Lecture Notes for January 8, 2014 — Part 1

See Chapter 5 of General Equilibrium Theory: An Introduction, 2nd ed.

N goods in the economy.

A typical array of prices is an N-dimensional vector

$$p = (p_1, p_2, p_3, \ldots, p_{N-1}, p_N) = (3, 1, 5, \ldots, 0.5, 10).$$

Assume only <u>relative prices</u> (price ratios) matter here, not the numerical values of prices. This is essentially assuming that there is no money, no monetary instrument held as wealth in which prices are denominated.

The price space: The unit simplex in $\mathbf{R}^N$, is

$$P = \left\{ p \mid p \in R^N, p_i \geq 0, i = 1, \ldots, N, \sum_{i=1}^{N} p_i = 1 \right\}. \quad (5.1)$$

The unit simplex is a (generalized) triangle in N-space. It's called "unit" because the co-ordinates add to 1. It's a "simplex" because it has that generalized triangle specification.

For each household $i \in H$, we define a demand function, $D^i : P \to \mathbf{R}^N$.

For each firm $j \in F$, a supply function, $S^j : P \to \mathbf{R}^N$.

Positive co-ordinates in $S^j(p)$ are outputs, negative co-ordinates are inputs.
$p \cdot S^j(p) \equiv \sum_{n=1}^{N} p_n S^j_n(p) \equiv$ profits of firm j.

The economy has an initial endowment of resources $r \in \mathbf{R}^N_+$ that is also supplied to the economy.

The market excess demand function is defined as

$$Z(p) = \sum_{i \in H} D^i(p) - \sum_{j \in F} S^j(p) - r, \tag{5.2}$$

$$Z : P \to \mathbf{R}^N \tag{5.3}$$

$Z(p) \equiv (Z_1(p), Z_2(p), Z_3(p), \ldots, Z_N(p))$, where $Z_k(p)$ is the excess demand for good k. When $Z_k(p)$, the excess demand for good k, is negative, we will say that good k is in excess supply.

There are two principal assumptions: Walras's Law and Continuity of $Z(p)$:

Walras's Law: For all $p \in P$,

$$p \cdot Z(p) = \sum_{n=1}^{N} p_n \cdot Z_n(p) = \sum_{i \in H} p \cdot D^h(p) - \sum_{j \in F} p \cdot S^j(p) - p \cdot r = 0.$$

The economic basis for Walras's Law involves the assumption of scarcity and the structure of household budget constraints. $\sum_{i \in H} p \cdot D^h(p)$ is the value of aggregate household expenditure. The term $\sum_{j \in F} p \cdot S^j(p) + p \cdot r$ is the value of aggregate household income (value of firm profits plus the value of endowment). Walras's Law says that expenditure equals income.

Continuity:

$Z : P \to \mathbf{R}^N, Z(p)$ is a continuous function for all $p \in P$.

That is, small changes in p result in small changes in $Z(p)$ everywhere in P.

We assume in this lecture that $Z(p)$ is well defined and fulfills Walras's Law and Continuity. As mathematical theorists, part of our job is to derive these properties from more elementary properties during the next few weeks (so that we can be sure of their generality).

Definition : $p^o \in P$ is said to be an equilibrium price vector if $Z(p^o) \leq 0$ (0 is the zero vector; the inequality applies coordinatewise) with $p^o_k = 0$ for k such that $Z_k(p^o) < 0$. That is, p^o is an equilibrium price vector if supply equals demand in all markets (with possible excess supply of free goods).

Theorem 5.1 (Brouwer Fixed-Point Theorem) : Let $f(\cdot)$ be a continuous function, $f : P \to P$. Then there is $x^* \in P$ so that $f(x^*) = x^*$.

Theorem 5.2 : Let Walras's Law and Continuity be fulfilled. Then there is $p^* \in P$ so that p^* is an equilibrium.

