Mathematical Logic

Logical Inference
Let A and B be two logical conditions, like A="it's sunny today" and B="the light outside is very bright"

\[ A \Rightarrow B \]
A implies B, if A then B

\[ A \iff B \]
A if and only if B, A implies B and B implies A, A and B are equivalent conditions

Proofs
Just like in high school geometry.

Concept of Proof by contradiction: Suppose we want to show that \( A \Rightarrow B \). Ordinarily, we'd like to prove this directly. But it may be easier to show that \( \neg (A \Rightarrow B) \) is false. How? Show that \( [A \& (\neg B)] \) leads to a contradiction. A: x = 1, B: x+3=4. Then \( [A \& (\neg B)] \) leads to the conclusion that 1+3 \( \neq \) 4 or equivalently 1 \( \neq \) 1, a contradiction. Hence \( [A \& (\neg B)] \) must fail so \( A \Rightarrow B \). (Yes, it does feel backwards, like your pocket is being picked, but it works).

Set Theory

Definition of a Set
\[
\{ \}
\{ x \mid x \text{ has property } P \}
\{1, 2, ..., 9, 10\} = \{ x \mid x \text{ is an integer, } 1 \leq x \leq 10 \}.
\]

Elements of a set
\[ x \in A ; \quad y \not\in A \]
\[ x \neq \{ x \} \]
\[ x \in \{ x \} \]
\[ \emptyset = \text{the empty set (null set), the set with no elements.} \]

Subsets
\[ A \subset B \text{ or } A \subseteq B \text{ if } x \in A \Rightarrow x \in B \]
\[ A \subset A \text{ and } \emptyset \subset A . \]

Set Equality
\[ A = B \text{ if } A \text{ and } B \text{ have precisely the same elements} \]
\[ A = B \text{ if and only if } A \subset B \text{ and } B \subset A . \]
Set Union
\[ A \cup B = \{ x \mid x \in A \text{ or } x \in B \} \quad ('or' \text{ includes 'and'}) \]

Set Intersection
\[ A \cap B = \{ x \mid x \in A \text{ and } x \in B \} \]
If \( A \cap B = \emptyset \) we say that A and B are disjoint.

**Theorem 6.1:** Let A, B, C be sets,
\[ a. \quad A \cap A = A, \quad A \cup A = A \quad \text{(idempotency)} \]
\[ b. \quad A \cap B = B \cap A, \quad A \cup B = B \cup A \quad \text{(commutativity)} \]
\[ c. \quad A \cap (B \cap C) = (A \cap B) \cap C \quad \text{(associativity)} \]
\[ A \cup (B \cup C) = (A \cup B) \cup C \]
\[ d. \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \text{(distributivity)} \]
\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

Complementation (set subtraction)
\[ A \setminus B = \{ x \mid x \in A, \ x \notin B \} \]

Cartesian Product
\[ ordered \ pairs \]
\[ A \times B = \{(x, y) \mid x \in A, \ y \in B \} \]
Note: If \( x \neq y \), then \((x, y) \neq (y, x)\).

**R** = The set of real numbers
**R**^N = N-fold Cartesian product of R with itself.
**R**^N = R x R x R x ... x R, where the product is taken N times.
The order of elements in the ordered N-tuple \((x, y, ...)\) is essential. If \( x \neq y, \ (x, y, ...) \neq (y, x, ...) \).

**R**^N, Real N-dimensional Euclidean space


**R**^2 = plane
**R**^3 = 3-dimensional space
**R**^N = N-dimensional Euclidean space

Definition of **R**:
- **R** = the real line
- \( \pm \infty \notin **R** \)
closed interval: \([a, b] \equiv \{x \mid x \in \mathbb{R}, a \leq x \leq b\}.

\(\mathbb{R}\) is complete. Nested intervals property: Let \(x^v < y^v\) and \([x^{v+1}, y^{v+1}] \subseteq [x^v, y^v]\), \(v = 1, 2, 3, \ldots\). Then there is \(z \in \mathbb{R}\) so that \(z \in [x^v, y^v]\), for all \(v\).

