Economic Theory **26**, 1–62 (2005)
DOI: 10.1007/s00199-004-0573-6

Economic Theory

Research Articles

'Expected utility / subjective probability' analysis without the sure-thing principle or probabilistic sophistication*

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Received: May 4, 2004; revised version: October 4, 2004

Summary. The basic analytical concepts, tools and results of the classical expected utility/subjective probability model of risk preferences and beliefs under subjective uncertainty can be extended to general "event-smooth" preferences over subjective acts that do not necessarily satisfy either of the key behavioral assumptions of the classical model, namely the Sure-Thing Principle or the Hypothesis of Probabilistic Sophistication. This is accomplished by a technique analogous to that used by Machina (1982) and others to generalize expected utility analysis under *objective* uncertainty, combined with an event-theoretic approach to the classical model and the use of a special class of subjective events, acts and mixtures that exhibit "almost-objective" like properties. The classical expected utility/subjective probability characterizations of outcome monotonicity, outcome derivatives, probabilistic sophistication, comparative and relative subjective likelihood, and comparative risk aversion are all globally "robustified" to general event-smooth preferences over subjective acts.

Keywords and Phrases: Subjective uncertainty, Almost-objective uncertainty, Robustness.

JEL Classification Numbers: D81.

* This paper presents a considerably improved version of the concept of event-differentiability from Machina (1992). An alternative definition has been independently developed by Epstein (1999) in his analysis of the concept of uncertainty aversion. I am grateful to Kenneth Arrow, Mark Durst, Jürgen Eichberger, Daniel Ellsberg, Clive Granger, Simon Grant, Edi Karni, Peter Klibanoff, David Kreps, Duncan Luce, Robert Nau, Uzi Segal, Peter Wakker, Joel Watson and especially Larry Epstein, Ted Groves and Joel Sobel for helpful discussions and comments. This material is based upon work supported by the National Science Foundation under Grants No. 9209012 and 9870894.

1 Introduction

This paper addresses the question:

> *"To what extent are the analytics of the classical expected utility/subjective probability model of choice under subjective uncertainty robust to departures from both the assumption of expected utility risk preferences and the assumption of probabilistic beliefs?"*

We approach this question by combining the following ideas:

The Calculus Approach to Robustness, which can be used to establish the local and global robustness of any model that exhibits "constant sensitivity" in its key variables, and which has already been used to establish the robustness of expected utility analysis under *objective* uncertainty.[1]

An Event-Theoretic Representation of the Classical Model, which shows that the expected utility/subjective probability model under subjective uncertainty exhibits constant sensitivity in the *events* attached to each uncertain outcome, and is therefore also amenable to the above approach to robustness.

While analogous to the earlier robustification of objective expected utility analysis in both its goals and its overall findings, the subjective analysis of this paper involves quite different mathematical tools, and does not assume prior knowledge of the earlier approach.

The following section motivates the analysis by a discussion of the classical model, its main empirical violations, existing responses to these violations, and the argument for robustification. Section 3 sketches out the calculus approach to robustness, and how it has been applied to expected utility analysis under objective uncertainty. Section 4 presents the tools needed to extend this approach to subjective uncertainty, namely an event-theoretic representation of the classical model, the notion of "event-smoothness," and a special class of subjective events, acts and mixtures that exhibit "almost-objective" like properties. Section 5 establishes the global robustness of the classical expected utility/subjective probability characterizations of outcome-monotonicity, outcome derivatives, probabilistic sophistication, comparative and relative subjective likelihood, and comparative risk aversion. Section 6 discusses related work of Epstein, and extensions of the present analysis. Proofs are in an Appendix.

2 The classical model of risk preferences and beliefs

By "classical model" we mean the *subjective expected utility* (*SEU*) model proposed by Ramsey (1931) and fully developed by Savage (1954), in which choice over subjectively uncertain prospects is characterized by expected utility risk preferences and standard probabilistic beliefs.[2] The key sense in which this

[1] E.g., Machina (1982, 1983, 1984, 1989, 1995), Dekel (1986), Allen (1987), Karni (1987, 1989), Chew, Epstein and Zilcha (1988), Chew and Nishimura (1992), Bardsley (1993) and Wang (1993).

[2] See Fishburn (1982, Chs. 9-12) or Kreps (1988, Chs. 8-10) for modern expositions of this approach.

model differs from the earlier *objective* expected utility model of Bernoulli (1738) and von Neumann-Morgenstern (1947) is the manner in which the two models represent uncertainty. In the objective framework, uncertainty comes prepackaged in terms of numerical probabilities, and the objects of choice are probability distributions over outcomes or *lotteries* $\mathbf{P} = (x_1, p_1; \ldots ; x_n, p_n)$, yielding outcome x_i with probability p_i. But in the more realistic *subjective* framework, uncertainty is represented by a set $\mathcal{S} = \{\ldots , s, \ldots\}$ of mutually exclusive and exhaustive *states of nature*, or by *events* E (subsets of $\mathcal{S}$), and the objects of choice are *bets* or *acts* $f(\cdot) = [x_1 \text{ on } E_1; \ldots ; x_n \text{ on } E_n]$, which specify the outcome $x = f(s)$ as a function of the state or event that occurs. The *SEU* model posits a *subjective probability measure* $\mu(\cdot)$ over events, and an *expected utility preference function* $V_{EU}(\mathbf{P}) \equiv \sum_{i=1}^{n} U(x_i) \cdot p_i$ over lotteries, such that preferences over subjective acts can be represented by the *SEU preference function*

$$
\begin{aligned}
W_{SEU}\big(x_1 \text{ on } E_1; \ldots ; x_n \text{ on } E_n\big) \quad &\equiv \quad V_{EU}\big(x_1, \mu(E_1); \ldots ; x_n, \mu(E_n)\big) \\
\equiv \quad \sum\nolimits_{i=1}^{n} U(x_i) \cdot \mu(E_i) \quad &\equiv \quad \int_{\mathcal{S}} U\big(f(s)\big) \cdot d\mu(s)
\end{aligned} \tag{1}
$$

The classical *SEU* form can be represented as the composition of two mappings:

$$
\big[x_1 \text{ on } E_1; \ldots ; x_n \text{ on } E_n\big] \to \big(x_1, \mu(E_1); \ldots ; x_n, \mu(E_n)\big) \to \sum\nolimits_{i=1}^{n} U(x_i) \cdot \mu(E_i)
$$

In the first mapping, the subjective probability measure $\mu(\cdot)$ is used to determine the act's *implied lottery over outcomes*. In the second mapping, the expected utility preference function $V_{EU}(\cdot)$ is used to determine the level of preference of this implied lottery, and by implication, the level of preference for the act. Thus, the classical model is often represented as the *joint* hypothesis of

Probabilistic Sophistication: Acts are evaluated solely on the basis of their implied lotteries over outcomes. This property only involves the individual's *beliefs.*

Expected Utility Risk Preferences: Preferences over implied outcome lotteries are linear in the probabilities. This property only involves the individual's *attitudes toward risk.*

This so-called "separation of beliefs from risk preferences" – with the use of subjective probabilities to represent the former and expected utility to represent the latter – is often viewed as the characteristic feature of the classical *SEU* model.

2.1 The two orthogonal hypotheses of the classical model

Although the separation of the classical model into the hypotheses of probabilistically sophisticated beliefs and expected utility risk preferences accords with both intuitive and normative ideals of individual and statistical decision making, it is less than perfect from a scientific or analytical point of view, for the simple reason that the two hypotheses are *nested:* the hypothesis of expected utility risk preferences is defined over the *output* of the probabilistic sophistication hypothesis, namely the implied lottery of each act. Accordingly, we shall organize our analysis in terms of an alternative pair of hypotheses, which also jointly characterize the classical model, but which are *orthogonal* in the sense that each is defined directly on preferences over subjective acts, and each can be satisfied independently of the other:

Event-Separability: Preferences over acts are separable across mutually exclusive events.

Probabilistic Sophistication: As before, acts are evaluated solely on the basis of their implied lotteries over outcomes, via some subjective probability measure $\mu(\cdot)$.

