Convexity

A set of points $S$ in $\mathbb{R}^n$ is said to be convex if the line segment between any two points of the set is completely included in the set.

$S$ is convex if $x, y \in S$, implies $\{z \mid z = \alpha x + (1 - \alpha) y, \ 0 \leq \alpha \leq 1\} \subseteq S$.

$S$ is said to be strictly convex if $x, y \in S$, $x \neq y$, $0 < \alpha < 1$, implies $\alpha x + (1 - \alpha) y \in \text{interior } S$.

The notion of convexity is that a set is convex if it is connected, has no holes on the inside and no indentations on the boundary. A set is strictly convex if it is convex and has a continuous strict curvature (no flat segments) on the boundary.

Economically, this notion corresponds to "diminishing marginal utility" "diminishing marginal rate of substitution" "diminishing marginal product".

Properties of Convex Sets

Let $C_1, C_2$ be convex subsets of $\mathbb{R}^n$. Then:

$C_1 \cap C_2$ is convex,

$C_1 + C_2$ is convex,

$\bar{C}_1$ is convex
The unit simplex in $\mathbb{R}^N$, is

$$P = \left\{ p \mid p \in \mathbb{R}^N, p_i \geq 0, i = 1, \ldots, N, \sum_{i=1}^{N} p_i = 1 \right\}. \quad (5.1)$$

The unit simplex is a (generalized) triangle in $N$-space.

Note that $P$ is compact (closed and bounded) and convex.

Theorem 5.1 (Brouwer Fixed-Point Theorem) Let $f(\cdot)$ be a continuous function, $f : P \rightarrow P$. Then there is $x^* \in P$ so that $f(x^*) = x^*$.

The four properties assumed in the Brouwer Fixed Point Theorem — continuity of $f$, closedness, boundedness, and convexity of $P$ — are all essential to the theorem. Omit any one of them and the theorem fails.

The result can be generalized from $P$ to any compact convex set.