\(\mathbb{R}^N\) = N-fold Cartesian product of \(\mathbb{R}\).
\(x \in \mathbb{R}^N, x = (x_1, x_2, \ldots, x_N)\)
\(x_i\) is the \(i\)th co-ordinate of \(x\).
\(x =\) point (or vector) in \(\mathbb{R}^N\)

Algebra of elements of \(\mathbb{R}^N\)
\(x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_N + y_N)\)
\(0 = (0, 0, 0, \ldots, 0)\), the origin in N-space
\(x - y \equiv x + (-y) = (x_1 - y_1, x_2 - y_2, \ldots, x_N - y_N)\)
\(t \in \mathbb{R}, x \in \mathbb{R}^N\), then \(tx \equiv (tx_1, tx_2, \ldots, tx_N)\)

\(x, y \in \mathbb{R}^N, x \cdot y = \sum_{i=1}^{N} x_i y_i\). If \(p \in \mathbb{R}^N\) is a price vector and \(y \in \mathbb{R}^N\) is an economic action, then \(p \cdot y = \sum_{n=1}^{N} p_n y_n\) is the value of the action \(y\) at prices \(p\).

Norm in \(\mathbb{R}^N\), the measure of distance
\(|x| \equiv \|x\| \equiv \sqrt{x \cdot x} \equiv \sqrt{\sum_{i=1}^{N} x_i^2}\).

Let \(x, y \in \mathbb{R}^N\). The distance between \(x\) and \(y\) is \(|x - y|\).

\(|x - y| = \sqrt{\sum_{i=1}^{N} (x_i - y_i)^2}\).
\(|x - y| \geq 0\) all \(x, y \in \mathbb{R}^N\)
\(|x - y| = 0\) if and only if \(x = y\).

Limits of Sequences
\(x^v, v = 1, 2, 3, \ldots\),
Example: \(x^v = 1/v\). \(1, 1/2, 1/3, 1/4, 1/5, \ldots\). \(x^v \to 0\).

Formally, let \(x^i \in \mathbb{R}, i = 1, 2, \ldots\). Definition: We say \(x^i \to x^0\) if for any \(\varepsilon > 0\), there is \(q(\varepsilon)\) so that for all \(q' > q(\varepsilon)\), \(|x^{q'} - x^0| < \varepsilon\).
So in the example \( x^\nu = 1/\nu, q(\epsilon) = 1/\epsilon \)

Let \( x^i \in R^N, i = 1, 2, \ldots \). We say that \( x^i \to x^0 \) if for each co-ordinate 

\[ n = 1, 2, \ldots, N, x^i_n \to x^0_n . \]

**Theorem 7.1:** Let \( x^i \in R^N, i = 1, 2, \ldots \). Then \( x^i \to x^0 \) if and only if for any \( \epsilon \) there is \( q(\epsilon) \) such that for all \( q' > q(\epsilon), \|x^{q'}-x^0\| < \epsilon \).

\( x^o \) is a *cluster point* of \( S \subseteq R^N \) if there is a sequence \( x^\nu \in R^N \) so that \( x^\nu \to x^o \).

**Open Sets**

Let \( X \subseteq R^N ; \) \( X \) is open if for every \( x \in X \) there is an \( \epsilon > 0 \) so that \( \|x-y\| < \epsilon \) implies \( y \in X \).

Open interval in \( R : \) \((a, b) = \{ x | x \in R, a < x < b \}\)

\( \emptyset \) and \( R^N \) are open.

**Closed Sets**

Example: Problem - Choose a point \( x \) in the closed interval \([a, b]\) (where \( 0 < a < b \)) to maximize \( x^2 \). Solution: \( x = b \).

Problem - Choose a point \( x \) in the open interval \((a, b)\) to maximize \( x^2 \). There is no solution in \((a, b)\) since \( b \notin (a, b) \).

A set is closed if it contains all of its cluster points.

**Definition:** Let \( X \subseteq R^N \). \( X \) is said to be a *closed* set if for every sequence \( x^\nu, \nu = 1, 2, 3, \ldots \), satisfying,

\[ \begin{align*}
(i) & \quad x^\nu \in X, \text{ and} \\
(ii) & \quad x^\nu \to x^0,
\end{align*} \]

it follows that \( x^0 \in X \).

Examples: A closed interval in \( R, [a, b] \) is closed.

A closed ball in \( R^N \) of radius \( r \), centered at \( c \) \( \in R^N \), \( \{ x \in R^N | |x-c| \leq r \} \) is a closed set.

A line in \( R^N \) is a closed set.

But a set may be neither open nor closed (for example the sequence \( \{1/\nu\}, \nu=1, 2, 3, 4, \ldots \) is not closed in \( R \), since \( 0 \) is a limit point of the sequence but is not an element of the sequence; it is not open since it consists of isolated points).

