (x, y) \leq \frac{1}{\nu}, (x, y) \geq 0, c \geq |(x, y)| \geq 0\right\},$$

$(c, 0) \in \tilde{B}^i(p^\circ)$, but there is no sequence $(x^\nu, y^\nu) \in \tilde{B}^i(p^\nu)$ so that $(x^\nu, y^\nu) \to (c, 0)$. On the contrary, for any sequence $(x^\nu, y^\nu) \in \tilde{B}^i(p^\nu)$ so that $(x^\nu, y^\nu) = \tilde{D}^i(p^\nu)$, $(x^\nu, y^\nu)$ will converge to some $(x^*, 0)$, where $0 \leq x^* \leq 1$. For suitably chosen $\succeq_i$, we may have $(c, 0) = \tilde{D}^i(p^\circ)$. Hence $\tilde{D}^i(p)$ need not be continuous at $p^\circ$. This completes the example.

Example 12.2 demonstrates that when the budget constraint coincides with the boundary of the consumption set, discontinuities in the budget set (a large change in the consumption choices available in response to a small change in prices) and corresponding discontinuity in demand behavior may result. Hence, to ensure continuity of demand, (C.VII) adequacy of income (sufficient income to stay off the boundary of the consumption set) may be required.

## 16.2 Household choice in an unbounded budget set

We will denote the household budget or income as a real number, $M^i(p) \geq 0$. Then the household budget constraint set is

$$B^i(p) \equiv \{x \mid x \in \mathbf{R}^N, p \cdot x \leq M^i(p)\}.$$

Lemma 16.1 $B^i(p)$ is a closed convex set.

$$D^i(p) \equiv \{y \mid y \in B^i(p) \cap X^i, y \succeq_i x \text{ for all } x \in B^i(p) \cap X^i\}$$
$$\equiv \{y \mid y \in B^i(p) \cap X^i, u^i(y) \geq u^i(x) \text{ for all } x \in B^i(p) \cap X^i\}.$$

We will restrict attention to models where $M^i(p)$ is homogeneous of degree one, that is, where $M^i(\lambda p) = \lambda M^i(p)$. It is immediate then that $B^i(p)$ is homogeneous of degree zero.

Lemma 16.2 Let $B^i(p)$ be homogeneous of degree 0. Then $D^i(p)$ is homogeneous of degree 0 also.

We will confine attention to price vectors on the set $P$, the unit simplex in $\mathbf{R}^N$,

$$P \equiv \left\{ p \mid p \in \mathbf{R}^N, p_i \geq 0, i = 1, 2, 3, \ldots, N, \sum_{i=1}^{N} p_i = 1 \right\}.$$

Even with a well-defined budget set, we still have a problem in defining demand behavior for typical $i \in H$. For some $p \in P$, household $i$'s opportunity set $(B^i(p) \cap X^i)$ may not be compact. Unbounded $B^i(p) \cap X^i$ will arise when some goods' prices are zero so that the budget constraint is consistent with unbounded consumption of some goods. In an economy with a bounded attainable set, such consumptions could never be equilibria, but during the process of price adjustment the Walrasian auctioneer should be free to search through the nil prices and households should be free to demand the unbounded consumption plans. It should be a conclusion – not an assumption – that such points are not equilibria, and this information should be communicated to agents in the economy through prices, not by assumption. As an intermediate step in characterizing household consumption behavior, we use the same technical device that we used on the production side in a similar setting. We create an artificially bounded budget set containing as a proper subset all of the economy's attainable points consistent with budget constraint. The strategy of proof will then be:

- to characterize demand behavior in the artificially bounded economy,
- to show that it coincides with demand of the unbounded economy throughout the attainable set,
- to find an equilibrium for the artificially bounded economy and show that the equilibrium is attainable, and finally
- to show that the artificial bound is not a binding constraint in equilibrium so that the equilibrium of the artificially bounded economy is also an equilibrium for the unbounded economy.

We wish now to characterize a bounded subset of $B^i(p)$ containing the consumption plans that are both within the budget $\tilde{M}^i(p) > 0$ (where $\tilde{M}^i(p)$

equals $M^i(p)$ when the latter derives from attainable firm production plans) and that are also attainable. We have not yet fully described this budget.

