Nonexpected Utility Theory

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Introduction

Since the early eighteenth century, the predominant model of individual behavior under uncertainty has been the *expected utility model* of preferences over uncertain prospects. This model was first introduced by Nicholas **Bernoulli** [5] in his famous resolution of the St. Petersburg Paradox; was formally axiomatized by von Neumann and Morgenstern [36] in their development of the game theory; was formally integrated with the theory of subjective probability by Savage [31] in his work on the foundations of statistics; and was shown to possess great analytical power by Arrow [4] and Pratt [27] in their work on **risk aversion**, as well as by Rothschild and Stiglitz [29, 30] in their work on comparative risk. It is fair to say that the expected utility model is the received analytical foundation for the great majority of work in the theory of insurance, theory of games, theory of **investment** and capital markets, theory of search, and most other branches of statistics, **decision theory**, and the economics of uncertainty.

However, beginning with the work of Allais [1] and Edwards [11, 12] in the early 1950s and continuing through the present, psychologists and economists have uncovered a growing body of evidence that individuals do not necessarily conform to many of the key assumptions or predictions of the expected utility model, and indeed, seem to depart from the model in predictable and systematic ways. This has led to the development of alternative or 'nonexpected utility' models of risk preferences, which seek to accommodate these systematic departures from the expected utility model, while retaining as much of its analytical power as possible.

The Expected Utility Model

In one of the simplest settings of choice under uncertainty, the objects of choice consist of finite-valued *objective* **lotteries**, each of the form $\mathbf{P} = (x_1, p_1; \ldots; x_n, p_n)$ and yielding a monetary payoff of x_i with probability p_i, where $p_1 + \cdots + p_n = 1$. In such a case the expected utility model assumes (or posits axioms sufficient to imply) that the individual ranks such risky prospects based on an *expected utility preference function* of the form

$$V_{\mathrm{EU}}(\mathbf{P}) = V_{\mathrm{EU}}(x_1, p_1; \ldots; x_n, p_n) \equiv U(x_1)p_1 + \cdots + U(x_n)p_n \qquad (1)$$

in the sense that the individual prefers some lottery $\mathbf{P}^* = (x_1^*, p_1^*; \ldots; x_{n^*}^*, p_{n^*}^*)$ over a lottery $\mathbf{P} = (x_1, p_1; \ldots; x_n, p_n)$ if and only if $V_{\mathrm{EU}}(\mathbf{P}^*) > V_{\mathrm{EU}}(\mathbf{P})$, and will be indifferent between them if and only if $V_{\mathrm{EU}}(\mathbf{P}^*) = V_{\mathrm{EU}}(\mathbf{P})$. $U(\cdot)$ is termed the individual's *von Neumann–Morgenstern utility function*, and its various mathematical properties serve to characterize various features of the individual's attitudes toward risk, for example,

$V_{\mathrm{EU}}(\cdot)$ will exhibit *first-order stochastic dominance preference* (a preference for shifting probability from lower to higher outcome values) if and only if $U(x)$ is an increasing function of x,

$V_{\mathrm{EU}}(\cdot)$ will exhibit **risk aversion** (an aversion to all mean-preserving increases in risk) if and only if $U(x)$ is a concave function of x,

$V_{\mathrm{EU}}^*(\cdot)$ will be *at least as risk averse as* $V_{\mathrm{EU}}(\cdot)$ (in several equivalent senses) if and only if its utility function $U^*(\cdot)$ is a concave transformation of $U(\cdot)$ (i.e. if and only if $U^*(x) \equiv \rho(U(x))$ for some increasing concave function $\rho(\cdot)$).

As shown by **Bernoulli** [5], Arrow [4], Pratt [27], Friedman and Savage [13], Markowitz [26], and others, this model admits to a tremendous flexibility in representing many aspects of attitudes toward risk.