**Note:** Closed and open are not antonyms among sets. \( \emptyset \) and \( R^N \) are each both closed and open.

For a YouTube reference: www.youtube.com/watch?v=SyD4p8_y8Kw
Let $X \subseteq \mathbb{R}^N$. The closure of $X$ is defined as
\[ \overline{X} \equiv \{ y \mid \text{there is } x^v \in X, v = 1, 2, 3, \ldots \text{, so that } x^v \to y \}. \]
For example the closure of the sequence in $\mathbb{R}$, \( \{1/v \mid v=1, 2, 3, 4, \ldots \} \) is \( \{0\} \cup \{1/v \mid v=1, 2, 3, 4, \ldots \} \).

**Theorem 7.2:** Let $X \subseteq \mathbb{R}^N$. $X$ is closed if $\mathbb{R}^N \setminus X$ is open.

**Proof:** Suppose $\mathbb{R}^N \setminus X$ is open. We must show that $X$ is closed. If $X = \mathbb{R}^N$ the result is trivially satisfied. For $X \neq \mathbb{R}^N$, let $x^v \in X$, $x^v \to x^o$. We must show that $x^o \in X$ if $\mathbb{R}^N \setminus X$ is open. Proof by contradiction. Suppose not. Then $x^o \in \mathbb{R}^N \setminus X$. But $\mathbb{R}^N \setminus X$ is open. Thus there is an $\varepsilon$ neighborhood about $x^o$ entirely contained in $\mathbb{R}^N \setminus X$. But then for $v$ large, $x^v \in \mathbb{R}^N \setminus X$, a contradiction. Therefore $x^o \in X$ and $X$ is closed. QED

**Theorem 7.3:**
1. \( X \subseteq \overline{X} \)
2. \( X = \overline{X} \) if and only if $X$ is closed.

**Bounded Sets**
Def: $K(k) = \{ x \mid x \in \mathbb{R}^N, |x_i| \leq k, i = 1, 2, \ldots, N \}$ = cube of side $2k$ (centered at the origin).

Def: $X \subseteq \mathbb{R}^N$. $X$ is **bounded** if there is $k \in \mathbb{R}$ so that $X \subseteq K(k)$.

**Compact Sets**
THE IDEA OF COMPACTNESS IS ESSENTIAL!
Def: $X \subseteq \mathbb{R}^N$. $X$ is **compact** if $X$ is closed and bounded.

Finite subcover property: An open covering of $X$ is a collection of open sets so that $X$ is contained in the union of the collection. It is a property of compact $X$ that for every open covering there is a finite subset of the open covering whose union also contains $X$. That is, every open covering of a compact set has a finite subcover.

**Boundary, Interior, etc.**
$X \subseteq \mathbb{R}^N$, Interior of $X = \{ y \mid y \in X, \text{ there is } \varepsilon > 0 \text{ so that } \|x - y\| < \varepsilon \text{ implies } x \in X\}$

Boundary $X = \overline{X} \setminus \text{Interior } X$

**Set Summation in $\mathbb{R}^N$**
Let $A \subseteq \mathbb{R}^N$, $B \subseteq \mathbb{R}^N$. Then
\[ A + B = \{ x \mid x = a + b, a \in A, b \in B \}. \]

The Bolzano-Weierstrass Theorem, Completeness of $\mathbb{R}^N$.

**Theorem 7.4** (Nested Intervals Theorem): By an interval in $\mathbb{R}^N$, we mean a set $I$ of the form $I = \{ (x_1, x_2, \ldots, x_N) \mid a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2, \ldots, a_N \leq x_N \leq b_N, a_i, b_i \in R \}$.
Consider a sequence of nonempty closed intervals $I_k$ such that
\[ I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_k \supseteq \ldots. \]
Then there is a point in $R^N$ contained in all the intervals. That is, $\exists x^o \in \bigcap_{i=1}^{\infty} I_i$ and therefore $\bigcap_{i=1}^{\infty} I_i \neq \phi$; the intersection is nonempty.

**Proof:** Follows from the completeness of the reals, the nested intervals property on $R$.

**Corollary** (Bolzano-Weierstrass theorem for sequences): Let $x^i$, $i = 1, 2, 3, ...$ be a bounded sequence in $R^N$. Then $x^i$ contains a convergent subsequence.

**Proof** 2 cases: $x^i$ assumes a finite number of values, $x^i$ assumes an infinite number of values.

It follows from the Bolzano-Weierstrass Theorem for sequences and the definition of compactness that an infinite sequence on a compact set has a convergent subsequence whose limit is in the compact set.
# 12:
# 13: