Lecture Notes for Econ 200B, January 8, 2015

See Chapter 5 of General Equilibrium Theory: An Introduction, 2nd ed.

N goods in the economy.

A typical array of prices is an N-dimensional vector

$$p = (p_1, p_2, p_3, \ldots, p_{N-1}, p_N) = (3, 1, 5, \ldots, 0.5, 10).$$

Assume only relative prices (price ratios) matter here, not the numerical values of prices. This is essentially assuming that there is no money, no monetary instrument held as wealth in which prices are denominated.

The price space: The unit simplex in $\mathbf{R}^N$, is

$$P = \left\{ p \mid p \in R^N, p_i \geq 0, i = 1, \ldots, N, \sum_{i=1}^{N} p_i = 1 \right\}. \quad (5.1)$$

The unit simplex is a (generalized) triangle in N-space. It's called "unit" because the co-ordinates add to 1. It's a "simplex" because it has that generalized triangle specification.

For each household $i \in H$, we define a demand function, $D^i : P \to \mathbf{R}^N$.

For each firm $j \in F$, a supply function, $S^j : P \to \mathbf{R}^N$.

Positive co-ordinates in $S^j(p)$ are outputs, negative co-ordinates are inputs.
$p \cdot S^j(p) \equiv \sum_{n=1}^{N} p_n S^j_n(p) \equiv$ profits of firm j.

The economy has an initial endowment of resources $r \in \mathbf{R}^N_+$ that is also supplied to the economy.

The market excess demand function is defined as

$$Z(p) = \sum_{i \in H} D^i(p) - \sum_{j \in F} S^j(p) - r, \tag{5.2}$$

$$Z : P \to \mathbf{R}^N \tag{5.3}$$

$Z(p) \equiv (Z_1(p), Z_2(p), Z_3(p), \ldots, Z_N(p))$, where $Z_k(p)$ is the excess demand for good k. When $Z_k(p)$, the excess demand for good k, is negative, we will say that good k is in excess supply.

There are two principal assumptions: Walras's Law and Continuity of $Z(p)$:

Walras's Law: For all $p \in P$,

$$p \cdot Z(p) = \sum_{n=1}^{N} p_n \cdot Z_n(p) = \sum_{i \in H} p \cdot D^h(p) - \sum_{j \in F} p \cdot S^j(p) - p \cdot r = 0.$$

The economic basis for Walras's Law involves the assumption of scarcity and the structure of household budget constraints. $\sum_{i \in H} p \cdot D^h(p)$ is the value of aggregate household expenditure. The term $\sum_{j \in F} p \cdot S^j(p) + p \cdot r$ is the value of aggregate household income (value of firm profits plus the value of endowment). Walras's Law says that expenditure equals income.

Continuity:

$Z : P \to \mathbf{R}^N, Z(p)$ is a continuous function for all $p \in P$.

That is, small changes in p result in small changes in $Z(p)$ everywhere in P.

We assume in this lecture that $Z(p)$ is well defined and fulfills Walras's Law and Continuity. As mathematical theorists, part of our job is to derive these properties from more elementary properties during the next few weeks (so that we can be sure of their generality).

Definition : $p^o \in P$ is said to be an equilibrium price vector if $Z(p^o) \leq 0$ (0 is the zero vector; the inequality applies coordinatewise) with $p^o_k = 0$ for k such that $Z_k(p^o) < 0$. That is, p^o is an equilibrium price vector if supply equals demand in all markets (with possible excess supply of free goods).

Theorem 5.1 (Brouwer Fixed-Point Theorem) : Let $f(\cdot)$ be a continuous function, $f : P \to P$. Then there is $x^* \in P$ so that $f(x^*) = x^*$.

Theorem 5.2 : Let Walras's Law and Continuity be fulfilled. Then there is $p^* \in P$ so that p^* is an equilibrium.

Proof

Let $T : P \to P$, where $T(p) = (T_1(p), T_2(p), \ldots, T_k(p), \ldots, T_N(p))$. $T_k(p)$ is the adjusted price of good k, adjusted by the auctioneer trying to bring supply and demand into balance. Let $\gamma^k > 0$. The adjustment process of the kth price can be represented as $T_k(p)$, defined as follows:

$$T_k(p) \equiv \frac{\max[0, p_k + \gamma^k Z_k(p)]}{\sum_{n=1}^{N} \max[0, p_n + \gamma^n Z_n(p)]}. \tag{5.4}$$

The function T is a price adjustment function. It raises the relative price of goods in excess demand and reduces the price of goods in excess supply while keeping the price vector on the simplex. In order for T to be well defined, the denominator must be nonzero, that is,

$$\sum_{n=1}^{N} \max[0, p_n + \gamma^n Z_n(p)] \neq 0. \tag{5.5}$$

(5.5)follows from Walras's Law. For the sum in the denominator to be zero or negative, all goods would have to be in excess supply simultaneously, which is contrary to our notions of scarcity and– it turns out– to Walras's Law as well.

