

Lecture Notes for February 3: Convergence of the core of a large economy

22.1 Replication; a large economy

We will treat a Q -fold replica economy, denoted Q - H . Q will be a positive integer; $Q = 1, 2, \dots$. In a Q -fold replica economy we take an economy consisting of households $i \in H$, with endowments r^i and preferences \succeq_i , and create a similar larger economy with Q times as many agents in it, totaling $\#H \times Q$ agents. There will be Q agents with preferences \succeq_1 and endowment r^1 , Q agents with preferences \succeq_2 and endowment r^2, \dots , and Q agents with preferences $\succeq_{\#H}$ and endowment $r^{\#H}$. Each household $i \in H$ now corresponds to a household type. There are Q individual households of type i in the replica economy Q - H . Note that the competitive equilibrium prices in the original H economy will be equilibrium prices of the Q - H economy. Household i 's competitive equilibrium allocation x^i in the original H economy will be a competitive equilibrium allocation to all type i households in the Q - H replica economy. Agents in the Q - H replica economy will be denoted by their type and a serial number. Thus, the agent denoted i, q will be the q th agent of type i , for each $i \in H, q = 1, 2, \dots, Q$.

22.2 Equal treatment

Theorem 22.1 (Equal treatment in the core) Assume C.IV*, C.V, and C.VI(SC). Let $\{x^{i,q}, i \in H, q = 1, \dots, Q\}$ be in the core of Q - H , the Q -fold replica of economy H . Then for each $i, x^{i,q}$ is the same for all q . That is, $x^{i,q} = x^{i,q'}$ for each $i \in H, q \neq q'$.

22.3 Core convergence in a large economy

Theorem 8.1, Bounding Hyperplane Theorem (Minkowski) Let K be convex, $K \subseteq \mathbf{R}^N$. There is a hyperplane H through z and bounding for K if z is not interior to K . That is, there is $p \in \mathbf{R}^N, p \neq 0$, so that for each $x \in K, p \cdot x \geq p \cdot z$.

Theorem 22.2 (Debreu-Scarff) Assume C.IV*, C.V, C.VI(SC). Let $X^i = \mathbf{R}_+^N$ and $r^i \gg 0$ for all $i \in H$. Let $\{x^{oi}, i \in H\} \in \text{core}(Q-H)$ for all $Q = 1, 2, 3, 4, \dots$. Then $\{x^{oi}, i \in H\}$ is a competitive equilibrium allocation for $Q-H$, for all Q .

Proof We must show that there is a price vector p so that for each household type $i, p \cdot x^{oi} \leq p \cdot r^i$ and that x^{oi} optimizes preferences \succeq_i subject to this budget. The strategy of proof is to create a set of net trades preferred to those that achieve $\{x^{oi}, i \in H\}$. We will show that it is a convex set with a supporting hyperplane through the origin. The normal to the supporting hyperplane will be designated p . We will then argue that p is a competitive equilibrium price vector supporting $\{x^{oi}, i \in H\}$.

For each $i \in H$, let $\Gamma^i = \{z \mid z \in \mathbf{R}^N, z + r^i \succ_i x^{oi}\}$. What is this set of vectors Γ^i ? Γ^i is defined as the set of net trades from endowment r^i so that an agent of type i strictly prefers these net trades to the trade $x^{oi} - r^i$, the trade that gives him the core allocation. We now define the convex hull (set of convex combinations) of the family of sets $\Gamma^i, i \in H$. Let $\Gamma = \{\sum_{i \in H} a_i z^i \mid z^i \in \Gamma^i, a_i \geq 0, \sum a_i = 1\}$, the set of convex combinations of preferred net trades. The set Γ is the convex hull of the union of the sets Γ^i . (See Figure 22.1.) Note that $(x^{oi} - r^i) \in \text{boundary}(\Gamma^i), (x^{oi} - r^i) \in \bar{\Gamma}^i$, and $(x^{oi} - r^i) \in \bar{\Gamma}$ for all i .

The strategy of proof now is to show that Γ and the constituent sets Γ^i are arrayed strictly above a hyperplane through the origin. The normal to the hyperplane will be the proposed equilibrium price vector.