Axiomatically, these two hypotheses are respectively equivalent to, and can be represented by:

P2 *Sure-Thing Principle* (Savage (1954)): For all events E and acts $f(\cdot)$, $f^*(\cdot)$, $g(\cdot)$, $h(\cdot)$:

$$\begin{bmatrix} f^*(\cdot) \text{ on } & E \\ g(\cdot) \text{ on } & \sim E \end{bmatrix} \succcurlyeq \begin{bmatrix} f(\cdot) \text{ on } & E \\ g(\cdot) \text{ on } & \sim E \end{bmatrix} \Rightarrow \begin{bmatrix} f^*(\cdot) \text{ on } & E \\ h(\cdot) \text{ on } & \sim E \end{bmatrix} \succcurlyeq \begin{bmatrix} f(\cdot) \text{ on } & E \\ h(\cdot) \text{ on } & \sim E \end{bmatrix}$$

P4* *Strong Comparative Probability Axiom* (Machina and Schmeidler, 1992): For all outcomes $x^* \succ x$, $y^* \succ y$, disjoint events A, B and acts $g(\cdot), h(\cdot)$:

$$\begin{bmatrix} x^* & \text{on} & A \\ x & \text{on} & B \\ g(\cdot) & \text{elsewhere} & \end{bmatrix} \succcurlyeq \begin{bmatrix} x & \text{on} & A \\ x^* & \text{on} & B \\ g(\cdot) & \text{elsewhere} & \end{bmatrix} \Rightarrow \begin{bmatrix} y^* & \text{on} & A \\ y & \text{on} & B \\ h(\cdot) & \text{elsewhere} & \end{bmatrix} \succcurlyeq \begin{bmatrix} y & \text{on} & A \\ y^* & \text{on} & B \\ h(\cdot) & \text{elsewhere} & \end{bmatrix}$$

Behaviorally, the two hypotheses can be respectively described as:

Stable Subact Preferences: The ranking of any pair of *subacts* $[f^*(\cdot) \text{ on } E]$ versus $[f(\cdot) \text{ on } E]$ is not affected by identical changes in statewise-identical payoffs *off* of the event E.

Stable Revealed Likelihood Rankings: The preferred assignment of greater versus lesser preferred payoffs to the events A versus B is not affected by identical changes in statewise-identical payoffs *off* of $A \cup B$, *or* by the values of the greater/lesser preferred payoffs themselves.

Theoretically, the two hypotheses can be shown to respectively characterize the following forms for the preference function over acts:

State-Dependent Expected Utility: The preference function $W_{SDEU}(f(\cdot)) \equiv \int_S U(f(s)|s) \cdot d\mu(s)$, for some subjective probability measure $\mu(\cdot)$ and *state-dependent utility function* $U(x|s)$.

Probabilistically Sophisticated Non-Expected Utility: The preference function $W_{PS}(f(\cdot)) \equiv V(x_1, \mu(E_1); \ldots ; x_n, \mu(E_n))$, for some subjective probability measure $\mu(\cdot)$ and *non-expected utility preference function* $V(\mathbf{P}) \equiv V(x_1, p_1; \ldots ; x_n, p_n)$ over lotteries.

2.2 Three violations of the classical model

The classical *SEU* model is violated by three well-known phenomena – Allais-type violations of event-separability, state-dependence type violations of probabilistic sophistication, and Ellsberg-type violations – which can be illustrated by the examples in Table 1.[3] Each example involves three mutually exclusive and exhaustive events, and four acts defined on these events. Outcomes are in dollars, and 1M = 1,000,000. The uncertainty in the Allais example comes from an urn known to contain 10 red, 1 black, and 89 yellow balls, and the rankings $\alpha_1 \succ \alpha_2$ and $\alpha_3 \prec \alpha_4$ are the well-known Allais (1953) preferences over these acts' implied lotteries. The uncertainty in the state-dependence example involves the individual's future health, with three events: staying healthy, having a debilitating tumor that is fatal without a \$50,000 operation, or getting the flu. The rankings $\beta_1 \succ \beta_2$ and $\beta_3 \prec \beta_4$ reflect the greater usefulness of \$10,000 when healthy than with the tumor, but the greater need for \$50,000 with the tumor than when healthy. The uncertainty in the Ellsberg example comes from an urn known to contain 30 red balls and 60 black and/or yellow balls in unknown proportion, and the rankings $\gamma_1 \succ \gamma_2$ and $\gamma_3 \prec \gamma_4$ are the well-known Ellsberg (1961) preferences. In each case, the stated preferences violate the classical *SEU* form (1).

Table 1. Three violations of the classical subjective expected utility model

Allais paradox[4]				State-dependence				Ellsberg paradox			
	10 balls	1 ball	89 balls		healthy	tumor	flu		30 balls: red	60 balls: black	60 balls: yellow
α_1	1M	1M	1M	β_1	10,000	1,000	100	γ_1	100	0	0
α_2	5M	0	1M	β_2	1,000	10,000	100	γ_2	0	100	0
α_3	1M	1M	0	β_3	50,000	5,000	500	γ_3	100	0	100
α_4	5M	0	0	β_4	5,000	50,000	500	γ_4	0	100	100
$\alpha_1 \succ \alpha_2$ yet $\alpha_3 \prec \alpha_4$				$\beta_1 \succ \beta_2$ yet $\beta_3 \prec \beta_4$				$\gamma_1 \succ \gamma_2$ yet $\gamma_3 \prec \gamma_4$			

2.3 Theoretical responses to the violations

Researchers have responded to Allais-type violations by the development and analysis of *non-expected utility preference functions over lotteries* – that is, preference functions $V(\mathbf{P}) = V(x_1, p_1; \ldots; x_n, p_n)$ which drop the expected utility property of linearity in the probabilities. Work in this area has proceeded along two lines: the development and analysis of specific functional forms for $V(x_1, p_1; \ldots; x_n, p_n)$,[5] and the analysis of general smooth $V(x_1, p_1; \ldots; x_n, p_n)$

[3] See the excellent reviews of these phenomena by Sugden (1986, 1991), Weber and Camerer (1987), Karni and Schmeidler (1991), Epstein (1992), Kelsey and Quiggin (1992), Camerer and Weber (1992) and Starmer (2000).

[4] I have reversed the usual labeling of the last two Allais acts in order to better highlight the structure of the violation.

[5] Most notably, the *rank-dependent* form of Quiggin (1982), the *weighted utility* form of Chew (1983), the *quadratic* form of Chew, Epstein and Segal (1991), and the *disappointment aversion* form of Gul (1991).

functions via their probability and/or outcome derivatives (as described below). In each case, non-expected utility *risk preferences* are completely compatible with probabilistically sophisticated *beliefs.* That is, for any non-expected utility preference function $V(x_1, p_1; \ldots; x_n, p_n)$ and any subjective probability measure $\mu(\cdot)$, the pair of mappings

$$\big[x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n\big] \to \big(x_1, \mu(E_1); \ldots; x_n, \mu(E_n)\big) \to V\big(x_1, \mu(E_1); \ldots; x_n, \mu(E_n)\big)$$

yields a *probabilistically sophisticated non-expected utility preference function*

$$W_{PS}\big(x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n\big) \;\equiv\; V\big(x_1, \mu(E_1); \ldots; x_n, \mu(E_n)\big) \qquad (2)$$

over subjective acts, which can be shown to be the generalization of the classical form (1) obtained by retaining the hypothesis of probabilistic sophistication but dropping the hypothesis of event-separability.

Researchers have primarily responded to state-dependence type violations of probabilistic sophistication by working with the well-known *state-dependent expected utility* preference function[6]

$$W_{SDEU}\big(x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n\big) \equiv \int_S U\big(f(s)|s\big) \cdot d\mu(s) \equiv \sum_{i=1}^{n} \int_{E_i} U(x_i|s) \cdot d\mu(s) \quad (3)$$

which can be shown to be the generalization of the classical form (1) obtained by retaining the hypothesis of event-separability but dropping the hypothesis of probabilistic sophistication.

One may ask why state-dependent preferences constitute a violation of probabilistic sophistication, since they can be represented by a form (3) that still contains a subjective probability measure $\mu(\cdot)$. The answer is that probabilistic sophistication does not simply mean that subjective probabilities *are somehow used* in the evaluation of an act, but rather, that they *serve to completely encode its uncertainty,* so that the *only* feature of an outcome's event that matters is its subjective probability. Thus, if E_i has a greater subjective probability than E_j, the individual would always prefer staking the more preferred outcome on E_i and the less preferred outcome on E_j rather than vice versa – such preferences are said to "reveal" the individual's comparative likelihood ranking of the two events. However, the rankings $\beta_1 \succ \beta_2$ and $\beta_3 \prec \beta_4$ are seen to violate this property, and constitute a *revealed likelihood reversal.* Although the form (3) involves a subjective probability measure $\mu(\cdot)$, it accommodates state-dependence type violations of probabilistic sophistication by allowing the events and outcomes to interact in an additional manner, via the function $U(x|s)$.

Since the Ellsberg rankings $\gamma_1 \succ \gamma_2$ and $\gamma_3 \prec \gamma_4$ are seen to violate *both* event separability *and* probabilistic sophistication (that is, to violate both P2 and P4*), they cannot be represented by the probabilistically sophisticated form (2) *or* by the state-dependent expected utility form (3).[7] Thus in contrast with the Allais

[6] E.g., Karni (1983, 1985).