Suppose, contrary to (5.5), $\sum_{n=1}^{N} \max[0, p_n + \gamma^n Z_n(p)] = 0$. Then $p_n + \gamma^n Z_n(p) \leq 0$ all $n = 1, \ldots, N$, and for each $p_n > 0$ we have $Z_n(p) < 0$. Then $\sum_{n=1}^{N} p_n Z_n(p) < 0$. But Walras's Law says $\sum_{n=1}^{N} p_n Z_n(p) = 0$. The contradiction proves (5.5).

Recall that $Z(\cdot)$ is a continuous function. The operations of max[ ], sum, and division by a nonzero continuous function maintain continuity. Hence, $T(p)$ is a continuous function from the simplex into itself.

By the Brouwer Fixed-Point Theorem there is $p^* \in P$ so that $T(p^*) = p^*$.

We must show that p^* is not just the stopping point of the price adjustment process, but that it actually does represent general equilibrium prices for the economy.

Since $T(p^*) = p^*$, for each good k, $T_k(p^*) = p_k^*$. That is, for all $k = 1, \ldots, N$,

$$p_k^* = \frac{\max[0, p_k^* + \gamma^k Z_k(p^*)]}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n Z_n(p^*)]}. \tag{5.6}$$

For each k, either

$$p_k^* = 0 \qquad \text{(Case 1)} \tag{5.7}$$

or

$$p_k^* = \frac{p_k^* + \gamma^k Z_k(p^*)}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n Z_n(p^*)]} > 0 \qquad \text{(Case 2)}. \tag{5.8}$$

CASE 1 $p_k^* = 0 = \max[0, p_k^* + \gamma^k Z_k(p^*)]$. Hence, $0 \geq p_k^* + \gamma^k Z_k(p^*) = \gamma^k Z_k(p^*)$ and $Z_k(p^*) \leq 0$. This is the case of free goods with market clearing or with excess supply in equilibrium.

CASE 2 To avoid repeated messy notation, let

$$\lambda = \frac{1}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n Z_n(p^*)]} \tag{5.9}$$

so that $T_k(p^*) = \lambda(p_k^* + \gamma^k Z_k(p^*))$. Note that $\lambda > 0$, by the argument demonstrating (5.5). Since p^* is the fixed point of T we have $p_k^* = \lambda(p_k^* + \gamma^k Z_k(p^*)) > 0$. This expression is true for all k with $p_k^* > 0$, and λ is the same for all k. Let's perform some algebra on this expression. We first combine terms in p_k^*:

$$(1 - \lambda)p_k^* = \lambda\gamma^k Z_k(p^*), \tag{5.10}$$

then multiply through by $Z_k(p^*)$ to get

$$(1 - \lambda)p_k^* Z_k(p^*) = \lambda\gamma^k (Z_k(p^*))^2, \tag{5.11}$$

and now sum over all k in Case 2, obtaining

$$(1 - \lambda) \sum_{k \in \text{Case2}} p_k^* Z_k(p^*) = \lambda \sum_{k \in \text{Case2}} \gamma^k (Z_k(p^*))^2. \tag{5.12}$$

Walras's Law says

$$0 = \sum_{k=1}^{N} p_k^* Z_k(p^*) = \sum_{k \in \text{Case1}} p_k^* Z_k(p^*) + \sum_{k \in \text{Case2}} p_k^* Z_k(p^*). \tag{5.13}$$

But for $k \in$ Case 1, $p_k^* Z_k(p^*) = 0$, and so

$$0 = \sum_{k \in \text{Case1}} p_k^* Z_k(p^*). \tag{5.14}$$

Therefore,

$$\sum_{k \in \text{Case2}} p_k^* Z_k(p^*) = 0. \tag{5.15}$$

Hence, from (5.11) we have

$$0 = (1-\lambda)\cdot \sum_{k\in\text{Case2}} p_k^* Z_k(p^*) = \lambda\cdot \sum_{k\in\text{Case2}} \gamma^k (Z_k(p^*))^2. \quad (5.16)$$

Using Walras's Law, we established that the left-hand side equals 0, but the right-hand side can be zero only if $Z_k(p^*) = 0$ for all k such that $p_k^* > 0$ (k in Case 2). Thus, p^* is an equilibrium. This concludes the proof.