We wish to show that $0 \notin \Gamma$. We will show that the possibility that $0 \in \Gamma$ corresponds to the possibility of forming a blocking coalition against the core allocation x^{oi} , a contradiction. The typical element of Γ can be represented as $\sum a_i z^i$, where $z^i \in \Gamma^i$. Suppose that $0 \in \Gamma$. Then there are $0 \leq a_i \leq 1$, $\sum_{i \in H} a_i = 1$ and $z^i \in \Gamma^i$ so that $\sum_{i \in H} a_i z^i = 0$. We'll focus on these values of a_i, z^i , and consider the k -fold replication of H , eventually letting k become arbitrarily large. Let the notation $[\cdot]$ represent the smallest integer greater than or equal to the argument \cdot . Consider the hypothetical net trade for a household of type i , $\frac{ka_i}{[ka_i]} z^i$. We have $\frac{ka_i}{[ka_i]} z^i \rightarrow z^i$ as $k \rightarrow \infty$. Therefore, by (C.V, continuity) for k sufficiently large,

$$\left[r^i + \frac{ka_i}{[ka_i]} z^i \right] \succ_i x^{oi} \quad (\dagger)$$

Further,

$$\sum_{i \in H} [ka_i] \frac{ka_i}{[ka_i]} z^i = k \sum_{i \in H} a_i z^i = 0 \quad (\ddagger).$$

It is now time to form a blocking coalition. We confine attention to those $i \in H$ so that $a_i > 0$. The blocking coalition is formed by $[\hat{k}a_i]$ households of type i where \hat{k} is the smallest integer so that (\dagger) is fulfilled for all $i \in H$ for $a_i > 0$. That is, let $\hat{k} \equiv \inf\{k \in \mathcal{N} | (\dagger) \text{ is fulfilled for all } i \in H \text{ such that } a_i > 0\}$ where \mathcal{N} is the set of positive integers. Consider Q larger than \hat{k} . Form the coalition S consisting of $[\hat{k}a_i]$ households of type i for all i so that $a_i > 0$. The blocking allocation to each household of type i

is $r^i + \frac{ka_i}{[ka_i]}z^i$. This allocation is attainable to the coalition by (\ddagger) and it is preferable to the coalition by (\dagger) . This is how replication with large Q overcomes the indivisibility of the individual agents. Thus S blocks x^{oi} , which is a contradiction. Hence, as claimed, $0 \notin \Gamma$.

Having established that 0 is not an element of Γ , we should recognize that 0 is nevertheless very close to Γ . Indeed $0 \in$ boundary of Γ . This occurs inasmuch as $0 = (1/\#H) \sum_{i \in H} (x^{oi} - r^i)$, and the right-hand side of this expression is an element of $\bar{\Gamma}$, the closure of Γ . Thus 0 represents just the sort of boundary point through which a supporting hyperplane may go in the Bounding Hyperplane Theorem. The set Γ is trivially convex. Hence we can invoke the Bounding Hyperplane Theorem. There is $p \in \mathbf{R}^N, p \neq 0$, so that for all $v \in \Gamma, p \cdot v \geq p \cdot 0 = 0$. Noting $X^i = \mathbf{R}_+^N$, C.IV* and C.VI(SC), we know that $p \geq 0$. Now $(x^{oi} - r^i) \in \bar{\Gamma}$ for each i , so $p \cdot (x^{oi} - r^i) \geq 0$. But $\sum_{i \in H} (x^{oi} - r^i) = 0$, so $p \cdot \sum_{i \in H} (x^{oi} - r^i) = 0$. Hence $p \cdot (x^{oi} - r^i) = 0$ each i . Equivalently, $p \cdot x^{oi} = p \cdot r^i$. This gives us

$$0 = p \cdot \sum_{i \in H} \frac{1}{\#H} (x^{oi} - r^i) = \inf_{x \in \Gamma} p \cdot x = \sum_{i \in H} \frac{1}{\#H} \left[\inf_{z^i \in \Gamma^i} p \cdot z^i \right],$$

so

$$p \cdot (x^{oi} - r^i) = \inf_{z^i \in \Gamma^i} p \cdot z^i.$$

We have then for each i , that $p \cdot (x^{oi} - r^i) = \inf p \cdot y$ for $y \in \Gamma^i$. Equivalently, x^{oi} minimizes $p \cdot (x - r^i)$ subject to $x \succeq_i x^{oi}$. In addition, $p \cdot x^{oi} = p \cdot r^i$. Further, by the specification of X^i and r^i , there is an ε -neighborhood of x^{oi} contained in X^i . By C.IV*, C.V, and C.VI(SC), and strict positivity of r^i , expenditure minimization subject to a utility constraint is equivalent to utility

maximization subject to budget constraint. Hence $x^{oi}, i \in H$, is a competitive equilibrium allocation. QED