But in spite of its flexibility, the expected utility model has testable implications that hold regardless of the shape of the utility function $U(\cdot)$, since they follow from the *linearity in the probabilities* property of the preference function $V_{\mathrm{EU}}(\cdot)$. These implications can be best expressed by the concept of an $\alpha{:}(1-\alpha)$ *probability mixture* of two lotteries $\mathbf{P}^* = (x_1^*, p_1^*; \ldots; x_{n^*}^*, p_{n^*}^*)$ and $\mathbf{P} = (x_1, p_1; \ldots; x_n, p_n)$, which is defined as the lottery $\alpha\mathbf{P}^* + (1-\alpha)\mathbf{P} \equiv (x_1^*, \alpha p_1^*; \ldots; x_{n^*}^*, \alpha p_{n^*}^*; x_1, (1-\alpha)p_1; \ldots; (1-\alpha)x_n, p_n)$, and which can be thought of as a coin flip that yields prize $\mathbf{P}^*$ with probability α and prize $\mathbf{P}$ with probability $(1-\alpha)$, where the uncertainty in the coin and in the subsequent prize are realized simultaneously, so that $\alpha\mathbf{P}^* + (1-\alpha)\mathbf{P}$ (like $\mathbf{P}^*$ and $\mathbf{P}$) is a single-stage lottery. Linearity in the probabilities is equivalent to the following property, which serves as the key foundational axiom of the expected utility model:

Independence Axiom. For any pair of lotteries $\mathbf{P}^*$ and $\mathbf{P}$, lottery $\mathbf{P}^*$ is preferred (resp. indifferent) to $\mathbf{P}$ if and only if the probability mixture $\alpha\mathbf{P}^* + (1-\alpha)\mathbf{P}^{**}$ is preferred (resp. indifferent) to $\alpha\mathbf{P} + (1-\alpha)\mathbf{P}^{**}$ for every lottery $\mathbf{P}^{**}$ and for every probability $\alpha \in (0, 1]$.

This axiom can be interpreted as saying 'given an α:$(1-\alpha)$ coin, one's preferences for receiving $\mathbf{P}^*$ versus $\mathbf{P}$ in the event of heads should not depend upon the prize $\mathbf{P}^{**}$ that will be received in the event of tails, nor upon the probability α of landing heads (so long as it is positive)'. The strong normative appeal of this axiom has contributed to the widespread adoption of the expected utility model. An insurance-based example would be if all insurance contracts had clauses that render them void (with a premium refund) in the event of an 'act of war', then an individual's ranking of such contracts ($\mathbf{P}^*$ versus $\mathbf{P}$) would not depend upon their perceived probability $1-\alpha$ of an act of war, or upon the distribution of wealth $\mathbf{P}^{**}$ that would result in such an event.

The property of linearity in the probabilities, as well as the senses in which it is empirically violated, can be illustrated in the special case of preferences over the family of all lotteries $\mathbf{P} = (\overline{x}_1, p_1; \overline{x}_2, p_2; \overline{x}_3, p_3)$ over a fixed set of outcome values $\overline{x}_1 < \overline{x}_2 < \overline{x}_3$. Since we must have $p_2 = 1 - p_1 - p_3$, each lottery in this set can be completely summarized by its pair of probabilities (p_1, p_3), as plotted in the 'probability triangle' of Figure 1. Since upward movements in the diagram (increasing p_3 for fixed p_1) represent shifting probability from outcome $\overline{x}_2$ up to $\overline{x}_3$, and leftward movements represent shifting probability from $\overline{x}_1$ up to $\overline{x}_2$, such movements both constitute first **order** stochastically dominating shifts and will thus always be referred. Expected utility indifference curves (loci of constant expected utility) are given by the formula

$$U(\overline{x}_1)p_1 + U(\overline{x}_2)[1 - p_1 - p_3] + U(\overline{x}_3)p_3 = \text{constant}$$

and are accordingly seen to be parallel straight lines of slope $[U(\overline{x}_2) - U(\overline{x}_1)]/[U(\overline{x}_3) - U(\overline{x}_2)]$, as indicated by the solid lines in the figure. The dashed lines

