Lecture Notes for January 13, 2015

An Economy with Unbounded Production Technology, Supply and Demand Correspondences

Principal characterization of firm and household behavior: maximization of a criterion function (profit or utility) subject to a constraint (technology or budget). Results in a well-defined outcome, a supply or demand function (point- or set-valued), if the criterion is a continuous function of its arguments and the constraint set is compact and hence bounded (Corollary 7.2). Unbounded production technology sets make sense and our theory should be able to deal with them; if a firm could acquire arbitrarily large inputs it would find it technically possible to produce arbitrarily large outputs. Scarcity — the limits of available inputs — should be communicated by prices, not by the modeler's assumptions. Price incentives should lead firms to choose finite inputs and outputs as an optimizing choice. On the household side, it should be prices, not an arbitrary constraint, that alert households that they cannot afford unbounded consumption.

Theory of production: The unbounded technology case

Production is organized in firms; these are represented by technology sets Y^j. The population of firms is the finite set F, indexed $j = 1, \ldots, \#F$. $Y^j \subseteq R^N$. The set Y^j represents the technical possibilities of firm j. $y \in Y^j$ is a possible combination of inputs and outputs. Negative coordinates of y are inputs; positive coordinates are outputs. For example, if $y \in Y^j$, $y = (-2, -3, 0, 0, 1)$, then an input of two units of good 1 and three units of good 2 will allow firm j to produce one unit of good 5. Y^j is like a list of recipes or a collection of blueprint plans for production, to be implemented as a matter of choice by the firm. There is no guarantee that the economy can provide the inputs $y \in Y^j$ specifies, either from endowment or from the output of other firms. Rather, $y \in Y^j$ represents the technical output possibilities of production by firm j if the specified inputs are provided.

Assumptions on Production Technology:

(P.I) Y^j is convex for each $j \in F$.
(P.II) $0 \in Y^j$ for each $j \in F$.
(P.III) Y^j is closed for each $j \in F$.

The aggregate technology set is $Y = \sum_{j \in F} Y^j$.

15.2 Boundedness of the attainable set

(P.IV) is designed as weak and economically meaningful technical assumptions under which a bounded attainable set is assured. P.IV(a) is the "no free lunch" postulate–there are no outputs without inputs. P.IV(b) is the irreversibility postulate–there exists no way to transform an output back to the original quantities of all inputs.

(P.IV)(a) if $y \in Y$ and $y \neq 0$, then $y_k < 0$ for some k.
(b) if $y \in Y$ and $y \neq 0$, then $-y \notin Y$.

P.IV is not an assumption about the individual firms; it treats the production sector of the whole economy. P.IV enunciates two quite reasonable sounding notions regarding production. P.IV(a) says we cannot expect outputs without inputs. There's no free lunch, a fundamental notion of scarcity appearing throughout economics. P.IV(b) says that production is irreversible. You can't unscramble an egg. You cannot take labor and capital to produce an output and then take the output and transform it back into labor and capital. Let $r \in \mathbf{R}^N_+$ be the vector of total initial resources or endowments. Finiteness of r and P.IV imply that there can never be an infinite production. We will demonstrate this below in Theorems 15.1 and 15.2.

Definition : Let $y \in Y$. Then y is said to be <u>attainable</u> if $y + r \geq 0$ (the inequality holds co-ordinatewise).

We will show that the set of attainable vectors y is bounded under P.I–P.IV.

In an attainable production plan $y \in Y$, $y = y^1 + y^2 + \ldots + y^{\#F}$, we have $y + r \geq 0$. But an individual firm's part of this plan, y^j , need not satisfy $y^j + r \geq 0$. Thus

Definition : We say that $y^j \in Y^j$ is <u>attainable in Y^j</u> if there exists a $y^k \in Y^k$ for each of the firms $k \in F$, $k \neq j$, such that $y^j + \sum_{k \in F, k \neq j} y^k$ is attainable.

y^j is attainable in Y^j if there is a plan for firm j and for all of the other firms in the economy so that, with available inputs, there is an attainable output for the economy as a whole, consistent with firm j producing y^j. We wish to show, in Theorem 15.1 below, that this definition and P.I–P.IV imply boundedness for the set of plans y^j attainable in Y^j.