Definition $x \in \boldsymbol{R}_+^N$ is an ***attainable*** consumption if $y + r \geq x \geq 0$, where $y \in Y$ and $r$ is the economy's initial resource endowment, so that y is an attainable production plan. The inequality is to be read co-ordinate-wise.

Note that Theorem 15.2 says that the set of attainable consumptions is bounded under P.I–P.IV.

Choose $c$ so that $|x| < c$ (a strict inequality) for all attainable consumptions $x$. Let

$$\tilde{B}^i(p) = \{x \mid x \in \mathbf{R}^N, p \cdot x \leq \tilde{M}_i(p)\} \cap \{x \mid |x| \leq c\}.$$

$$\begin{aligned}\tilde{D}^i(p) &\equiv \{x \mid x \in \tilde{B}^i(p) \cap X^i, x \succeq_i y \text{ for all } y \in \tilde{B}^i(p) \cap X^i\}\\ &\equiv \{x \mid x \in \tilde{B}^i(p) \cap X^i, x \text{ maximizes } u^i(y) \text{ for all } y \in \tilde{B}^i(p) \cap X^i\}.\end{aligned}$$

Sets $\tilde{B}^i(\cdot)$ and $\tilde{D}^i(\cdot)$ are homogeneous of degree 0 as are $B^i(\cdot)$ and $D^i(\cdot)$. Let $D(p) = \sum_{i \in H} D^i(p)$ and $\tilde{D}(p) = \sum_{i \in H} \tilde{D}^i(p)$.

## 24.3 Households; with set-valued demand

For each firm $j$, there is a list of households that are shareholders in $j$. We let $\alpha^{ij} \in \mathbf{R}, 0 \leq \alpha^{ij} \leq 1$, represent $i$'s share of firm $j$. We assume $\sum_{i \in H} \alpha^{ij} = 1$ for each $j \in F$. That is, we assume that every firm is 100% owned by some one or several shareholders and that there is no negative ownership of firms (no short sales). A household $i \in H$ is characterized by its endowment of goods $r^i \in \mathbf{R}_+^N$, by its endowed shares $\alpha^{ij} \in \mathbf{R}_+$ of firms $j \in F$, by $\succeq_i$, and its possible consumption set $X^i$. The initial resource endowment of the economy, designated $r \in \mathbf{R}_+^N$ is

$$r \equiv \sum_{i \in H} r^i.$$

The convexity assumption on household preferences, C.VI(C), admits the possibility of set-valued linear segments in demand behavior, occurring, for example, in the case of perfect substitutes in consumption. To see how this might arise, consider Example 24.3.

Example 24.3 Convex set-valued household demand. Let household $i$'s possible consumption set $X^i$ be $\mathbf{R}_+^2$, the nonnegative quadrant in $\mathbf{R}^2$. Let the household endowment be (1, 1) with no ownership of shares of firms. At prices $p \in \mathbf{R}^2_+$, the household income is $p \cdot (1,1) = p_x + p_y$. Let household preferences be described by the utility function $u(x,y) = [ax + by]$. Then household demand can be characterized as

$$D^i(p) = \begin{cases} ([p_x + p_y]/p_x, 0) \text{ for } \frac{p_x}{p_y} < \frac{a}{b} \\ (0, [p_x + p_y]/p_y) \text{ for } \frac{p_x}{p_y} > \frac{a}{b} \\ \{(x, [p_x + p_y - p_x x]/p_y) | x \in [0, (p_x + p_y)/p_x]\} \text{ for } \frac{p_x}{p_y} = \frac{a}{b} \\ \text{ undefined for } p_x = 0 \text{ or } p_y = 0. \end{cases}$$

Note that $D^i(p)$ is convex set valued for $p_x/p_y = a/b$. This simply reflects the idea that if goods $x$ and $y$ are perfect substitutes at the ratio $a/b$ then, when their prices occur in this ratio, the household will be indifferent among a whole set of linear combinations of $x$ and $y$ in the inverse of this ratio. After all, if the goods $x$ and $y$ are perfect substitutes then it really doesn't matter in what proportion they are used. The demand behavior, $D^i(p)$, is described as upper hemicontinuous and convex valued for all $p$ so that $p_x \neq 0$ and $p_y \neq 0$.