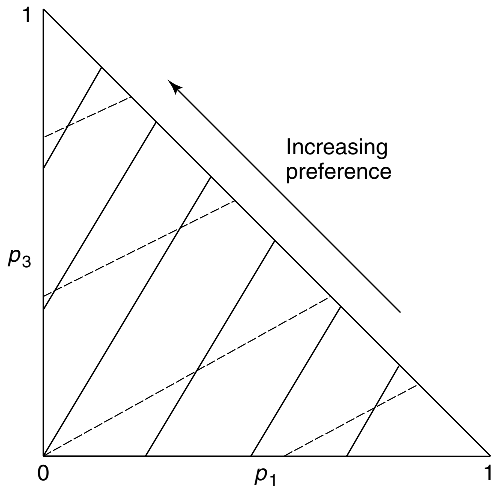

Figure 1 Expected utility indifference curves in the probability triangle

in Figure 1 are loci of constant expected value, given by the formula $\overline{x}_1 p_1 + \overline{x}_2[1 - p_1 - p_3] + \overline{x}_3 p_3 =$ constant, with slope $[\overline{x}_2 - \overline{x}_1]/[\overline{x}_3 - \overline{x}_2]$. Since northeast movements along the constant expected value lines shift probability from $\overline{x}_2$ both down to $\overline{x}_1$ and up to $\overline{x}_3$ in a manner that preserves the mean of the distribution, they represent simple increases in risk. When $U(\cdot)$ is concave (i.e. risk averse), its indifference curves will have a steeper slope than these constant expected value lines, and such increases in risk are seen to move the individual from more preferred to less preferred indifference curves, as illustrated in the figure. It is straightforward to show that the indifference curves of any expected utility maximizer with a more risk-averse (i.e. more concave) utility function $U^*(\cdot)$ will be even steeper than those generated by $U(\cdot)$.

Systematic Violations of the Expected Utility Hypothesis

In spite of its normative appeal, researchers have uncovered two forms of widespread systematic violations of the Independence Axiom. The first type of violation can be exemplified by the following example, known as the *Allais Paradox* [1], where individuals are asked to rank the lotteries a_1 versus a_2, and separately, a_3 versus a_4, where \$1M denotes $1 000 000

$$a_1: \{1.00 \text{ chance of \$1M}$$

versus

$$a_2: \begin{cases} 0.10 \text{ chance of \$5M} \\ 0.89 \text{ chance of \$1M} \\ 0.01 \text{ chance of \$0} \end{cases}$$

$$a_3: \begin{cases} 0.10 \text{ chance of \$5M} \\ 0.90 \text{ chance of \$0} \end{cases}$$

versus

$$a_4: \begin{cases} 0.11 \text{ chance of \$1M} \\ 0.89 \text{ chance of \$0} \end{cases}$$

Most individuals express a preference for a_1 over a_2 in the first pair, and a_3 over a_4 in the second pair. However, such preferences violate expected utility theory, since the first ranking implies the inequality $U(\$1M) > 0.10U(\$5M) + 0.89U(\$1M) + 0.01U(\$0)$, whereas the second ranking implies the inconsistent inequality $0.10U(\$5M) + 0.90U(\$0) > 0.11U(\$1M) + 0.89U(\$0)$. By setting $\bar{x}_1 = \$0$, $\bar{x}_2 = \$1M$, and $\bar{x}_3 = \$5M$, the four prospects can be plotted in the probability triangle as in Figure 2, where they are seen to form a parallelogram. The typical preference for a_1 over a_2 and a_3 over a_4 suggests that indifference curves depart from the expected utility property of parallel straight lines in the direction of 'fanning out', as in the figure. The Allais Paradox is merely one example of a widely observed phenomenon termed the *Common Consequence Effect*,

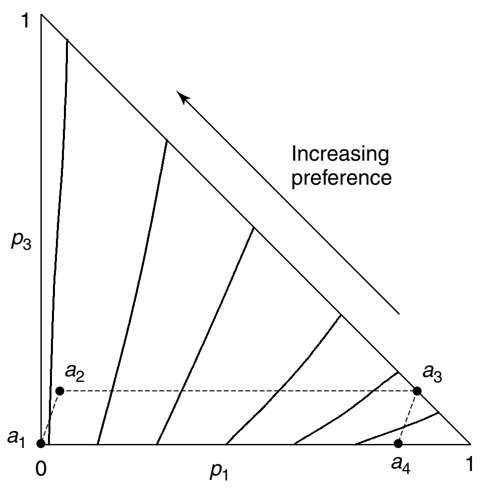

Figure 2 Allais Paradox choices and indifference curves that 'fan out'

which states that when $\mathbf{P}^*$ is riskier than $\mathbf{P}$, individuals tend to depart from the Independence Axiom by preferring $\alpha\mathbf{P}^* + (1-\alpha)\mathbf{P}^{**}$ to $\alpha\mathbf{P} + (1-\alpha)\mathbf{P}^{**}$ when $\mathbf{P}^{**}$ involves low payoffs, but reverse this preference when $\mathbf{P}^{**}$ instead involves high payoffs.