Here is the strategy of proof. The argument is by contradiction. We use the convexity of Y and each Y^j to concentrate on a subset of Y^j (for suitably chosen j) contained in a sphere of radius 1. How could there be an attainable plan in Y^j that is unbounded? We will show that this could occur only in two possible ways: Either firm j could be producing outputs without inputs (contradicting P.IV(a)) or firm j's unbounded production plan could be partly reversed by the plans of the other firms, so that the net effect is a bounded attainable sum even though there is an unbounded attainable sequence in Y^j. We map back into a bounded set and take a limit–using both convexity and closedness of Y^j. Then, in the limit, it follows that other firms' production plans precisely reverse those of firm j. But this contradicts the assumption of irreversibility, P.IV(b). The contradiction completes the proof.

Lemma 15.1 : Assume P.II and P.IV. Let $y = \sum_{j \in F} y^j$, $y^j \in Y^j$ for all $j \in F$. Let $y \in Y$, $y = \mathbf{0}$. Then $y^j = \mathbf{0}$ for all $j \in F$.

Proof Let $k \in F$. By P.II,

$$\sum_{j \in F, j \neq k} y^j \in Y, \quad \text{and} \quad y^k \in Y.$$

But

$$y^k + \sum_{j \in F, j \neq k} y^j = \mathbf{0}.$$

So

$$y^k = -\sum_{j \in F, j \neq k} y^j.$$

But under P.IV(b), this occurs only if

$$\mathbf{0} = y^k = -\sum_{j \in F, j \neq k} y^j = \mathbf{0}.$$

But this holds for all $k \in F$. QED

QED

Theorem 15.1 : For each $j \in F$, under P.I, P.II, P.III, and P.IV, the set of vectors attainable in Y^j is bounded.

Proof : We will use a proof by contradiction. Suppose contrary to the theorem that the set of vectors attainable in $Y^{j'}$ is not bounded for some $j' \in F$. Then, for each $j \in F$, there exists a sequence $\{y^{\nu j}\} \subset Y^j, \nu = 1, 2, 3, \ldots,$ such that:

(1) $|y^{\nu j'}| \to +\infty$, for some $j' \in F$,
(2) $y^{\nu j} \in Y^j$, for all $j \in F$, and
(3) $y^\nu = \sum_{j \in F} y^{\nu j}$ is attainable; that is, $y^\nu + r \geq 0$.

We show that this contradicts P.IV. Recall P.II, $0 \in Y^j$, for all j. Let $\mu^\nu = \max_{j \in F} |y^{\nu j}|$. For ν large, $\mu^\nu \geq 1$. By (1) we have $\mu^\nu \to +\infty$. Consider the sequence $\tilde{y}^{\nu j} \equiv \frac{1}{\mu^\nu} y^{\nu j} = \frac{1}{\mu^\nu} y^{\nu j} + (1 - \frac{1}{\mu^\nu})0$. By P.I, $\tilde{y}^{\nu j} \in Y^j$. Let $\tilde{y}^\nu = \frac{1}{\mu^\nu} y^\nu = \sum_{j \in F} \tilde{y}^{\nu j}$. By (3) and P.I we have

(4) $\tilde{y}^{\nu} + \frac{1}{\mu^{\nu}} r \geq 0$.

The sequences $\tilde{y}^{\nu j}$ and $\tilde{y}^{\nu}$ are bounded ($\tilde{y}^{\nu}$ as the finite sum of vectors of length less than or equal to 1). Without loss of generality, take corresponding convergent subsequences so that $\tilde{y}^{\nu} \to \tilde{y}^{\circ}$ and $\tilde{y}^{\nu j} \to \tilde{y}^{\circ j}$ for each j, and $\sum_j \tilde{y}^{\nu j} \to \sum_j \tilde{y}^{\circ j} = \tilde{y}^{\circ}$. Of course, $\frac{1}{\mu^{\nu}} r \to 0$. Taking the limit of (4), we have

$$\tilde{y}^{\circ} + 0 = \sum_{j \in F} \tilde{y}^{\circ j} + 0 \geq 0 \text{ (the inequality holds co-ordinatewise) .}$$

By P.III, $\tilde{y}^{\circ j} \in Y^j$, so $\sum_{j \in F} \tilde{y}^{\circ j} = \tilde{y}^{\circ} \in Y$. But, by P.IV(a), we have that $\sum_{j \in F} \tilde{y}^{\circ j} = 0$. Lemma 15.1 says then that $\tilde{y}^{\circ j} = \mathbf{0}$ for all j, so $|\tilde{y}^{\circ j}| \neq 1$.