Household $i$'s income is defined as

$$M^i(p) = p \cdot r^i + \sum_{j \in F} \alpha^{ij} \pi^j(p).$$

For the model with restricted firm supply behavior, household income is

$$\tilde{M}^i(p) = p \cdot r^i + \sum_{j \in F} \alpha^{ij} \tilde{\pi}^j(p).$$

Note that $M^i(p)$ may not be everywhere well defined since $\pi^j(p)$ may not be well defined for some $j \in F, p \in P$. Conversely, $\tilde{M}^i(p)$ is continuous, real valued, nonnegative, and well defined for all $p \in \mathbf{R}^N_+, p \neq 0$. $\tilde{B}^i(p)$ and $\tilde{D}^i(p)$ are homogeneous of degree 0 in $p$. This allows us to confine attention in prices to the unit simplex in $\mathbf{R}^N$, denoted $P$.

$$B^i(p) \equiv \{x | x \in \mathbf{R}^N, p \cdot x \leq M^i(p)\}.$$

The demand correspondence (possibly set-valued) is

$$\begin{aligned} D^i \ &: \ \mathbf{R}^N_+ \to \mathbf{R}^N, \\ D^i(p) &\equiv \{y | y \in B^i(p) \cap X^i, y \succeq_i x \text{ for all } x \in B^i(p) \cap X^i\} \\ &\equiv \{y | y \in B^i(p) \cap X^i, u^i(y) \geq u^i(x) \text{ for all } x \in B^i(p) \cap X^i\}. \end{aligned}$$

Choose $c$ so that $|x|<c$ (a strict inequality) for all attainable consumptions $x$. Theorem 15.1 assures us that $c$ exists under P.I - P.IV. The artificially restricted budget set is then defined as

$$\tilde{B}^i(p) = \{x | x \in \mathbf{R}^N, p \cdot x \leq \tilde{M}^i(p), |x| \leq c\}.$$

$\tilde{B}^i(\cdot)$ is homogeneous of degree 0, just as is $B^i(\cdot)$. We now define the artificially restricted demand correspondence,

$$\tilde{D}^i(p) \equiv \{x | x \in \tilde{B}^i(p) \cap X^i, x \succeq_i y \text{ for all } y \in \tilde{B}^i(p) \cap X^i\}.$$

Under convexity (C.VI(C)), $\tilde{D}^i(p)$ may be set valued.

Firm $j$'s profit function is $\pi^j(p) = \max_{y \in Y^j} p \cdot y$. Since $Y^j$ need not be compact, $\pi^j(p)$ may not be well defined. Firm $j$'s profit function in the artificially restricted firm technology set $\tilde{Y}^j$ is $\tilde{\pi}^j(p) = \max_{y \in \tilde{Y}^j} p \cdot y$. The function $\tilde{\pi}^j(p)$ is always well defined, since $\tilde{Y}^j$ is compact by definition and P.III.

We want to show that the (artificially restricted) demand correspondence of household $i$, $\tilde{D}^i(p)$, is upper hemicontinuous and convex valued. To demonstrate upper hemicontinuity, we will use the Theorem of the Maximum, Theorem 23.3. That theorem requires that the opportunity set, in this case $\tilde{B}^i(p) \cap X^i$, be continuous, both upper and lower hemicontinuous. Continuity of $\tilde{B}^i(p) \cap X^i$ is the message of Theorem 24.2.

Theorem 24.2 Assume P.I - P.IV, C.I, C.II, C.III, and C.VII. Then $\tilde{B}^i(p) \cap X^i$ is continuous (lower and upper hemicontinuous), compact valued, and nonnull for all $p \in P$.

Proof P.I - P.IV and Theorem 15.1 ensure that $c$ is well-defined. Continuity of $\tilde{B}^i(p) \cap X^i$ depends on continuity of $\tilde{M}^i(p)$. This follows from definition and Theorem 24.1 (continuity of $\tilde{\pi}^j(p)$). Upper hemicontinuity of $\tilde{B}^i(p) \cap X^i$ is left as an exercise. Nonnullness follows directly from C.VII. Compactness follows from closedness and the restriction to $\{x | |x| \leq c\}$. To demonstrate lower hemicontinuity, we will use adequacy of income, C.VII, and the convexity of $\tilde{B}^i(p) \cap X^i$. Consider a sequence $p^\nu \in P, p^\nu \to p^\circ, y^\circ \in \tilde{B}^i(p^\circ) \cap X^i$. To establish lower hemicontinuity we need to show that there is a sequence $y^\nu$, so that $y^\nu \in \tilde{B}^i(p^\nu) \cap X^i$ and $y^\nu \to y^\circ$. We will consider two cases depending on the cost of $y^\circ$ at price vector $p^\circ$.