(In the Allais example, $\mathbf{P}$ =($1M,1), $\mathbf{P}^*$ =($5M, 10/11;$0,1/11), $\alpha = 0.11$, and $\mathbf{P}^{**}$ is ($1M,1) in the first pair and ($0,1) in the second pair.)

The second broad class of systematic violations, also originally noted by Allais, can be illustrated by the following example from Kahneman and Tversky [18]:

$$b_1: \begin{cases} 0.45 \text{ chance of \$6000} \\ 0.55 \text{ chance of \$0} \end{cases}$$

versus

$$b_2: \begin{cases} 0.90 \text{ chance of \$3000} \\ 0.10 \text{ chance of \$0} \end{cases}$$

$$b_3: \begin{cases} 0.001 \text{ chance of \$6000} \\ 0.999 \text{ chance of \$0} \end{cases}$$

versus

$$b_4: \begin{cases} 0.002 \text{ chance of \$3000} \\ 0.998 \text{ chance of \$0} \end{cases}$$

Most individuals express a preference for b_2 over b_1 and for b_3 over b_4, which again violates expected utility theory since they imply the inconsistent inequalities $0.45U(\$6000) + 0.55U(\$0) < 0.90U(\$3000) + 0.10U(\$0)$ and $0.001U(\$6000) + 0.999U(\$0) > 0.002U(\$3000) + 0.998U(\$0)$. Setting $\bar{x}_1 = \$0$, $\bar{x}_2 = \$3000$ and $\bar{x}_3 = \$6000$, the prospects can be plotted as in Figure 3, where the above preference rankings again suggest that indifference curves depart from expected utility by fanning out. This is one example of a widely observed phenomenon termed the *Common Ratio Effect*, which states that for positive-outcome prospects $\mathbf{P}^*$ which is risky and $\mathbf{P}$ which is certain, with $\mathbf{P}^{**}$ being a sure chance of $0, individuals tend to depart from the Independence Axiom by preferring $\alpha\mathbf{P}^* + (1-\alpha)\mathbf{P}^{**}$ to $\alpha\mathbf{P} + (1-\alpha)\mathbf{P}^{**}$ for low values of α, but reverse this preference for high values of α. Kahneman and Tversky [18] observed that when the positive outcomes $3000 and $6000 are replaced by *losses* of $3000 and $6000 to create the prospects b_1', b_2', b_3' and b_4', then preferences typically 'reflect', to prefer b_1' over b_2' and b_4' over b_3'. Setting $\bar{x}_1 = -\$6000$, $\bar{x}_2 = -\$3000$ and $\bar{x}_3 = \$0$ (to preserve the ordering $\bar{x}_1 < \bar{x}_2 < \bar{x}_3$) and plotting as in Figure 4, such preferences again

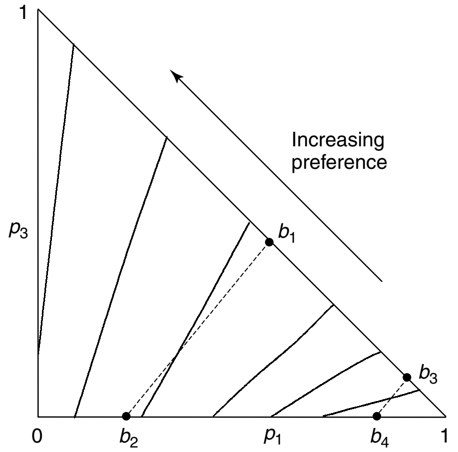

Figure 3 Common ratio effect and indifference curves that fan out

7 suggest that indifference curves in the probability triangle fan out.

Further descriptions of these and other violations of the expected utility hypothesis can be found in [6, 18, 21, 23, 24, 34, 35, 38].

Nonexpected Utility Functional Forms

Researchers have responded to the above types of departures from linearity in the probabilities in two manners. The first consists of replacing the expected utility preference function $V_{EU}(\cdot)$ by some more general functional form for $V(\mathbf{P}) = V(x_1, p_1; \ldots; x_n, p_n)$, such as the forms in the Table below.

Most of these forms have been axiomatized, and given proper curvature assumptions on their component functions $\upsilon(\cdot)$, $\tau(\cdot)$, $G(\cdot)$, $K(\cdot, \cdot)$,..., most are capable of exhibiting first-order stochastic dominance preference, risk aversion, and the common consequence and common ratio effects. The rank-dependent expected utility form, in particular, has been widely applied to the analysis of many standard questions in economic choice under uncertainty.