Proof

Let $T : P \to P$, where $T(p) = (T_1(p), T_2(p), \ldots, T_k(p), \ldots, T_N(p))$. $T_k(p)$ is the adjusted price of good k, adjusted by the auctioneer trying to bring supply and demand into balance. Let $\gamma^k > 0$. The adjustment process of the kth price can be represented as $T_k(p)$, defined as follows:

$$T_k(p) \equiv \frac{\max[0, p_k + \gamma^k Z_k(p)]}{\sum_{n=1}^{N} \max[0, p_n + \gamma^n Z_n(p)]}. \tag{5.4}$$

The function T is a price adjustment function. It raises the relative price of goods in excess demand and reduces the price of goods in excess supply while keeping the price vector on the simplex. In order for T to be well defined, the denominator must be nonzero, that is,

$$\sum_{n=1}^{N} \max[0, p_n + \gamma^n Z_n(p)] \neq 0. \tag{5.5}$$

(5.5)follows from Walras's Law. For the sum in the denominator to be zero or negative, all goods would have to be in excess supply simultaneously, which is contrary to our notions of scarcity and– it turns out– to Walras's Law as well. Recall that $Z(\cdot)$ is a continuous function. The operations of max[ ], sum, and division by a nonzero continuous function maintain continuity. Hence, $T(p)$ is a continuous function from the simplex into itself.

By the Brouwer Fixed-Point Theorem there is $p^*\in P$ so that $T(p^*)=p^*$.

We must show that p^* is not just the stopping point of the price adjustment process, but that it actually does represent general equilibrium prices for the economy.

Since $T(p^*) = p^*$, for each good k, $T_k(p^*) = p^*_k$. That is, for all $k = 1, \ldots, N$,

$$p^*_k = \frac{\max[0, p^*_k + \gamma^k Z_k(p^*)]}{\sum_{n=1}^{N} \max[0, p^*_n + \gamma^n Z_n(p^*)]}. \tag{5.6}$$

For each k, either

$$p^*_k = 0 \qquad \text{(Case 1)} \tag{5.7}$$

or

$$p^*_k = \frac{p^*_k + \gamma^k Z_k(p^*)}{\sum_{n=1}^{N} \max[0, p^*_n + \gamma^n Z_n(p^*)]} > 0 \qquad \text{(Case 2)}. \tag{5.8}$$

CASE 1 $p_k^* = 0 = \max[0, p_k^* + \gamma^k Z_k(p^*)]$. Hence, $0 \geq p_k^* + \gamma^k Z_k(p^*) = \gamma^k Z_k(p^*)$ and $Z_k(p^*) \leq 0$. This is the case of free goods with market clearing or with excess supply in equilibrium.

CASE 2 To avoid repeated messy notation, let

$$\lambda = \frac{1}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n Z_n(p^*)]} \tag{5.9}$$

so that $T_k(p^*) = \lambda(p_k^* + \gamma^k Z_k(p^*))$. Assume (without proof at this point) $\lambda > 0$. Since p^* is the fixed point of T we have $p_k^* = \lambda(p_k^* + \gamma^k Z_k(p^*)) > 0$. This expression is true for all k with $p_k^* > 0$, and λ is the same for all k. Let's perform some algebra on this expression. We first combine terms in p_k^*:

$$(1 - \lambda)p_k^* = \lambda \gamma^k Z_k(p^*), \tag{5.10}$$

then multiply through by $Z_k(p^*)$ to get

$$(1 - \lambda)p_k^* Z_k(p^*) = \lambda \gamma^k (Z_k(p^*))^2, \tag{5.11}$$

and now sum over all k in Case 2, obtaining

$$(1 - \lambda) \sum_{k \in \text{Case2}} p_k^* Z_k(p^*) = \lambda \sum_{k \in \text{Case2}} \gamma^k (Z_k(p^*))^2. \tag{5.12}$$

Walras's Law says

$$0 = \sum_{k=1}^{N} p_k^* Z_k(p^*) = \sum_{k \in \text{Case1}} p_k^* Z_k(p^*) + \sum_{k \in \text{Case2}} p_k^* Z_k(p^*). \tag{5.13}$$

But for $k \in$ Case 1, $p_k^* Z_k(p^*) = 0$, and so

$$0 = \sum_{k \in \text{Case1}} p_k^* Z_k(p^*). \tag{5.14}$$

Therefore,

$$\sum_{k \in \text{Case2}} p_k^* Z_k(p^*) = 0. \tag{5.15}$$

Hence, from (5.11) we have

$$0 = (1-\lambda)\cdot \sum_{k \in \text{Case2}} p_k^* Z_k(p^*) = \lambda \cdot \sum_{k \in \text{Case2}} \gamma^k (Z_k(p^*))^2. \quad (5.16)$$

Using Walras's Law, we established that the left-hand side equals 0, but the right-hand side can be zero only if $Z_k(p^*) = 0$ for all k such that $p_k^* > 0$ (k in Case 2). Thus, p^* is an equilibrium. This concludes the proof.

QED

Lecture Notes, January 8, 2014 --- Part 2

The Arrow-Debreu Model of General Competitive Equilibrium

General Equilibrium Theory: Who was Prof. Debreu and why did he have his own parking space in Berkeley's Central Campus??