CASE 1 $p^\circ \cdot y^\circ > 0$ and

$$p^\circ \cdot y^\circ > \min_{x \in X^i \cap \{y | y \in \mathbf{R}^N, c \geq |y|\}} p^\circ \cdot x.$$

The strategy of proof in this case is to create the required sequence $y^\nu$ in the following way. Find a minimum expenditure point, $x^\circ$ in $X^i \cap \{x||x| \le c\}$. We extend a ray from $x^\circ$ through $y^\circ$. We then take a sequence of points on the ray chosen to fulfill the budget constraint at $p^\nu$ and to converge to $y^\circ$. That sequence is $y^\nu$. This construction is depicted in Figure 24.3.

For $\nu$ large, we have

$$p^\nu \cdot y^\circ > \min_{x \in X^i \cap \{y|y \in \mathbf{R}^N, c \ge |y|\}} p^\circ \cdot x.$$

We choose $x^\circ$ as a cost-minimizing element of $X^i \cap \{x||x| \le c\}$ at prices $p^\circ$. Let $x^\circ \in X^i \cap \{x||x| \le c\}$ and

$$p^\circ \cdot x^\circ = \min_{x \in X^i \cap \{y|y \in \mathbf{R}^N, c \ge |y|\}} p^\circ \cdot x.$$

We now construct $y^\nu$ as a convex combination of $x^\circ$ and $y^\circ$, fulfilling budget constraint at $p^\nu$.

$$\text{Let } \alpha^\nu = \min\left[1, \frac{[\tilde{M}^i(p^\nu) - p^\nu \cdot x^\circ]}{p^\nu \cdot (y^\circ - x^\circ)}\right],$$
$$y^\nu = \alpha^\nu y^\circ + (1 - \alpha^\nu) x^\circ.$$

For $\nu$ large, $\alpha^\nu$ is well defined. $y^\nu$ is chosen here so that it fulfills budget constraint and converges to $y^\circ$. We have $p^\nu \cdot y^\nu = p^\nu \cdot ((1 - \alpha^\nu)x^\circ + \alpha^\nu y^\circ) \le \tilde{M}^i(p^\nu)$. $\alpha^\nu \to 1$ as $\nu$ becomes large. By convexity of $X^i$ (C.III), $y^\nu \in X^i \cap \{x||x| \le c\}$. For $\nu$ large, $p^\nu \cdot x^\circ < p^\nu \cdot y^\circ$ and $p \cdot y^\nu \le \tilde{M}^i(p^\nu)$. So $y^\nu \in \tilde{B}^i(p^\nu) \cap X^i$ and $y^\nu \to y^\circ$. Hence the sequence $y^\nu$ demonstrates lower hemicontinuity of $\tilde{B}^i(p) \cap X^i$.

CASE 2 $p^\circ \cdot y^\circ = 0 < \tilde{M}^i(p^\circ)$ or

$$p^\circ \cdot y^\circ = \min_{x \in X^i \cap \{y|y \in \mathbf{R}^N, c \ge |y|\}} p^\circ \cdot x.$$

Once again we need to construct a sequence $y^\nu$ with the required convergence properties. In this case it is trivial. By continuity of the dot product, for large $\nu$, $p^\nu \cdot y^\circ < \tilde{M}^i(p^\nu)$. By hypothesis we have $y^\circ \in \tilde{B}^i(p^\circ) \cap X^i$. Thus we can set $y^\nu = y^\circ$; then for $\nu$ large, we have $y^\nu \in \tilde{B}^i(p^\nu) \cap X^i$ and hence $y^\nu \to y^\circ$ trivially.

Cases 1 and 2 exhaust the possibilities. In each case we have demonstrated the presence of sequence $y^\nu$, so that $y^\nu \in \tilde{B}^i(p^\nu) \cap X^i$ and $y^\nu \to y^\circ$. This is precisely what lower hemicontinuity of $\tilde{B}^i(p) \cap X^i$ requires.