Generalized Expected Utility Analysis

An alternative approach to nonexpected utility, which yields a direct extension of most of the basic results

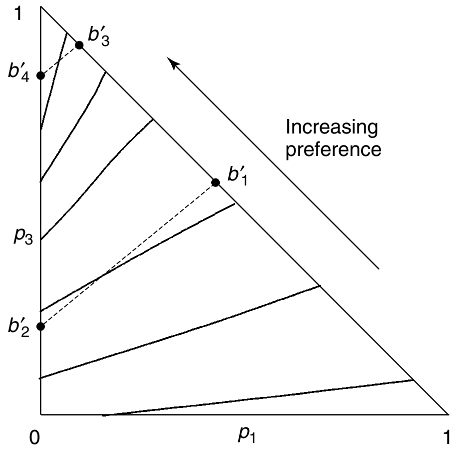

Figure 4 Common ratio effect for negative payoffs and indifference curves that fan out

'Prospect theory'	$\sum_{i=1}^{n} \upsilon(x_i)\pi(p_i)$	[11, 12, 18]
Moments of utility	$f\left(\sum_{i=1}^{n} \upsilon(x_i)p_i, \sum_{i=1}^{n} \upsilon(x_i)^2 p_i, \ldots\right)$	[17]
Weighted utility	$\dfrac{\sum_{i=1}^{n} \upsilon(x_i)p_i}{\sum_{i=1}^{n} \tau(x_i)p_i}$	[7]
Dual expected utility	$\sum_{i=1}^{n} x_i \left[G\left(\sum_{j=1}^{i} p_j\right) - G\left(\sum_{j=1}^{i-1} p_j\right)\right]$	[39]
Rank-dependent expected utility	$\sum_{i=1}^{n} \upsilon(x_i) \left[G\left(\sum_{j=1}^{i} p_j\right) - G\left(\sum_{j=1}^{i-1} p_j\right)\right]$	[28]
Quadratic in probabilities	$\sum_{i=1}^{n} \sum_{j=1}^{n} K(x_i, x_j)p_i p_j$	[8]
Ordinal independence	$\sum_{i=1}^{n} h\left(x_i, \sum_{j=1}^{i} p_j\right)\left[G\left(\sum_{j=1}^{i} p_j\right) - G\left(\sum_{j=1}^{i-1} p_j\right)\right]$	[33]

of expected utility analysis, applies calculus to a general 'smooth' nonexpected utility preference function $V(\mathbf{P}) \equiv V(x_1, p_1; \ldots; x_n, p_n)$, concentrating in particular on its *probability derivatives* $\partial V(\mathbf{P})/\partial\ \text{prob}(x)$. This approach is based on the observation that for the expected utility function $V_{\text{EU}}(x_1, p_1; \ldots; x_n, p_n) \equiv U(x_1)p_1 + \cdots + U(x_n)p_n$, the value $U(x)$ can be interpreted as the coefficient of prob(x), and that many theorems relating to a linear function's *coefficients* continue to hold when generalized to a nonlinear function's *derivatives*. By adopting the notation $U(x;\mathbf{P}) \equiv \partial V(\mathbf{P})/\partial\ \text{prob}(x)$ for the probability derivative, the above expected utility characterizations of first-order stochastic dominance preference, risk aversion, and comparative risk aversion generalize to any smooth preference function $V(\mathbf{P})$ in the following manner:

- $V(\cdot)$ will exhibit *first-order stochastic dominance preference* if and only if at each lottery $\mathbf{P}$, $U(x;\mathbf{P})$ is an increasing function of x
- $V(\cdot)$ will exhibit *risk aversion* (an aversion to all mean-preserving increases in risk) if and only if at each lottery $\mathbf{P}$, $U(x;\mathbf{P})$ is a concave function of x
- $V^*(\cdot)$ will be *at least as risk averse as* $V(\cdot)$ if and only if at each lottery $\mathbf{P}$, $U^*(x;\mathbf{P})$ is a concave transformation of $U(x;\mathbf{P})$

It can be shown that the indifference curves of a smooth preference function $V(\cdot)$ will fan out in the probability triangle and its multiple-outcome generalizations if and only if $U(x;\mathbf{P}^*)$ is a concave transformation of $U(x;\mathbf{P})$ whenever $\mathbf{P}^*$ first order stochastically dominates $\mathbf{P}$; see [3, 9, 19, 22, 23, 37] for the development and further applications of generalized expected utility analysis.