Nobel Prizes: Arrow, Debreu

What does mathematical general equilibrium theory do? Tries to put microeconomics on same basis of logical precision as algebra or geometry. Axiomatic method: allows generalization; clearly distinguishes assumptions from conclusions and clarifies the links between them.

Four ideas about writing an economic theory:

Ockam's razor (KISS - Keep it simple, stupid.), improves generality

Testable assumptions (logical positivism), avoids vacuity

Link with experience, robustness, Solow "All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive. A "crucial" assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic. When the results of a theory seem to flow specifically from a special crucial assumption, then if the assumption is dubious, the results are suspect. " (Contribution to the Theory of Economic Growth, 1956)

Precision, reliable results, Hugo Sonnenschein: "In 1954, referring to the first and second theorems of classical welfare economics, Gerard wrote 'The contents of both Theorems ... are old beliefs in economics. Arrow and Debreu have recently treated these questions with techniques permitting proofs.' This statement is precisely correct; once there were beliefs, now there was knowledge.

"But more was at stake. Great scholars change the way that we think about the world, and about what and who we are. The Arrow-Debreu model, as communicated in *Theory of Value* changed basic thinking, and it quickly became the standard model of price theory. It is the 'benchmark' model in Finance, International Trade, Public Finance, Transportation, and even macroeconomics. ... In rather short order it was no longer 'as it is' in Marshall, Hicks, and Samuelson; rather it became 'as it is' in *Theory of Value*." (remarks at the Debreu conference, Berkeley, 2005).

The Market, Commodities and Prices

N commodities

$x = (x_1, x_2, x_3, ..., x_N) \in \mathbf{R}^N$, a commodity bundle

The market takes place at a single instant, prior to the rest of economic activity.
commodity = good or service completely specified
description
location
date (of delivery)

Time: A futures market: no reopening of trade. This issue can be complex. We'll deal with it more thoroughly in Chapter 20.

Price system : $p = (p_1, p_2, ..., p_N) \neq 0$.
$p_i \geq 0$ for all i = 1, ..., N.
Value of a bundle $x \in R^N$ at prices p is $p \cdot x$.

Bounded and Unbounded Firm Technologies

Prices should communicate scarcity (and the boundedness of attainable outputs) to firms. Firms should be able to think: "If we had unbounded inputs we could produce unbounded outputs." So ideally we'd like a model where the firm could decide on arbitrarily large inputs and outputs --- then the price system would communicate that such a plan is unprofitable.

But a firm trying to plan a profit-maximizing production plan on an unbounded technology set may result in no well-defined plan. There may be no profit maximum since arbitrarily large plans may appear to produce arbitrarily large profits.

Modeling strategy:

1. Model production with bounded (and closed) firm technology, $\mathcal{Y}^j$. Then there will surely be a maximum profit achievable.

2. Demonstrate that with finite inputs and convex unbounded technology, Y^j, only finite outputs are possible.

3. Based on 2, consider the an artificial model economy with unbounded technology constrained to a bounded subset, $\tilde{Y}^j$, which then fulfills 1. Find market-clearing prices.

4. Show that the profit maximizing plan does not change when $\tilde{Y}^j$ is replaced by Y^j. Hence prices are still market-clearing. A mathematician's trick:

rearrange the problem to one you know how to solve, (reduce it to the previous --- already solved --- case).

Firms and Production Technology

F , $j \in F$, j = 1, ... , #F. Fixed finite number of firms.

Production technology: $\mathcal{Y}^j \subset R^N$. $y \in \mathcal{Y}^j$ (the script Y notation is to emphasize that $\mathcal{Y}^j$ is bounded).

Negative co-ordinates of y are inputs; positive co-ordinates are outputs.

$y \in \mathcal{Y}^j$, $y = (-2, -3, 0, 0, 1)$

This is a more general specification than a production function. The relationship is $f^j(x) \equiv \max \{ w \mid (-x, w) \in \mathcal{Y}^j \}$.

The Form of Production Technology

P.II. $0 \in \mathcal{Y}^j$.