QED

Theorem 24.2 demonstrates the continuity of the consumer's opportunity set $\tilde{B}^i(p) \cap X^i$ as a function of $p$. We are not really interested in $\tilde{B}^i(p) \cap X^i$ on its own. Rather, we are interested in the household demand behavior, $\tilde{D}^i(p)$. In order to apply the Kakutani Fixed-Point Theorem and find a general equilibrium we would like $\tilde{D}^i(p)$ to be upper hemicontinuous and convex valued. Upper hemicontinuity follows from Theorem 24.2 and the Maximum Theorem (Theorem 23.3). This is demonstrated in Theorem 24.3.

Theorem 24.3 Assume P.I - P.IV, C.I,C.II, C.III, C.V, and C.VII. Then $\tilde{D}^i(p)$ is an upper hemicontinuous nonnull correspondence for all $p \in P$.

Proof By Theorem 24.2 above, $\tilde{B}^i(p)$ is continuous with $\tilde{B}^i(p) \cap X^i$ nonempty, compact, continuous for all $p \in P$. By Theorem 12.1, $u^i(\cdot)$ is a continuous real-valued function. $\tilde{D}^i(p)$ is defined as the set of maximizers of $u^i(\cdot)$ on $\tilde{B}^i(p) \cap X^i$. Nonnullness follows since a continuous function achieves its maximum on a compact set. Upper hemicontinuity of $\tilde{D}^i(p)$ follows from the Maximum Theorem (Theorem 23.3). QED

Recall the convexity assumption

(C.VI(C)) $x \succ_i y$ implies $((1-\alpha)x + \alpha y) \succ_i y,$ for $0 < \alpha < 1$.

Under C.VI(C), we have convexity of $\tilde{D}^i(p)$. This is formalized as Theorem 24.4.

Theorem 24.4 Assume P.I - P.IV, C.I, C.II, C.III, C.V, C.VI(C), and C.VII. Then $\tilde{B}^i(p)$ and $\tilde{D}^i(p)$ are convex-valued.

Proof Exercise 24.3.

Under nonsatiation (C.IV), continuity (C.V), and convexity (C.VI(C)), given the geometry of $X^i$, we can rely on households spending all of their available income subject to constraint. This is the implication of Lemmas 24.4 and 24.5 below.

Lemma 24.4 Under C.I - C.V, C.VI(C), $x \in D^i(p)$ implies $p \cdot x = M^i(p)$.

Proof Exercise 24.4.

Lemma 24.5 Under C.I - C.V, C.VI(C), $x \in \tilde{D}^i(p)$ implies $p \cdot x \leq \tilde{M}^i(p)$. Further, if $p \cdot x < \tilde{M}^i(p)$, then $|x| = c$.

Proof Exercise 24.5. The proof follows from non-satiation, C.IV, and convexity C.VI(C). (See proof of Lemma 12.3.)

Lemma 24.6 Under P.I - P.IV, C.I - C.V, C.VI(C), and C.VII, $\tilde{D}^i(p)$ is upper hemicontinuous, convex, nonnull, and compact for all $p \in P$. If $M^i(p)$ is well defined and $M^i(p) = \tilde{M}^i(p)$, and if $x \in \tilde{D}^i(p)$ and $x$ is attainable, then $x \in D^i(p)$.

Proof Upper hemicontinuity follows from Theorem 23.2. Convexity follows from convexity of preferences (C.VI(C)) and convexity of $\tilde{B}^i(p)$ summarized in Theorem 24.4.

If $x \in \tilde{D}^i(p)$ and $x$ is attainable then $|x| < c$. Note the strict inequality. We now wish to show that $x \in D^i(p)$. Suppose not. Then there is $x' \in B^i(p) \cap X^i$ so that $x' \succ_i x$. But then by C.VI(C) convexity of preferences, for all $\alpha$, $0 < \alpha < 1$, $(1-\alpha)x + \alpha x' \succ_i \; x$. For $\alpha$ sufficiently small, then $(1-\alpha)x + \alpha x' \in \tilde{B}^i(p)$, but this is a contradiction since $x$ is the optimizer of $\succeq_i$ in $\tilde{B}^i(p)$. QED