[7] That is, if $\{E_1, E_2, E_3\}$ are the Ellsberg events and we assume monotonicity in wealth, the revealed likelihood reversal $\gamma_1 \succ \gamma_2$ and $\gamma_3 \prec \gamma_4$ cannot be represented by any preference function of the form $V(x_1, \mu(E_1); x_2, \mu(E_2); x_3, \mu(E_3))$, or of the form $U(x_1|E_1) \cdot \mu(E_1) + U(x_2|E_2) \cdot \mu(E_2) + U(x_3|E_3) \cdot \mu(E_3)$.

Paradox (which can be accommodated by non-expected utility risk preferences) and state dependence (which can be accommodated by state-outcome interactions), the Ellsberg Paradox seems to strike at the heart of the classical probability-theoretic notion of *beliefs.* Although the “explanation” of the paradox in terms of a general aversion to ambiguity is well known, this observation does not “solve” the problem so much as underscore its key point, namely that, even if we allow for non-expected utility risk preferences or state-dependence, *subjective probabilities do not suffice to encode all aspects of uncertain beliefs.*

Responses to Ellsberg-type violations have primarily consisted of alternative formulas for the subjective preference function $W(\cdot)$ which replace its subjective probability measure $\mu(\cdot)$ by some more general structure for beliefs. One of these is the *Choquet expected utility* form

$$W_{Choquet}\big(x_1 \text{ on } E_1; \ldots ; x_n \text{ on } E_n\big) \equiv \sum_{i=1}^{n} U(x_i)\cdot\Big[C\big(\bigcup_{j=1}^{i} E_j\big) - C\big(\bigcup_{j=1}^{i-1} E_j\big)\Big] \quad (4)$$

for some utility function $U(\cdot)$ and *capacity* (i.e. monotonic non-additive measure) $C(\cdot)$, where the outcomes are labeled so that $x_1 \preccurlyeq \ldots \preccurlyeq x_n$ (e.g., Gilboa, 1987; Schmeidler, 1989; Wakker, 1989, 1990; Gilboa and Schmeidler, 1994). Another is the *maxmin expected utility* form

$$\begin{aligned} W_{maxmin}\big(x_1 \text{ on } E_1; \ldots ; x_n \text{ on } E_n\big) &\equiv \min_{\tau\in\mathcal{T}} \int_{\mathcal{S}} U\big(f(s)\big)\cdot d\mu_\tau(s) \\ &\equiv \min_{\tau\in\mathcal{T}} \sum_{i=1}^{n} U(x_i)\cdot\mu_\tau(E_i) \end{aligned} \quad (5)$$

for some utility function $U(\cdot)$ and *family* $\{\mu_\tau(\cdot)\,|\,\tau\in\mathcal{T}\}$ of probability measures on $\mathcal{S}$ (e.g., Gärdenfors and Sahlin (1982, 1983), Cohen and Jaffray (1985), Gilboa and Schmeidler (1989)).[8] Several other forms for representing preferences in the Ellsberg Paradox have been proposed. [9]

The above theoretical and empirical relationships can be summarized as in Table 2.

8 Although both $W_{Choquet}(\cdot)$ and $W_{maxmin}(\cdot)$ can accommodate the joint departure from event-separability and probabilistic sophistication in the Ellsberg example of Table 1, each form implies that preferences are event-separable and probabilistically sophisticated over *large subregions* of the space of subjective acts, so any departure from event-separability or probabilistic sophistication within any of these regions will constitute a violation of these forms.

9 Some models represent beliefs by *decision weights* which depend on the assignment of outcomes to events (Hazen, 1987, 1989; Luce, 1988, 1991), by *lexicographic probabilities* (Blume, Brandenberger and Dekel, 1991), by *second order probabilities* (Nau, 2001), or by *indeterminate probabilities* (Whalley, 1991; Nau, 1989, 1992, 2002). Other models allow the incorporation of objective information on some events but not others (Eichberger and Kelsey, 1999), have ordinal but not cardinal likelihoods (Kelsey, 1993), allow the level of ambiguity to affect the utility of an outcome (Sarin and Winkler, 1992), evaluate acts just on the basis of their outcomes (Barberá and Jackson, 1988), or represent uncertainty aversion by incomplete preferences (Bewley, 1986, 1987). Sarin and Wakker (1992) derive a belief measure which is only additive over a prespecified subalgebra of events, Fishburn (1991, 1993) axiomatizes a primitive *degree of ambiguity* relation, and Fishburn (1989) and Fishburn and LaValle (1987) consider *skew-symmetric additive* act preferences.

Table 2. Orthogonal hypotheses, theoretical forms and empirical phenomena

Theoretical forms *characterized by:*

	event separability	∼ event separability
probabilistic sophistication	$W_{SEU}(\cdot)$	$W_{PS}(\cdot)$
∼probabilistic sophistication	$W_{SDEU}(\cdot)$	

Empirical phenomena *consistent with:*

	event separability	∼ event separability
probabilistic sophistication		Allais preferences
∼probabilistic sophistication	state-dependence	Ellsberg preferences

2.4 The argument for robustification

Although each of the forms $W_{PS}(\cdot), W_{SDEU}(\cdot), W_{Choquet}(\cdot)$ and $W_{maxmin}(\cdot)$ succeeds in accommodating its own respective departure from the classical model (Allais, state-dependence or Ellsberg), none of them can be considered a satisfactory alternative to the classical model, for two reasons.

The first reason is that each form constitutes a *partial* rather than a *general* response to the above set of violations, in the sense that it can accommodate *some* of the violations but not others. Thus, $W_{PS}(\cdot)$ cannot accommodate departures from probabilistic sophistication, $W_{SDEU}(\cdot)$ cannot accommodate departures from event-separability, and while $W_{Choquet}(\cdot)$ and $W_{maxmin}(\cdot)$ can accommodate the specific Allais and Ellsberg examples of Table 1, they cannot accommodate more general departures from event separability or probabilistic sophistication (see Note 8), nor can they accommodate state-dependence type departures without additional modification.

The second reason is that each of the above alternatives consists of a *specific functional form* for the preference function $W(\cdot)$. Each such form comes with its own component function and component measure (or set of component measures), in addition to (or in place of) the classical von Neumann-Morgenstern utility function $U(\cdot)$ and subjective probability measure $\mu(\cdot)$. Thus, each new form involves its own analytical "startup costs" – namely the costs of determining conditions on its component functions or measures that characterize the standard behavioral properties of comparative risk aversion, comparative likelihood, etc. A *second* round of analytical costs is incurred by the need to revisit each of the important applied topics in choice under uncertainty (such as portfolio allocation, insurance demand, search, or auctions) under each new form. Finally, if more than one form is deemed acceptable, this leads to a *third* round of costs, namely the costs of deriving conditions for comparative risk aversion, comparative beliefs, etc. *across* such forms. Given the classical model's tremendous head start in each of these topics, the analytical costs of "starting over" with any new functional form seem overwhelming.

Although functional forms are indispensable for *empirical* estimation, calibration and testing, it is hard to think of any other branch of economics that has conducted so much of its *theoretical* analysis by the study of functional forms. Besides the above analytical costs, this approach carries another modeling danger, which can be illustrated by a hypothetical example from regular consumer theory over nonstochastic commodity bundles: Say we observe violations of "linearity in the commodities" in the form of diminishing marginal rates of substitution, and respond by developing and analyzing (say) the Cobb-Douglas functional form for utility. Among the theoretical implications that emerge are zero cross-price elasticities of demand. But should we really be predicting *zero cross-price elasticities* from the empirical phenomenon of *diminishing MRS?* Such unintended implications have already occurred at least once in choice under uncertainty: For many years [Edwards (1955) through Kahneman and Tversky (1979)], the main form used to represent nonlinearities in the probabilities was the non-expected utility form $V(\mathbf{P}) \equiv \sum_{i=1}^{n} U(x_i) \cdot \pi(p_i)$. This form was eventually dropped when it was found to imply violations of first order stochastic dominance preference in *every neighborhood of every lottery,* a phenomenon which was not in the data it sought to represent, and has not been observed in the data since.

The above considerations call for an approach which can accommodate *general* departures from *both* event-separability and probabilistic sophistication, which does not rely on a functional form, and which can retain as much of the classical analytics of risk preferences and beliefs as possible – in other words, an approach which "robustifies" the classical model and its analytics.

3 The calculus approach to robustness

3.1 Local and global robustification: A three-step approach

The following approach is simply an extension of the standard manner in which calculus is used to derive local and global generalizations of linear algebra results. In its most general terms, this approach to robustifying a given model – call it the "classical model" – involves three steps:

Step 1. Represent the classical model so that it exhibits "constant sensitivity" to changes in its key variables. Then represent the model's analytical concepts, tools and results in terms of its "sensitivity rates" to these variables.

Step 2. Define "smoothness" to establish local robustness. Define differentiability so that non-classical models that are "smooth" in these key variables will have "local classical approximations" and "local sensitivity rates," thus establishing the *local* robustness of the classical analytics.

Step 3. Use line integrals to establish global robustness. Construct line integrals in the key variables, so that a smooth non-classical model's response to any global change will be the integrated sum of its "local classical" responses to this change. Those classical results that do not actually require constant sensitivity rates will then be found to be globally robust.

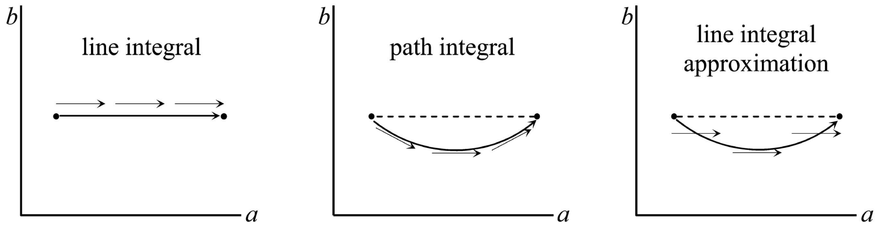

Figure 1. Line integral, path integral and line integral approximation

We can illustrate this approach – and an important mathematical issue we shall encounter – for a simple result in ordinal utility theory over nonstochastic (apple, banana) commodity bundles, namely the result that a "classical" (i.e. linear) utility function $\mathcal{U}_{Lin}(a,b) \equiv c_a \cdot a + c_b \cdot b$ will exhibit a global weak preference for additional apples if and only if its sensitivity rate to apples satisfies $c_a \geq 0$. This result can be *locally* robustified to a general smooth $\mathcal{U}(\cdot,\cdot)$ at a given bundle (a_0, b_0) by means of the analogous condition on its *local* sensitivity rate to apples, namely that $\mathcal{U}_a(a_0,b_0) \equiv \partial\mathcal{U}(a_0,b_0)/\partial a \geq 0$. It can also be *globally* robustified to a general smooth $\mathcal{U}(\cdot,\cdot)$, to take the form:

$$\begin{array}{c}\mathcal{U}(\cdot,\cdot) \text{ exhibits a } \textit{global} \text{ weak} \\ \text{preference for additional apples}\end{array} \quad \Leftrightarrow \quad \begin{array}{c}\mathcal{U}_a(a,b) \geq 0 \\ \text{at } \textit{each} \text{ bundle } (a,b)\end{array} \tag{6}$$

The above-mentioned mathematical issue, which poses no problem for this example but will prove substantial when working with subjective uncertainty, involves the *method of proving* a global robustness result such as (6). The left-hand condition in (6) is simply the property that $\mathcal{U}(a+\Delta a,b) \geq \mathcal{U}(a,b)$ for all (a,b) and all $\Delta a \geq 0$. If $\mathcal{U}(\cdot,\cdot)$ is smooth, then $\mathcal{U}(a+\Delta a,b) - \mathcal{U}(a,b)$ can be exactly expressed in terms of $\mathcal{U}(\cdot,\cdot)$'s local sensitivity rates (i.e. partial derivatives), *either* by a straight line integral from the bundle (a,b) to $(a+\Delta a,b)$, *or* by any smooth path integral between these bundles, as illustrated by the left and middle diagrams of Figure 1. But of the two, *only the line integral* can be used to *prove* the global robustness result (6). The reason is that the right-hand condition of (6) ensures that each differential movement along the straight line path has a nonnegative differential effect upon $\mathcal{U}(\cdot,\cdot)$, so that the integral of these effects (that is, the line integral in the figure) will be nonnegative. But even when the right-hand condition of (6) holds, the *curved middle path* cannot be used to prove $\mathcal{U}(a+\Delta a,b) \geq \mathcal{U}(a,b)$, since the condition $\mathcal{U}_a(a,b) \geq 0$ at each (a,b) is not enough to ensure that these differential effects are everywhere nonnegative along the downward sloping portion of the path, or that any such negative effects are guaranteed to be outweighed by positive effects along the upward sloping portion. Intuitively, if a global robustness result like (6) is to be proven by an exact path integral, it must be along a *constant-direction path,* such as the straight-line path in the figure, and not along any curved path in the space of commodity bundles.

There is, however, another way to prove an exact global robustness result like (6). Consider the integral in the right diagram in Figure 1. Like the middle diagram, it involves a curved path, but unlike that diagram, it does not integrate the differential effect upon $\mathcal{U}(\cdot,\cdot)$ of *moving in the direction of the path,* but rather, integrates the differential effect of *moving in the direction of the global change.* Thus if $\{(a(t),b(t))\,|\,t\in[0,1]\}$ is the curved path in the middle and right diagrams, the middle path integral is given by $\int_0^1 [\mathcal{U}_a(a(t),b(t))\cdot a'(t)+\mathcal{U}_b(a(t),b(t))\cdot b'(t)]\cdot dt$, whereas the right *line integral approximation* is given by $\int_0^1[\mathcal{U}_a(a(t),b(t))\cdot\Delta a]\cdot dt$. Since it is not a true path integral, this expression will not exactly equal $\mathcal{U}(a+\Delta a,b)-\mathcal{U}(a,b)$. But with sufficient regularity, as the path $\{(a(t),b(t))\,|\,t\in[0,1]\}$ becomes *arbitrarily close* to a straight line path, the line integral approximation $\int_0^1[\mathcal{U}_a(a(t),b(t))\cdot\Delta a]\cdot dt$ will *converge* to $\mathcal{U}(a+\Delta a,b)-\mathcal{U}(a,b)$. Since the right condition of (6) ensures this integral is nonnegative for any path, and since any limit of nonnegative values must be nonnegative, we obtain $\mathcal{U}(a+\Delta a,b)-\mathcal{U}(a,b)\geq 0$. Line integral approximations will prove essential for robustifying the classical model of risk preferences and beliefs under subjective uncertainty, where the key variables will be the *events,* and constant-direction paths will not exist.

3.2 Robustifying expected utility analysis under objective uncertainty

As noted, this approach has already been used to establish the robustness of much of standard expected utility analysis under *objective* uncertainty.[10] Since the expected utility form $V_{EU}(\mathbf{P})=\sum_{i=1}^n U(x_i)\cdot p_i$ exhibits constant sensitivity in the *probabilities,* we choose probabilities rather than outcomes as the key variables, and accordingly represent each objective lottery in the form

$$\mathbf{P} \;=\; \{\underbrace{\ldots,p_{x'},p_{x''},p_{x'''},\ldots}_{x\in\mathcal{X}}\} \;=\; \{p_x\,|\,x\in\mathcal{X}\} \tag{7}$$

that is, as an *outcome-indexed list of probabilities* (summing to unity). The *change* between any pair of lotteries **P** and **P*** can be represented in terms of the changes in these probabilities:

$$\mathbf{P}^*–\mathbf{P}=\{\underbrace{\ldots,\,p^*_{x'}-p_{x'},\,p^*_{x''}-p_{x''},\,p^*_{x'''}-p_{x'''},\,\ldots}_{x\in\mathcal{X}}\} \;=\; \{\Delta p_x\,|\,x\in\mathcal{X}\} \tag{8}$$

and *changes in expected utility* can be represented in terms of these probability changes and $V_{EU}(\cdot)$'s constant sensitivity rates $\{\ldots,U(x'),U(x''),U(x'''),\ldots\}$ to these probability changes:

$$V_{EU}(\mathbf{P}^*)-V_{EU}(\mathbf{P}) \;=\; \sum_{x\in\mathcal{X}} U(x)\cdot(p^*_x-p_x) \;=\; \sum_{x\in\mathcal{X}} U(x)\cdot\Delta p_x \tag{9}$$

Given the choice of probabilities as the key variables, a *smooth non-expected utility preference function* $V(\cdot)$ is defined as one that is differentiable in the probabilities, i.e. one that satisfies

$$V(\mathbf{P}^*)-V(\mathbf{P}) \;=\; \sum_{x\in\mathcal{X}} U(x;\mathbf{P})\cdot(p^*_x-p_x) \;+\; o\big(||\mathbf{P}^*-\mathbf{P}||\big) \tag{10}$$

[10] See the references in Note 1. The following results are from Machina (1982).

where $\{U(x;\mathbf{P})\,|\,x \in \mathcal{X}\} \equiv \{\partial V(\mathbf{P})/\partial p_x\,|\,x \in \mathcal{X}\}$ are $V(\cdot)$'s partial derivatives (*local sensitivity rates*) with respect to the variables $\{p_x\,|\,x \in \mathcal{X}\}$ at $\mathbf{P}$, for some choice of norm $||\cdot||$. Because of the close correspondence of (9) and (10), $U(\cdot;\mathbf{P})$ is termed the *local utility function* of $V(\cdot)$ at the lottery $\mathbf{P}$.

By replacing standard conditions on the expected utility sensitivity rates $\{U(x)\,|\,x \in \mathcal{X}\}$ with the corresponding conditions on the *local* sensitivity rates $\{U(x;\mathbf{P})\,|\,x \in \mathcal{X}\}$ of a smooth non-expected utility $V(\cdot)$ at each lottery $\mathbf{P}$, we can obtain the following global robustness results:

$V(\cdot)$ exhibits global weak first order stochastic dominance preference	$\Leftrightarrow$	$U(x;\mathbf{P})$ is nondecreasing in x at each $\mathbf{P}$
$V(\cdot)$ exhibits global weak risk aversion (weak aversion to mean-preserving spreads)	$\Leftrightarrow$	$U(x;\mathbf{P})$ is concave in x at each $\mathbf{P}$
$V^*(\cdot)$ is at least as risk averse as $V(\cdot)$ (see Machina (1982, Thm.4) for specifics)	$\Leftrightarrow$	$U^*(x;\mathbf{P})$ is at least as concave in x as $U(x;\mathbf{P})$ at each $\mathbf{P}$

As with the global robustness result (6), such results are proven by means of line integrals along constant-direction paths, which in this setting consist of the *probability mixture path* $\{\mathbf{P}_\alpha = (\ldots, \alpha\cdot p^*_x + (1-\alpha)\cdot p_x, \ldots)\,|\,\alpha \in [0,1]\}$ from $\mathbf{P}$ to $\mathbf{P}^*$, and which generate the line integral formula

$$V(\mathbf{P}^*)-V(\mathbf{P}) = \int_0^1 \frac{dV(\mathbf{P}_\alpha)}{d\alpha}\cdot d\alpha = \int_0^1 \Big[\sum_{x\in\mathcal{X}} U\big(x;\mathbf{P}_\alpha\big)\cdot\big(p^*_x - p_x\big)\Big]\cdot d\alpha \qquad (11)$$

Researchers have applied and extended this approach to establish the global robustness of several other key results of objective expected utility theory, including first order conditions for optimization, comparative statics with respect to changes in risk, and many results from insurance theory.

The following section prepares for our application of this robustness approach to subjective uncertainty, by (i) showing that the classical model exhibits constant sensitivity in the *events* attached to each outcome; (ii) defining a notion of "smoothness in the events"; and (iii) showing that while it is impossible to construct "constant-direction paths" in the events, we can work with *line-integral approximation paths,* analogous to those in the right diagram of Figure 1.

4 Robustifying the classical model: Preliminary steps

Although we shall consider more general settings in Section 6.2, the formal analysis of this paper will be conducted within the following framework:

$\mathcal{X} = \{\ldots, x, \ldots\}$	arbitrary space of *outcomes* or *consequences*
$\mathcal{S} = [\underline{s}, \overline{s}] \subset R^1$	set of *states of nature,* with uniform Lebesgue measure $\lambda(\cdot)$
$\mathcal{E} = \{\ldots, E, \ldots\}$	algebra of *events* (each a finite union of intervals) in $\mathcal{S}$

$f(\cdot) = [x_1 \text{ on } E_1; \ldots ; x_n \text{ on } E_n]$	finite-outcome *act,* for some $\mathcal{E}$-measurable[11] partition $\{E_1, \ldots , E_n\}$ of $\mathcal{S}$
$\mathcal{A} = \{\ldots , f(\cdot), \ldots\}$	set of all finite-outcome, $\mathcal{E}$-measurable acts
$W(\cdot)$ and $\succcurlyeq$	*preference function* and its corresponding *preference relation* on $\mathcal{A}$

We define $W(\cdot)$'s *outcome ranking* by $x^* \succcurlyeq/\succ/\sim x$ if and only if $W(x^* \text{ on } \mathcal{S}) \geq/>/= W(x \text{ on } \mathcal{S})$. An event E is said to be *null* for preference function $W(\cdot)$ if $W(\cdot)$ is always indifferent to the payoff assigned to E, so that $W(x^* \text{ on } E; f(\cdot) \text{ elsewhere}) = W(x \text{ on } E; f(\cdot) \text{ elsewhere})$ for all x^*, x and $f(\cdot)$.

4.1 Event-theoretic representation of the classical model

Although a subjective act is typically viewed as a mapping $f(\cdot)\colon \mathcal{S} \to \mathcal{X}$ from states to outcomes, it can be also represented in the form $[x_1 \text{ on } f^{-1}(x_1); \ldots ; x_n \text{ on } f^{-1}(x_n)]$, that is, in terms of its distinct outcomes $x_1, \ldots , x_n$ and their associated *events* $f^{-1}(x_1), \ldots , f^{-1}(x_n)$. More generally, we can represent each act in $\mathcal{A}$ as an event-valued mapping $f^{-1}(\cdot) : \mathcal{X} \to \mathcal{E}$ over the *entire* outcome set $\mathcal{X}$, or by analogy with (7), as an *outcome-indexed partition* of $\mathcal{S}$*:*

$$f(\cdot) \;=\; \{\underbrace{\ldots , E_{x'}, E_{x''}, E_{x'''}, \ldots}_{x \in \mathcal{X}}\} \;=\; \{E_x \,|\, x \in \mathcal{X}\} \qquad (12)$$

where each event $E_x = f^{-1}(x)$ is $\mathcal{E}$-measurable, and $E_x = \emptyset$ for all but a finite number of $x \in \mathcal{X}$. Figure 2 illustrates this notation for an act $f(\cdot)$ (the solid line) over the state space $\mathcal{S}$.

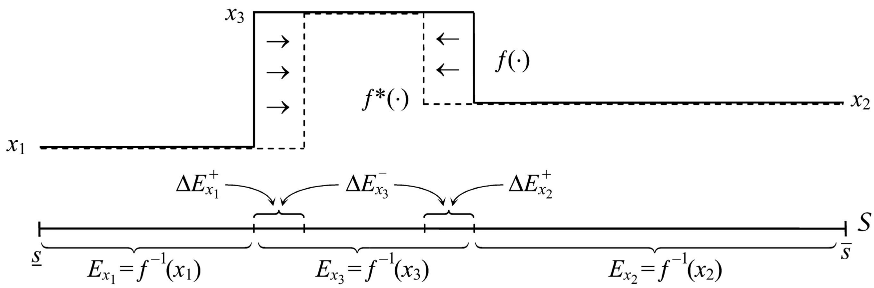

Figure 2. Event-theoretic representation of an act $f(\cdot)$ and the change $f(\cdot) \to f^*(\cdot)$

We can represent the *change* from any act $f(\cdot)$ to another act $f^*(\cdot)$ in terms of the changes – the growth and/or shrinkage – in each outcome's event E_x. For each $x \in \mathcal{X}$, we define the

[11] A partition $\{E_1, \ldots , E_n\}$ is $\mathcal{E}$*-measurable* if $E_i \in \mathcal{E}$ for each i.

$$\begin{aligned} \textit{growth set of } x: \ \Delta E_x^+ &= E_x^* - E_x = f^{*-1}(x) - f^{-1}(x) \\ \textit{shrinkage set of } x: \ \Delta E_x^- &= E_x - E_x^* = f^{-1}(x) - f^{*-1}(x) \end{aligned} \tag{13}$$

and represent the change $f(\cdot) \to f^*(\cdot)$ in terms of this family $\{(\Delta E_x^+, \Delta E_x^-) | x \in \mathcal{X}\}$ of growth and shrinkage sets, which we to refer to collectively as *change sets*. Since each state in the growth set of one outcome must also be in the shrinkage set of some other outcome, and vice versa, we have

$$\bigcup_{x\in\mathcal{X}} \Delta E_x^+ = \bigcup_{x\in\mathcal{X}} \Delta E_x^- = \{s \in \mathcal{S} \,|\, f^*(s) \neq f(s)\} \tag{14}$$

which we term the *total change set* between $f(\cdot)$ and $f^*(\cdot)$. Figure 2 also illustrates the growth and shrinkage sets for the change from $f(\cdot)$ to the (dashed) act $f^*(\cdot)$.

Since we retain Savage's approach of imposing no structure on the outcome space $\mathcal{X}$, we cannot define the "distance" between a pair of acts $f(\cdot)$ and $f^*(\cdot)$ in terms of the "closeness" or "distance" between their outcomes. Instead, we base it on the size (Lebesgue measure) of the total change set (14), and define the *distance function*[12]

$$\delta(f^*, f) \equiv \lambda\{s \in S \,|\, f^*(s) \neq f(s)\} = \lambda\Big\{\bigcup_{x\in\mathcal{X}} \Delta E_x^+\Big\} = \lambda\Big\{\bigcup_{x\in\mathcal{X}} \Delta E_x^-\Big\} \tag{15}$$

and we assume each preference function $W(\cdot)$ is *event-continuous* in the sense that

$$\lim_{\delta(f^*,f)\to 0} W(f^*(\cdot)) = W(f(\cdot)) \qquad \text{all } f(\cdot) \in \mathcal{A} \tag{16}$$

Like Savage's own continuity axiom P6 (1954, p. 39), event-continuity implies indifference between any two acts that differ only on a finite set of states, which for the classical form $W_{SEU}(\cdot)$ serves to rule out any atoms (mass points) in its subjective probability measure $\mu(\cdot)$. However, neither Savage's P6 nor the event-continuity condition (16) is strong enough to ensure that $\mu(\cdot)$ will be *absolutely continuous*[13] with respect to Lebesgue measure $\lambda(\cdot)$, which is required for it to possess a density function $\nu(\cdot)$ such that $\mu(E) = \int_E \nu(s) \cdot ds$ for all E. We shall ensure absolute continuity of all measures and signed measures considered in this paper by assuming that every preference function $W(\cdot)$ over $\mathcal{A}$ is *absolutely event-continuous* in the sense that $W(x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n)$ is an absolutely continuous function of the boundary points of each interval or finite interval union E_i, and similarly for the first order term in (19) below.