P.III. $\mathcal{Y}^j$ is closed. (continuity)

P.VI $\mathcal{Y}^j$ is a bounded set for each $j \in F$. (We'll dispense with this evenutally)

P.III and P.VI $\Rightarrow$ $\mathcal{Y}^j$ is compact

Compactness of $\mathcal{Y}^j$ is needed to be sure that profit maximization is well-defined, but P.VI is an ugly assumption: boundedness of a firm's attainable production possibilities should be communicated by the price system --- not by assumption. Chapter 15 of Starr's book weakens the assumption by showing that --- even when the firm's technology set is unbounded --- under weak assumptions, the set of attainable plans is bounded. Then circumscribe the unbounded technology set by a ball strictly containing the attainable plans. Apply the analysis of chaps. 11 - 14 to the artificially circumscribed production technology --- there will be an equilibrium (theorem 14.1) and an equilibrium is necessarily attainable, so the circumscibing ball is not a binding constraint in equilibrium. Then delete the artificial circumscribing ball; the prices and allocation remain an equilibrium. Conclusion: P.VI can be eliminated but it's a bit of work to do it.

Strictly Convex Production Technology

P.V. For each $j \in$ F, $\mathcal{Y}^j$ is strictly convex.

Convexity implies no scale economies, no indivisibilities.

$p \in R_+^N$, $p = (p_1, p_2, ..., p_N)$, $p \neq 0$.

$\tilde{S}^j(p) \equiv \{y^{*j} \mid y^{*j} \in \mathcal{Y}^j, \quad p \cdot y^{*j} \geq p \cdot y \text{ for all } y \in \mathcal{Y}^j\}$.

Theorem 11.1: Assume P.II, P.III, P.V, and P.VI. Let $p \in R_+^N, p \neq 0$. Then $\tilde{S}^j(p)$ is a well defined continuous point-valued function.

Proof:
Well defined: $\tilde{S}^j(p)$ = maximizer of a continuous real-valued function on a compact set.
Point-valued: Strict convexity of $\mathcal{Y}^j$, P.V. Point valued-ness implies that $\tilde{S}^j(p)$ is a function.
Continuity: Let $p^\nu \in R_+^N; \nu = 1, 2, \ldots; p^\nu \neq 0$, $p^\nu \to p^o \neq 0$. Show $\tilde{S}^j(p^\nu) \to \tilde{S}^j(p^o)$. Note: this is a consequence of the Maximum Theorem (see Berge, *Topological Spaces*), but we can provide a direct proof here, by contradiction. Suppose not. Then there is a cluster point of the sequence $\tilde{S}^j(p^\nu)$, y* so that $y^* \neq \tilde{S}(p^o)$ and $p^o \cdot \tilde{S}^j(p^o) > p^o \cdot y^*$ (why does this inequality hold? by definition of $\tilde{S}^j(p^o)$). That is there is a subsequence p^ν so that $\tilde{S}^j(p^\nu) \to y^*$. Note that $p^\nu \cdot \tilde{S}^j(p^o) \to p^o \cdot \tilde{S}^j(p^o)$. We have $p^\nu \cdot \tilde{S}^j(p^\nu) \to p^o \cdot y^*$ and $p^o \cdot \tilde{S}^j(p^o) > p^o \cdot y^*$. But the dot product is a continuous function of its arguments, so for ν large, $p^\nu \cdot \tilde{S}^j(p^o) > p^\nu \cdot \tilde{S}^j(p^\nu)$, a contradiction. This is a contradiction since $\tilde{S}^j(p^\nu)$ is the profit maximizer at p^ν. Thus, we must have $\tilde{S}^j(p^\nu) \to \tilde{S}^j(p^o)$. Q.E.D.

Lemma 1: (homogeneity of degree 0) Assume P.II, P.III, and P.VI. Let $\lambda > 0$, $p \in R_+^N$. Then $\tilde{S}^j(\lambda p) = \tilde{S}^j(p)$.

$$\tilde{S}(p) \equiv \sum_{j \in F} \tilde{S}^j(p)$$

4.4 Attainable Production Plans

Definition: A sum of sets $\mathcal{Y}^j$ in R^N, is defined as $\mathcal{Y} = \sum_j \mathcal{Y}^j$ is the set $\{y \mid y = \sum_j y^j \text{ for some } y^j \in \mathcal{Y}^j\}$.

Aggregate technology set:

$\mathcal{Y} \equiv \sum_{j \in F} \mathcal{Y}^j$.

Initial inputs to production $r \in R^N_+$

Definition: Let $y \in \mathcal{Y}$. Then y is said to be attainable if $y + r \geq 0$.

$y \in \mathcal{Y}$ is attainable if $(y + r) \in [\mathcal{Y} + \{r\}] \cap R^N_+$.

Note that under this definition, and P.II, P.III, P.V, P.VI the attainable set of outputs is compact and convex.