QED

Mathematics: Analysis of point to set mappings

23.1 Correspondences

We will call a point-to-set mapping a correspondence. Let A and B be nonempty sets. For each $x \in A$ we associate a nonempty set $\beta \subset B$ by a rule φ. Then we say $\beta = \varphi(x)$ and φ is a correspondence; $\varphi : A \to B$. Note that if $x \in A$ and $y \in B$ it is meaningless or false to say $y = \varphi(x)$, rather we say $y \in \varphi(x)$. The graph of the correspondence is a subset of $A \times B : \{(x, y) \mid x \in A, y \in B \text{ and } y \in \varphi(x)\}$.

23.2 Upper hemicontinuity (also known as upper semicontinuity)

Definition Let $\varphi : S \to T$, φ be a correspondence, and S and T be closed subsets of $\mathbf{R}^N$ and $\mathbf{R}^K$, respectively. Let $x^\nu, x^\circ \in S, \nu = 1, 2, 3, \ldots$; let $x^\nu \to x^\circ$, $y^\nu \in \varphi(x^\nu)$, for all $\nu = 1, 2, 3, \ldots$, and $y^\nu \to y^\circ$. Then φ is said to be ***upper hemicontinuous*** (also known as upper semicontinuous) at x° if and only if $y^\circ \in \varphi(x^\circ)$.

Example 23.1 An upper hemicontinuous correspondence. Let $\varphi(x)$ be defined as follows. $\varphi : \mathbf{R} \to \mathbf{R}$. For

$$x < 0, \varphi(x) = \{y \mid x - 4 \leq y \leq x - 2\}$$
$$x = 0, \varphi(x) = \{y \mid -4 \leq y \leq +4\}$$
$$x > 0, \varphi(x) = \{y \mid x + 2 \leq y \leq x + 4\}.$$

Note that $\varphi(\cdot)$ is convex valued. For each $x \in \mathbf{R}, \varphi(x)$ is a convex set.

Example 23.2 A correspondence not upper hemicontinuous at 0. Let $\varphi(x)$ be defined much as in Example 23.1 but with a discontinuity at 0. $\varphi : \mathbf{R} \to \mathbf{R}$. For

$$x < 0, \varphi(x) = \{y \mid x - 4 \leq y \leq x - 2\}$$
$$x = 0, \varphi(0) = \{0\}$$
$$x > 0, \varphi(x) = \{y \mid x + 2 \leq y \leq x + 4\}.$$

Note that $\varphi(\cdot)$ is convex valued. For each $x \in \mathbf{R}, \varphi(x)$ is a convex set.

Theorem 23.1 φ is upper hemicontinuous if and only if its graph is closed in $S \times T$.

23.3 Lower hemicontinuity (also known as lower semicontinuity)

Definition Let $\varphi : S \to T$, where S and T are closed subsets of $\mathbf{R}^N$ and $\mathbf{R}^K$, respectively. Let $x^\nu \in S$, $x^\nu \to x^\circ, y^\circ \in \varphi(x^\circ), q = 1, 2, 3, \ldots$. Then φ is said to be lower hemicontinuous (also known as lower semicontinuous) at x° if and only if there is $y^\nu \in \varphi(x^\nu),\ y^\nu \to y^\circ$. Lower hemicontinuity asserts the presence of a sequence of points in the correspondence evaluated at a convergent sequence of points in the domain.

Intuitively, φ is lower hemicontinuous if it has caught a value, φ must be able to sneak up on it.

Example 23.3 A lower hemicontinuous correspondence. Let $\varphi(x)$ be defined as follows. $\varphi : \mathbf{R} \to \mathbf{R}$. For

$$x \neq 0,\ \varphi(x) = \{y \mid x - 4 \leq y \leq x\}$$
$$x = 0,\ \varphi(x) = \{y \mid -3 \leq y \leq -1\}.$$

The graph of $\varphi(\cdot)$ is shown in Figure 23.4. Note that $\varphi(\cdot)$ is convex valued. For each $x \in \mathbf{R}$, $\varphi(x)$ is a convex set. For all $x^\circ \in \mathbf{R}$, $\varphi(\cdot)$ is lower hemicontinuous at x°. The only point where this requires some care is at $x^\circ = 0$. Let $x^\nu \to 0,\ y^\circ \in \varphi(0)$. To demonstrate lower hemicontinuity, we must show that there is $y^\nu \in \varphi(x^\nu)$ so that $y^\nu \to y^\circ$. Note that $-3 \leq y^\circ \leq -1$. But for ν large, there is $y^\nu \in \varphi(x^\nu)$, so that $y^\nu = y^\circ$. Hence, trivially, $y^\nu \to y^\circ$. Note that $\varphi(\cdot)$ is not upper hemicontinuous at $x^\circ = 0$. This follows simply because $y = -4$ is the limit of a sequence of values in $\varphi(x^\nu)$ but $-4 \notin \varphi(0)$.

Example 23.4 An upper hemicontinuous correspondence that is not lower hemicontinuous. This example is merely Examples 23.1 and 23.2 revisited. $\varphi(\cdot)$ in both Examples 23.1 and 23.2 is not lower hemicontinuous at $x^\circ = 0$. In both cases $0 \in \varphi(0)$ but for a typical sequence $x^\nu \to 0$, there is no $y^\nu \in \varphi(x^\nu)$ so that $y^\nu \to 0$.

23.4 Continuous correspondence

Definition Let $\varphi : A \to B$, with φ a correspondence. $\varphi(\cdot)$ is said to be continuous at x° if $\varphi(\cdot)$ is both upper and lower hemicontinuous at x°.

Example 23.5 A continuous correspondence. The following correspondence, $\varphi(\cdot)$, is both upper and lower hemicontinuous throughout its range and hence is a continuous correspondence. For

$$\begin{aligned}
&x < 0, \varphi(x) = \{y \mid 2x \le y \le -x\} \\
&x = 0, \varphi(x) = \{0\} \\
&x > 0, \varphi(x) = \{y \mid -2x \le y \le -x\} \cup \{y \mid 3x \le y \le 4x\}.
\end{aligned}$$

Note that if φ is point valued (i.e., a function) with a compact range then upper hemicontinuity, continuity (in the sense of a function), and lower hemicontinuity are equivalent.

23.6 Optimization subject to constraint: Composition of correspondences; the Maximum Theorem

We formalize this notion in the following way. Let $f(\cdot)$ be a real-valued function, and let $\varphi(\cdot)$ be a correspondence intended to represent an opportunity set. Then we let $\mu(\cdot)$ represent the correspondence consisting of the maximizers of $f(\cdot)$ subject to choosing the maximizer in the opportunity set $\varphi(\cdot)$. Formally, we state

The Maximum Problem Let $T \subseteq \mathbf{R}^N, S \subseteq \mathbf{R}^M, f : T \to \mathbf{R}$, and $\varphi : S \to T$, where φ is a correspondence, and let $\mu : S \to T$, where $\mu(x) \equiv \{y^\circ \mid y^\circ$ maximizes $f(y)$ for $y \in \varphi(x)\}$.

Theorem 23.3 (The Maximum Theorem) Let $f(\cdot), \varphi(\cdot)$, and $\mu(\cdot)$ be as defined in the Maximum Problem. Let f be continuous on T and let φ be continuous (both upper and lower hemicontinuous) at x° and compact-valued in a neighborhood of x°. Then μ is upper hemicontinuous at x°.

Example 23.6 Applying the Maximum Theorem. Let $S = T = \mathbf{R}$. Let $f(y) = y^2$. Let

$$\varphi(x) = \{y \mid -x \leq y \leq x\} \text{ for x} \geq 0$$
$$\varphi(x) = \{y \mid x \leq y \leq -x\} \text{ for x} < 0.$$

Then $\mu(x) = \{x, -x\}$, since $\mu(x)$ is the set of maximizers of y^2 for $y \in \varphi(x)$. Note that $\varphi(x)$ is both upper and lower hemicontinuous throughout $\mathbf{R}$ and is convex valued. $\mu(x)$ is upper hemicontinuous by the Maximum Theorem. It is not, however, convex valued.

23.7 Kakutani Fixed-Point Theorem

Theorem 23.4 (Kakutani Fixed-Point Theorem) Let S be an N-simplex. Let $\varphi : S \to S$ be a correspondence that is upper hemicontinuous everywhere on S. Further, let $\varphi(x)$ be a convex set for all $x \in S$. Then there is $x^* \in S$ so that $x^* \in \varphi(x^*)$.

Example 23.7 Applying the Kakutani Fixed-Point Theorem. Let $\varphi : [0, 1] \to [0, 1]$. Let

$$\varphi(x) = \{1 - x/2\} \text{ for } 0 \leq x < .5$$
$$\varphi(0.5) = [.25, .75]$$
$$\varphi(x) = \{x/2\} \text{ for } 1 \geq x > .5,$$

where φ is upper hemicontinuous and convex valued. The fixed point is $x^\circ = 0.5$. (See Figure 23.10.)

Corollary 23.1 Let $K \subseteq \mathbf{R}^M, K \neq \emptyset$, be compact and convex. Let $\Psi : K \to K$, with $\Psi(x)$ upper hemicontinuous and convex valued for all $x \in K$. Then there is $x^* \in K$ so that $x^* \in \Psi(x^*)$.