[12] Since $\delta(f^*, f) = 0$ for any acts $f(\cdot)$ and $f^*(\cdot)$ that differ only on a set of Lebesgue measure zero, $\delta(\cdot, \cdot)$ is not a metric. It is, however, a *pseudometric*, since it will satisfy the triangle inequality $\delta(f^{**}, f) \leq \delta(f^{**}, f^*) + \delta(f^*, f)$.

[13] A measure or signed measure on $\mathcal{S}$ is *absolutely continuous* with respect to Lebesgue measure $\lambda(\cdot)$ if it assigns zero measure to every set $E \subseteq \mathcal{S}$ for which $\lambda(E) = 0$, which rules out atoms as well as nonatomic "singular continuous" measures such as the Cantor measure (e.g. Billingsley, 1986, pp. 427–429; Romano and Siegel, 1986, pp. 27–28). Although Savage's formulation implied only finitely-additive subjective probability (1954, Sect. 3.4), we follow Arrow (1970, Ch. 2) and others and assume countable additivity for all measures and signed measures in this paper.

Given the above, we can represent the classical preference function as[14]

$$\begin{aligned} W_{SEU}\big(f(\cdot)\big) \;&\equiv\; \int_{\mathcal{S}} U\big(f(s)\big)\cdot d\mu(s) \;\equiv\; \sum_{x\in\mathcal{X}} \int_{f^{-1}(x)} U(x)\cdot d\mu(s) \\ &\equiv\; \sum_{x\in\mathcal{X}} \int_{E_x} U(x)\cdot \nu(s)\cdot ds \;\equiv\; \sum_{x\in\mathcal{X}} \int_{E_x} \phi(x,s)\cdot ds \end{aligned} \tag{17}$$

for what Myerson (1979) has termed the outcome-state *evaluation function* $\phi(x,s) \equiv U(x)\cdot\nu(s)$. The function $\phi(x,s)$ – which gives the effect of receiving outcome x in state s – is seen to fully characterize $W_{SEU}(\cdot)$'s risk attitudes (when viewed as a function of x) as well as its beliefs (when viewed as a function of s). Both of these types of characterizations will be shown to be robust.

Since $W_{SEU}(\cdot)$ is seen to evaluate each outcome's event E_x by an additive function (i.e. signed measure) $\Phi_x(E_x) \equiv \int_{E_x}\phi(x,s)\cdot ds$ and then sum these evaluations, it is said to be *event-additive.* Event-additivity is the subjective analogue of linearity in the probabilities, and is seen to imply *constant sensitivity in the events* – that is, the property that $W_{SEU}(\cdot)$'s response to any change $f(\cdot) \to f^*(\cdot)$ will only depend upon $f(\cdot)$ and $f^*(\cdot)$ through their change sets $\{(\Delta E_x^+, \Delta E_x^-)\,|\,x\in\mathcal{X}\}$:

$$\begin{aligned} W_{SEU}\big(f^*(\cdot)\big) - W_{SEU}\big(f(\cdot)\big) \;&\equiv\; \sum_{x\in\mathcal{X}} \Big[\int_{E_x^*}\phi(x,s)\cdot ds - \int_{E_x}\phi(x,s)\cdot ds\Big] \\ &\equiv\; \sum_{x\in\mathcal{X}} \Big[\int_{\Delta E_x^+}\phi(x,s)\cdot ds - \int_{\Delta E_x^-}\phi(x,s)\cdot ds\Big] \end{aligned} \tag{18}$$

where the second line follows by subtracting $\int_{E_x^*\cap E_x}\phi(x,s)\cdot ds$ from each integral in the first line. The Allais changes $\alpha_1 \to \alpha_2$ and $\alpha_3 \to \alpha_4$ in Table 1 both involve the same family of change sets, as do the Ellsberg changes $\gamma_1 \to \gamma_2$ and $\gamma_3 \to \gamma_4$, which is why each example constitutes a test of event additivity/constant sensitivity in the events.

The *state-dependent* expected utility form $W_{SDEU}(\cdot)$ is also seen to be event-additive, with evaluation function $\phi(x,s) \equiv U(x|s)\cdot\nu(s)$:

$$\begin{aligned} W_{SDEU}\big(f(\cdot)\big) \;&\equiv\; \int_{\mathcal{S}} U\big(f(s)|s\big)\cdot d\mu(s) \;\equiv\; \sum_{x\in\mathcal{X}} \int_{f^{-1}(x)} U(x|s)\cdot d\mu(s) \\ &\equiv\; \sum_{x\in\mathcal{X}} \int_{E_x} U(x|s)\cdot \nu(s)\cdot ds \;\equiv\; \sum_{x\in\mathcal{X}} \int_{E_x} \phi(x,s)\cdot ds \end{aligned} \tag{17$'$}$$

and accordingly, $W_{SDEU}(\cdot)$ also exhibits constant sensitivity in the events:

$$\begin{aligned} W_{SDEU}\big(f^*(\cdot)\big) - W_{SDEU}\big(f(\cdot)\big) \;&\equiv\; \sum_{x\in\mathcal{X}} \Big[\int_{E_x^*}\phi(x,s)\cdot ds - \int_{E_x}\phi(x,s)\cdot ds\Big] \\ &\equiv\; \sum_{x\in\mathcal{X}} \Big[\int_{\Delta E_x^+}\phi(x,s)\cdot ds - \int_{\Delta E_x^-}\phi(x,s)\cdot ds\Big] \end{aligned} \tag{18$'$}$$

[14] Since we restrict attention to finite-outcome acts $f(\cdot)$, all sums of the form $\sum_{x\in\mathcal{X}}$ in this and the following sections will only involve a finite number of nonzero terms.

The following result shows that, under absolute event-continuity, the forms $W_{SEU}(\cdot)$ and $W_{SDEU}(\cdot)$ in fact characterize the properties of event-additivity and constant-sensitivity in the events:

Theorem 1 (Characterization of event-additivity/constant sensitivity in the events). The following conditions on an absolutely event-continuous preference function $W(\cdot)$ over $\mathcal{A}$ are equivalent:

(a) $W(\cdot)$ takes the classical form $W_{SEU}(f(\cdot)) \equiv \int_S U(f(s))\cdot d\mu(s)$, or the state-dependent expected utility form $W_{SDEU}(f(\cdot)) \equiv \int_S U(f(s)|s)\cdot d\mu(s)$, for some absolutely continuous subjective probability measure $\mu(\cdot)$.

(b) $W(\cdot)$ is *event-additive:* there exists a family of absolutely continuous signed measures $\{\Phi_x(\cdot)\,|\,x\in\mathcal{X}\}$ such that $W(f(\cdot)) \equiv \sum_{x\in\mathcal{X}}\Phi_x(f^{-1}(x)) \equiv \sum_{x\in\mathcal{X}}\Phi_x(E_x)$.

(c) $W(\cdot)$ exhibits *constant sensitivity in the events:* if the changes $f_1(\cdot)\to f_2(\cdot)$ and $f_3(\cdot)\to f_4(\cdot)$ involve the same change sets $\{(\Delta E_x^+,\Delta E_x^-)\,|\,x\in\mathcal{X}\}$, then $W(f_2(\cdot))-W(f_1(\cdot)) = W(f_4(\cdot))-W(f_3(\cdot))$.

4.2 Event-smoothness and the local evaluation function

Since constant sensitivity in the events is equivalent to event-additivity, a natural definition of "differentiability in the events" is *local event-additivity,* that is:

Definition. An absolutely event-continuous preference function $W(\cdot)$ over $\mathcal{A}$ is said to be *event-differentiable* at an act $f(\cdot)$ if there exists a family $\{\Phi_x(\cdot;f)\,|\,x\in\mathcal{X}\}$ of absolutely continuous signed measures such that

$$W\big(f^*(\cdot)\big)-W\big(f(\cdot)\big)=\sum_{x\in\mathcal{X}}\Phi_x\big(f^{*-1}(x);f\big)-\sum_{x\in\mathcal{X}}\Phi_x\big(f^{-1}(x);f\big)+o\big(\delta(f^*,f)\big) \tag{19}$$

where $o(\cdot)$ denotes a function that is zero at zero and of higher order than its argument.