Applications to Insurance

The above analysis also permits the extension of many expected utility-based results in the theory of insurance to general smooth nonexpected utility preferences. The key step in this extension consists of observing that the formula for the expected utility *outcome derivative* – upon which most insurance results are based – is linked to its probability coefficients via the relationship $\partial V_{\text{EU}}(\mathbf{P})/\partial x \equiv \text{prob}(x)\cdot U'(x)$, and that this relationship generalizes to any smooth nonexpected utility preference function $V(\cdot)$, where it takes the form $\partial V(\mathbf{P})/\partial x \equiv \text{prob}(x)\cdot U'(x;\mathbf{P})$ (where $U'(x)$ and $U'(x;\mathbf{P})$ denote derivatives with respect to the variable x). One important expected utility result that does *not* directly extend is the equivalence of risk aversion to the property of *outcome-convexity*, which states that for any set of probabilities $(p_1, \ldots, p_n)$, if the individual is indifferent between the lotteries $(x_1, p_1; \ldots; x_n, p_n)$ and $(x_1^*, p_1; \ldots; x_n^*, p_n)$ then they will strictly prefer any outcome mixture of the form $(\beta x_1^* + (1-\beta)x_1, p_1; \ldots; \beta x_n^* + (1-\beta)x_1, p_n)$ (for $0 < \beta < 1$). Under nonexpected utility, outcome-convexity still implies risk aversion but is no longer *implied* by it, so for some nonexpected utility insurance results, both risk aversion and outcome-convexity must be explicitly specified. (Nevertheless, the hypothesis that the individual is risk averse and outcome-convex is still weaker than the hypothesis that they are a risk averse expected utility maximizer.)

One standard insurance problem – termed the *demand for* **coinsurance** – involves an individual with initial wealth w who faces a random loss $\tilde{\ell}$ with probability distribution $(l_1, p_1; \ldots; l_n, p_n)$ ($l_i \geq 0$), who can insure fraction γ of this loss by paying a premium of $\lambda\gamma\ \text{E}[\tilde{\ell}]$, where $\text{E}[\tilde{\ell}]$ is the expected loss and the *load factor* λ equals or exceeds unity. The individual thus selects the most preferred option from the family of random variables $\{w - (1-\gamma)\tilde{\ell} - \lambda\gamma\text{E}[\tilde{\ell}] | 0 \leq \gamma \leq 1\}$. Another standard problem – termed the *demand for deductible insurance* – involves fully covering any loss over some amount d, so the individual receives a payment of $\max\{\tilde{\ell} - d, 0\}$, and paying a premium of $\lambda\text{E}[\max\{\tilde{\ell} - d, 0\}]$. The individual thus selects the most preferred option from the family of random variables $\{w - \min\{\tilde{\ell}, d\} - \lambda\text{E}[\max\{\tilde{\ell} - d, 0\}] | d \geq 0\}$. In each case, insurance is said to be *actuarially fair* if the load factor λ equals unity, and *actuarially unfair* if λ exceeds unity. Even without the assumption of outcome-convexity, the following results from the classic expected utility-based theory of insurance can be shown to extend to any risk averse smooth nonexpected utility preference function $V(\cdot)$:

- $V(\cdot)$ will purchase *complete* coinsurance (will choose to set the fraction γ equal to one) if and only if such insurance is actuarially fair
- $V(\cdot)$ will purchase *complete* deductible insurance (will choose to set the deductible d equal to zero) if and only such insurance is actuarially fair
- If $V^*(\cdot)$ is at least as risk averse as $V(\cdot)$, then whenever they each have the same initial wealth and

face the same random loss, $V^*(\cdot)$ will purchase at least as much coinsurance as $V(\cdot)$

If $V^*(\cdot)$ is at least as risk averse as $V(\cdot)$, then whenever they each have the same initial wealth and face the same random loss, $V^*(\cdot)$ choose at least as low a deductible level as $V(\cdot)$

If we do assume outcome-convexity, many additional results in the theory of insurance and **optimal risk-sharing** can be similarly extended from expected utility to smooth nonexpected utility preferences; see [25] for the above and additional extensions, and [10, 14, 16, 20, 32, 40] as well as the papers in [15], for additional results in the theory of insurance under nonexpected utility risk preferences.

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(*See also* **Risk Measures**)

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