The contradiction proves the theorem. QED

We have shown that under P.I–P.IV, the set of production plans attainable in Y^j is bounded. We can now conclude that the attainable subset of Y is compact (closed and bounded).

Theorem 15.2 : Under P.I–P.IV, the set of attainable vectors in Y is compact, that is, closed and bounded.

Proof We will demonstrate the result in two steps.

Boundedness: $y \in Y$ attainable implies $y = \sum_{j \in F} y^j$ where $y^j \in Y^j$ is attainable in Y^j. However, by Theorem 15.1, the set of such y^j is bounded for each j. Attainable y then is the sum of a finite number ($\#F$) of vectors, y^j, each taken from a bounded subset of Y^j, so the set of attainable

y in Y is also bounded.

Closedness: Consider the sequence $y^\nu \in Y$, y^ν attainable, $\nu = 1, 2, 3, \ldots$. We have $y^\nu + r \geq 0$. Suppose $y^\nu \to y^\circ$. We wish to show that $y^\circ \in Y$ and that y° is attainable. We write the sequence as $y^\nu = y^{\nu 1} + y^{\nu 2} + \ldots + y^{\nu j} + \ldots + y^{\nu \# F}$, where $y^{\nu j} \in Y^j$, $y^{\nu j}$ attainable in Y^j for all $j \in F$.

Since the attainable points in Y^j constitute a bounded set (by Theorem 15.1), without loss of generality, we can find corresponding convergent subsequences $y^\nu, y^{\nu 1}, y^{\nu 2}, \ldots, y^{\nu j}, \ldots, y^{\nu \# F}$ so that for all $j \in F$ we have $y^{\nu j} \to y^{\circ j} \in Y^j$, by P.III. We have then $y^\circ = y^{\circ 1} + y^{\circ 2} + \ldots + y^{\circ j} + \ldots + y^{\circ \# F}$ and $y^\circ + r \geq 0$. Hence, $y^\circ \in Y$ and y° is attainable. QED

15.3 An artificially bounded supply function

We wish to describe firm supply behavior as profit maximization subject to technology constraint. Since Y^j may not be bounded, maximizing behavior may not be well defined. However, we have shown above that attainable production plans do lie in a bounded set. We can, of course, describe well-defined profit-maximizing behavior subject to technology and boundedness constraints , where the bound includes all attainable plans. Eventually, we will wish to eliminate the boundedness constraint–not because we are interested in firms producing at unattainable levels but rather because the resource constraints that define attainability should be communicated to firms in prevailing prices rather than in an additional constraint on firm behavior.

Assume P.I, P.II, P.III, and P.IV. Choose a positive real number c, sufficiently large so that for all $j \in F$, $|y^j| < c$ (a strict inequality) for all y^j attainable in Y^j. Let $\tilde{Y}^j = Y^j \cap \{y \in \mathbf{R}^N \| y| \leq c\}$. Note the weak inequality in the definition of $\tilde{Y}^j$ and the strong inequality in the definition of c. That combination means that $\tilde{Y}^j$ includes all of the points attainable in Y^j and a surrounding band of larger points in Y^j that are too big to be attainable. Note that $\tilde{Y}^j$ is closed, bounded (hence compact), and convex. Restricting attention to $\tilde{Y}^j$ in describing firm j's production plans allows us to remain in a bounded set so that profit maximization will be well defined. A typical artificially bounded technology set, $\tilde{Y}^j$, is depicted in Figure 15.1.

24.2 Production with a (weakly) convex production technology

We will show that supply behavior of the firm is convex-set-valued (possibly including the empty set, ϕ) when the production technology is convex but not strictly convex. This includes the cases of constant returns to scale, linear production technology, and perfect substitutes among inputs to production. In each of these cases there may be a (linear) range of equally profitable production plans differing by scale of output or by the input mix. The purpose of developing a theory of set-valued supply behavior is to accommodate this range of indeterminacy.

<u>Supply correspondence with a weakly convex production technology</u>: Under P.I–P.IV profit maximization for firm j may yield no solution, a point-valued solution, or a convex-set-valued so-

lution.

Define the restricted supply correspondence of firm j as

$$\tilde{S}^j(p) = \{y^{*j} \mid y^{*j} \in \tilde{Y}^j, p \cdot y^{*j} \geq p \cdot y^j \text{ for all } y^j \in \tilde{Y}^j\}.$$

Define the (unrestricted) supply correspondence of firm j as

$$S^j(p) = \{y^{*j} \mid y^{*j} \in Y^j, p \cdot y^{*j} \geq p \cdot y \text{ for all } y \in Y^j\}.$$

Taking price vector $p \in \mathbf{R}^N_+, p \neq 0$, as given, each firm j "chooses" y^j in Y^j. Profit maximization guides the choice of y^j. Firm j chooses y^j to maximize $p \cdot y$ subject to $y \in Y^j$. We will consider two cases:

- the restricted supply correspondence where the supply behavior of firm j is required to be in the compact convex set $\tilde{Y}^j \subseteq Y^j$, which includes the plans attainable in Y^j as a proper subset, and
- the unrestricted supply correspondence where the only requirement is that the chosen supply behavior lie in Y^j. Of course, Y^j need not be compact. Hence, in this case, profit-maximizing supply behavior may not be well defined. Further, Y^j may include unattainable production plans. When the profit-maximizing production plan is unattainable, it cannot, of course, be fulfilled and cannot represent a market equilibrium.

Recall Theorems 15.1 and 15.2. They demonstrated that under assumptions P.I, P.II, P.III, and P.IV the set of attainable production plans for the economy and for firm j

were bounded. We then defined $\tilde{Y}^j$ as the bounded subset of Y^j containing production plans of Euclidean length c or less, where c was chosen as a strict upper bound on all attainable plans in Y^j. That is, choose c such that $|y^j| < c$ (a strict inequality) for y^j attainable in Y^j. Let $\tilde{Y}^j = Y^j \cap \{y||y| \leq c\}$. Note the weak inequality in the definition of $\tilde{Y}^j$. Restricting attention to $\tilde{Y}^j$ in describing firm j's production plans allows us to remain in a bounded set so that profit maximization will be well defined. Note that $\tilde{Y}^j$ is nonempty, closed, bounded (hence compact), and convex.

Lemma 24.1 Under P.I–P.IV, $\tilde{S}^j(p)$ is convex (a convex set).

Proof Let $y^1 \in \tilde{S}^j(p)$ and $y^2 \in \tilde{S}^j(p)$. For fixed $p, p \cdot y^1 = p \cdot y^2 \geq p \cdot y$ for all $y \in Y^j$. For $0 \leq \lambda \leq 1$, consider

$$p \cdot [\lambda y^1 + (1-\lambda)y^2] = \lambda p \cdot y^1 + (1-\lambda)p \cdot y^2 = p \cdot y^2 \geq p \cdot y$$

for all $y \in Y^j$.

But $(\lambda y^1 + (1-\lambda)y^2) \in Y^j$ by P.I. QED

Lemma 24.2 Under P.I–P.IV, $\tilde{S}^j(p)$ is nonempty and upper hemicontinuous for all $p \in \mathbf{R}^N_+, p \neq 0$.

Proof The set $\tilde{S}^j(p)$ consists of the maximizers of a continuous real-valued function on a compact set. The maximum is hence well defined and the set is nonempty.

Upper hemicontinuity follows from the Maximum Theorem (Theorem 23.3). To demonstrate upper hemicontinuity

directly, let $p^\nu \to p^\circ; p^\nu, p^\circ \in \mathbf{R}^N_+; p^\nu, p^\circ \neq 0; \nu = 1, 2, \ldots;$ and $y^\nu \in \tilde{S}^j(p^\nu), y^\nu \to y^\circ$.
We must show that $y^\circ \in \tilde{S}^j(p^\circ)$. Suppose not. Then there is $y' \in \tilde{Y}^j$ so that $p^\circ \cdot y' > p^\circ \cdot y^\circ$. The dot product is a continuous function:

$$p^\nu \cdot y' \to p^\circ \cdot y'$$
$$p^\nu \cdot y^\nu \to p^\circ \cdot y^\circ.$$

Therefore, for ν sufficiently large, $p^\nu \cdot y' > p^\nu \cdot y^\nu$. But this contradicts the definition of $\tilde{S}(p^\nu)$. The contradiction proves the lemma. QED

Lemma 24.3 (homogeneity of degree 0) Assume P.I–P.IV. Let $\lambda > 0$, $p \in \mathbf{R}^N_+$. Then $\tilde{S}^j(\lambda p) = \tilde{S}^j(p)$ and $S^j(\lambda p) = S^j(p)$.

Proof Exercise 24.1.

Under Lemma 24.3 only the relative prices matter, and not their numerical values. Hence without loss of generality we can represent price vectors restricted to the unit simplex in $\mathbf{R}^\mathbf{N}$. The unit simplex in $\mathbf{R}^N$, is

$P = \left\{ p \mid p \in R^N, p_i \geq 0, i = 1, \ldots, N, \Sigma_{i=1}^N p_i = 1 \right\}.$

Firm j's profit function is

$\pi^j(p) = \max_{y \in Y^j} p \cdot y = p \cdot y$ for $y \in S^j(p)$.

Note that $\pi^j(p)$ may not be well defined (may not exist) for some values of p. This reflects that since $\pi^j(p)$ is defined as the maximum of a real-valued function on the domain Y^j, a well-defined value of $\pi^j(p)$ depends on that maximum existing. Since Y^j may not be compact, the maximum may not exist.

Considering the artificially restricted firm technology sets $\tilde{Y}^j$, it is convenient to have a concept of the profit function for the firm so restricted,

$$\tilde{\pi}^j(p) = \max_{y \in \tilde{Y}^j} p \cdot y = p \cdot y \text{ for } y \in \tilde{S}^j(p).$$

Theorem 24.1 : Assume P.I - P.IV. Let $p \in \mathbf{P}$. Then

(a) $\tilde{S}^j(p)$ is an upper hemicontinuous correspondence throughout P. For each $p, \tilde{S}^j(p)$ is closed, convex, bounded, and nonnull;
(b) $\tilde{\pi}^j(p)$ is a well-defined continuous function for all $p \in \mathbf{P}$;
(c) if y^j is attainable in Y^j and $y^j \in \tilde{S}^j(p)$, then $y^j \in S^j(p)$.

Proof Part (a). Upper hemicontinuity and nonemptiness are established in Lemma 24.2. $\tilde{S}^j(p)$ is bounded since $\tilde{Y}^j$ is bounded. Closedness follows from upper hemicontinuity. Convexity is established in Lemma 24.1.

Part (b): For each $p \in \mathbf{P}$, $\tilde{S}^j(p)$ is nonempty and for any two $y', y'' \in \tilde{S}^j(p)$, $p \cdot y' = p \cdot y'' = \tilde{\pi}^j(p)$, so $\tilde{\pi}^j(p)$ is well defined. Let $p^\nu \in \mathbf{P}$, $\nu = 1, 2, ...,$ $p^\nu \to p^o$. Let $y^\nu \in \tilde{S}^j(p^\nu)$. Without loss of generality — since $\tilde{Y}^j$ is compact — let $y^\nu \to y^o$. The dot product is a continuous function of its arguments so $\tilde{\pi}^j(p^\nu) = p^\nu \cdot y^\nu \to p^o \cdot y^o = \tilde{\pi}^j(p^o)$. Thus $\tilde{\pi}^j(p)$ is continuous throughout P.

Part (c): Proof by contradiction. Suppose y^j attainable and $y^j \in \tilde{S}^j(p)$ but $y^j \notin S^j(p)$. Then there is $\hat{y}^j \in Y^j$ so that $p \cdot \hat{y}^j > p \cdot y^j$. Furthermore,

$$p \cdot [\alpha\hat{y}^j + (1-\alpha)y^j] > p \cdot y^j \text{ for any } \alpha, 0 < \alpha \leq 1.$$

But for α sufficiently small,

$$|\alpha\hat{y}^j + (1-\alpha)y^j| \leq c,$$

so that

$$\alpha\hat{y}^j + (1-\alpha)y^j \in \tilde{Y}^j.$$

But then $p \cdot (\alpha\hat{y}^j + (1-\alpha)y^j) > p \cdot y^j$ and $\alpha\hat{y}^j + (1-\alpha)y^j \in \tilde{Y}^j$; thus y^j is not the maximizer of $p \cdot y$ in $\tilde{Y}^j$ and $y^j \notin \tilde{S}^j(p)$ as was assumed. The contradiction proves the theorem. QED