Since the signed measures $\{\Phi_x(\cdot;f)\,|\,x\in\mathcal{X}\}$ are absolutely continuous, they can be represented by a family of signed density functions $\{\phi(x,\cdot;f)\,|\,x\in\mathcal{X}\}$, so we can express (19) as

$$\begin{aligned} W\big(f^*(\cdot)\big)-W\big(f(\cdot)\big) &\equiv \sum_{x\in\mathcal{X}}\Big[\int_{E_x^*}\phi(x,s;f)\cdot ds-\int_{E_x}\phi(x,s;f)\cdot ds\Big]+o\big(\delta(f^*,f)\big) \\ &\equiv \sum_{x\in\mathcal{X}}\Big[\int_{\Delta E_x^+}\phi(x,s;f)\cdot ds-\int_{\Delta E_x^-}\phi(x,s;f)\cdot ds\Big]+o\big(\delta(f^*,f)\big) \end{aligned} \tag{20}$$

Comparison with (18)/(18)$'$ yields that an event-differentiable $W(\cdot)$ will evaluate differential changes from an act $f(\cdot)$ in *precisely the same manner* as the expected utility forms $W_{SEU}(\cdot)$ and $W_{SDEU}(\cdot)$, with respect to its *local evaluation function* $\phi(x,s;f)$ at $f(\cdot)$. Thus, the differential effect upon $W(\cdot)$ of replacing outcome

x_0 by x at a state s is given by $\phi(x,s;f)-\phi(x_0,s;f)$, just as it is given by the expression $\phi(x,s)-\phi(x_0,s)$ for $W_{SEU}(\cdot)$ or $W_{SDEU}(\cdot)$. Since the local evaluation function will thus only enter into (20) through its statewise differences, we can select arbitrary $\underline{x}\in\mathcal{X}$ and additively normalize it, so that without loss of generality we shall assume $\phi(\underline{x},s;f)=0$ for all $s\in\mathcal{S}$ and all $f(\cdot)\in\mathcal{A}$.

To obtain our formal results we must impose some regularity on how much the local evaluation function $\phi(x,s;f)$ can vary in its arguments s and $f(\cdot)$. Although the space of acts $\mathcal{A}$ is not compact with respect to the distance function $\delta(\cdot,\cdot)$, our regularity conditions include the properties that a continuous $\phi(x,\cdot;\cdot)$ function would exhibit if its domain $\mathcal{S}\times\mathcal{A}$ *was* compact:

for each outcome x	:	$\phi(x,s;f)$ is *uniformly continuous* over $S\times A$
for each outcome x	:	$\phi(x,s;f)$ is *bounded above and below* on $S\times A$ (21)
for each pair $x^*\succ x$ and nonnull event E	:	$\int_E[\phi(x^*,s;f)-\phi(x,s;f)]\cdot ds$ is both *bounded above* and *bounded above* 0, uniformly in $f(\cdot)$

We thus define a general event-differentiable preference function $W(\cdot)$ on $\mathcal{A}$ to be *event-smooth* if it satisfies these properties, which can be stated more formally as:

for each $x\in\mathcal{X}$ and $\varepsilon>0$ there exists $\delta_{x,\varepsilon}>0$ such that $|s'-s|<\delta_{x,\varepsilon}$ and $\delta(f',f)<\delta_{x,\varepsilon}$ implies $|\phi(x,s';f')-\phi(x,s;f)|<\varepsilon$

for each $x\in\mathcal{X}$ there exist $\overline{\phi}_x$ and $\underline{\phi}_x$ such that $\overline{\phi}_x>\phi(x,s;f)>\underline{\phi}_x$ for all $s\in\mathcal{S}$ and all $f(\cdot)\in\mathcal{A}$ (21)′

for each pair $x^*\succ x$ and nonnull $E\in\mathcal{E}$ there exist $\overline{\Phi}_{x^*,x,E}>\underline{\Phi}_{x^*,x,E}>0$ such that $\overline{\Phi}_{x^*,x,E}>\int_E[\phi(x^*,s;f)-\phi(x,s;f)]\cdot ds>\underline{\Phi}_{x^*,x,E}$ for all $f(\cdot)\in\mathcal{A}$

Under event-smoothness, the local evaluation function $\phi(x,s;f)$ can be obtained by differentiating $W(\cdot)$ in the appropriate event-theoretic manner. Defining the ε-ball $B_{s,\varepsilon}=[s-\varepsilon,s+\varepsilon]$ about any interior state $s\in\mathcal{S}$, equation (20), event-smoothness and the normalization $\phi(\underline{x},s;f)\equiv 0$ imply[15]

$$\phi(x,s;f)=\lim_{\varepsilon\to 0}\frac{W\begin{pmatrix}x & \text{on} & B_{s,\varepsilon}\\ f(\cdot) & \text{on} & \mathcal{S}-B_{s,\varepsilon}\end{pmatrix}-W\begin{pmatrix}\underline{x} & \text{on} & B_{s,\varepsilon}\\ f(\cdot) & \text{on} & \mathcal{S}-B_{s,\varepsilon}\end{pmatrix}}{2\cdot\varepsilon} \tag{22}$$

The (pre-normalized) local evaluation function for the form $W_{SEU}(\cdot)$ at any act $f(\cdot)$ is simply $\phi(x,s;f)\equiv U(x)\cdot\nu(s)$, and for $W_{SDEU}(\cdot)$ it is $\phi(x,s;f)\equiv U(x|s)\cdot\nu(s)$. The "constant sensitivity in the events" property of these forms is reflected in the fact that their local evaluation functions do not depend upon the act $f(\cdot)$. When the other forms we have considered are event-smooth, their pre-normalized local evaluation functions are given by:

[15] Not surprisingly, (22) is similar to the classic *derivative of a set-function* $\lim_{\varepsilon\to 0}[\Omega(E\cup B_{s,\varepsilon})-\Omega(E-B_{s,\varepsilon})]/\lambda(B_{s,\varepsilon})$, for a function $\Omega(\cdot)$ defined over sets $E\subseteq R^n$ (e.g., Hahn and Rosenthal, 1948, Ch.V; Jeffery, 1953, pp. 125–127).

Probabilistically Sophisticated Non-Expected Utility: Writing $W_{PS}(f(\cdot)) \equiv V(\mathbf{P}_{f,\mu})$ where $\mathbf{P}_{f,\mu} \equiv (\ldots ; x, \mu(f^{-1}(x)); \ldots)$ is the lottery implied by $f(\cdot)$ under the subjective probability measure $\mu(\cdot)$, we have

$$\phi(x,s;f) \;\equiv\; U(x;\mathbf{P}_{f,\mu})\cdot\nu(s) \tag{23}$$

where $U(x;\mathbf{P})$ is the local utility function of $V(\cdot)$ (from (10)).

Choquet Expected Utility:[16] Writing $W_{Choquet}(f(\cdot)) \equiv \sum_{x\in\mathcal{X}} U(x)\cdot[C(E^{\preccurlyeq}_{x,f}) - C(E^{\prec}_{x,f})]$ for $f(\cdot)$'s *weak and strict cumulative payoff sets* $E^{\preccurlyeq}_{x,f} = \{s \in \mathcal{S} \,|\, f(s) \preccurlyeq x\}$ and $E^{\prec}_{x,f} = \{s \in \mathcal{S} \,|\, f(s) \prec x\}$, and defining $C(\cdot)$'s *generalized density* $c(s;E) \equiv \lim_{\varepsilon\to 0}[C(E \cup B_{s,\varepsilon}) - C(E - B_{s,\varepsilon})]/(2\cdot\varepsilon)$, we have[17]

$$\phi(x,s;f) \;\equiv\; U(x)\cdot c\big(s;E^{\preccurlyeq}_{x,f}\big) \,+\, \sum_{y\succ x} U(y)\cdot\big[c\big(s;E^{\preccurlyeq}_{y,f}\big) - c\big(s;E^{\prec}_{y,f}\big)\big] \tag{24}$$

Maxmin Expected Utility: Writing $W_{maxmin}(f(\cdot)) \equiv \sum_{x\in\mathcal{X}} U(x)\cdot\mu_{\tau^*(f(\cdot))}(f^{-1}(x))$ where $\tau^*(f(\cdot)) \equiv \text{argmin}_{\tau\in\mathcal{T}} \sum_{x\in\mathcal{X}} U(x)\cdot\mu_\tau(f^{-1}(x))$, $W_{maxmin}(f(\cdot))$ is seen to have a structure similar to that of a *support function* from convex analysis.[18] Thus if the measures $\{\mu_\tau(\cdot)\,|\,\tau\in\mathcal{T}\}$ have densities $\{\nu_\tau(\cdot)\,|\,\tau\in\mathcal{T}\}$, $W_{maxmin}(\cdot)$'s smoothness properties about any act $f(\cdot)$ will depend upon the properties of $\tau^*(\cdot)$ at that act as follows:

If $\tau^*(\cdot)$ is constant over some neighborhood of $f(\cdot)$, $W_{maxmin}(\cdot)$ takes the classical form over that region, and we will have $\phi(x,s;f) \equiv U(x)\cdot\nu_{\tau^*(f(\cdot))}(s)$

If $\tau^*(\cdot)$ is not constant about $f(\cdot)$ but at least continuous there, a standard envelope theorem argument implies that we will continue to have $\phi(x,s;f) \equiv U(x)\cdot\nu_{\tau^*(f(\cdot))}(s)$

If $\tau^*(\cdot)$ is not continuous at $f(\cdot)$, $W_{maxmin}(\cdot)$ is "kinked in the events" and not event-differentiable at $f(\cdot)$, although it may have directional event-derivatives

Given a pair of constant acts $f(\cdot) = [x \text{ on } \mathcal{S}]$ and $f^*(\cdot) = [x^* \text{ on } \mathcal{S}]$, we define the *single-sweep path* $\{f_\omega(\cdot)\,|\,\omega\in[\underline{s},\overline{s}]\}$ from $f(\cdot)$ to $f^*(\cdot)$ by[19]

$$f_\omega(\cdot) \;\equiv\; \big[x^* \text{ on } [\underline{s},\omega]; x \text{ on } (\omega,\overline{s}\,]\big] \qquad \omega\in[\underline{s},\overline{s}] \tag{25}$$

This path is illustrated in Figure 3. As ω runs from $\underline{s}$ to $\overline{s}$, the outcome x is replaced by x^* over an expanding interval $[\underline{s},\omega]$ whose right edge sweeps across the state space $\mathcal{S}$ at a uniform rate. Since $\delta(f_\omega, f_{\omega'}) \equiv |\omega-\omega'|$, event-continuity implies that $W(f_\omega(\cdot))$ is continuous in ω.

[16] For notational simplicity, this derivation will assume $U(x^*) \neq U(x)$ whenever $x^* \neq x$.

[17] When working with (24), recall that since $C(\cdot)$ is nonadditive we will typically have $\int_E c(s;E)\cdot ds \neq C(E)$, in other words, the generalized densities $c(s;E^{\preccurlyeq}_{x,f})$ and $c(s;E^{\prec}_{x,f})$ typically *do not* "integrate back" to $C(E^{\preccurlyeq}_{x,f})$ or $C(E^{\prec}_{x,f})$.

[18] A function on R^n is a *support function* if it is the pointwise minimum or the pointwise maximum of some family of linear functions (e.g., Rockafellar (1970, Ch.13)).

[19] Note that the initial act on this path is not actually $f(\cdot) = [x \text{ on } \mathcal{S}]$, but rather $f_{\underline{s}}(\cdot) = [x^* \text{ on } [\underline{s},\underline{s}]; x \text{ on } (\underline{s},\overline{s}\,]]$, which by event-continuity will satisfy $W(f_{\underline{s}}(\cdot)) = W(f(\cdot))$.

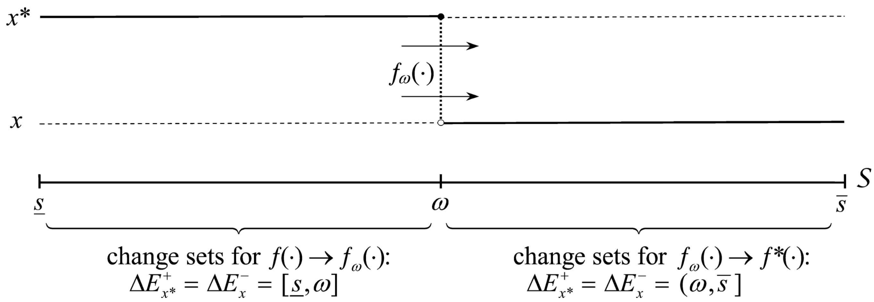

Figure 3. Single-sweep path from the constant act $f(\cdot) = [x \text{ on } \mathcal{S}]$ to $f^*(\cdot) = [x^* \text{ on } \mathcal{S}]$

Given an arbitrary $\hat{\omega} \in [\underline{s}, \overline{s}]$, event-differentiability implies

$$W\big(f_\omega(\cdot)\big) - W\big(f_{\hat{\omega}}(\cdot)\big) = \int_{\hat{\omega}}^{\omega} \phi(x^*, s; f_{\hat{\omega}}) \cdot ds - \int_{\hat{\omega}}^{\omega} \phi(x, s; f_{\hat{\omega}}) \cdot ds + o\big(|\omega - \hat{\omega}|\big) \quad (26)$$

for all $\omega > \hat{\omega}$, with a corresponding formula for $\omega < \hat{\omega}$, so that $W(f_\omega(\cdot))$ is differentiable in ω, with

$$\left.\frac{dW\big(f_\omega(\cdot)\big)}{d\omega}\right|_{\omega=\hat{\omega}} = \phi(x^*, \hat{\omega}; f_{\hat{\omega}}) - \phi(x, \hat{\omega}; f_{\hat{\omega}}) \quad (27)$$

By event-smoothness this derivative is seen to be continuous in $\hat{\omega}$, so that

$$W\big(f^*(\cdot)\big) - W\big(f(\cdot)\big) = \int_{\underline{s}}^{\overline{s}} \frac{dW\big(f_\omega(\cdot)\big)}{d\omega} \cdot d\omega = \int_{\underline{s}}^{\overline{s}} \big[\phi(x^*, \omega; f_\omega) - \phi(x, \omega; f_\omega)\big] \cdot d\omega \quad (28)$$

which illustrates how $W(\cdot)$’s ranking of $f(\cdot)$ versus $f^*(\cdot)$ can be exactly expressed in terms of its local evaluation function along the path $\{f_\omega(\cdot) \,|\, \omega \in [\underline{s}, \overline{s}]\}$.

Single-sweep paths can also be constructed between pairs of *nonconstant* acts $f(\cdot)$ and $f^*(\cdot)$, by defining $f_\omega(\cdot) \equiv [f^*(\cdot) \text{ on } [\underline{s}, \omega]; f(\cdot) \text{ on } (\omega, \overline{s}]]$. In this case, as ω runs from $\underline{s}$ to $\overline{s}$ the interval $[\underline{s}, \omega]$ sweeps rightward, replacing the outcome $f(\omega)$ by $f^*(\omega)$ at its right edge, and similar derivations yield that $W(f_\omega(\cdot))$ is continuous in ω, with $dW(f_\omega(\cdot))/d\omega = \phi(f^*(\omega), \omega; f_\omega) - \phi(f(\omega), \omega; f_\omega)$ at each point $\omega \in [\underline{s}, \overline{s}]$ where both $f^*(\cdot)$ and $f(\cdot)$ are continuous. Since $f(\cdot)$ and $f^*(\cdot)$ can each have only a finite number of discontinuities, we obtain the general comparison formula

$$W\big(f^*(\cdot)\big) - W\big(f(\cdot)\big) = \int_{\underline{s}}^{\overline{s}} \big[\phi\big(f^*(\omega), \omega; f_\omega\big) - \phi\big(f(\omega), \omega; f_\omega\big)\big] \cdot d\omega \quad (28)'$$

Although single-sweep paths suffice to establish the exact comparison formulas (28) and (28)′, they cannot serve as a general engine for establishing global robustness results, for the simple reason that they are not “constant-direction paths” in the space of subjective acts. To see this, recall that an inherent property of any constant-direction path (such as the left-hand path in Figure 1) is that the variable changes in going from the starting point to the midpoint of the path are *identical* to the variable changes in going from the midpoint to the final point, which implies that any *constant-sensitivity* function will exhibit an equal response along each half of the path.[20]

However for single-sweep paths, say the path $f_\omega(\cdot) \equiv [x^* \text{ on } [\underline{s},\omega]; x \text{ on } (\omega,\overline{s}]]$ of Figure 3, the variable changes in going from its starting point $f_{\underline{s}}(\cdot)$ to its midpoint $f_{(\underline{s}+\overline{s})/2}(\cdot)$ are the change sets $\Delta E^+_{x^*} = \Delta E^-_x = [\underline{s}, (\underline{s}+\overline{s})/2]$, whereas the variable changes in going from $f_{(\underline{s}+\overline{s})/2}(\cdot)$ to its final point $f_{\overline{s}}(\cdot)$ are the distinct (in fact, *disjoint*) change sets $\Delta E^+_{x^*} = \Delta E^-_x = ((\underline{s}+\overline{s})/2, \overline{s}]$. Accordingly, the response of any constant-sensitivity function $W_{SEU}(f(\cdot)) \equiv \int_{\mathcal{S}} U(f(s))\cdot d\mu(s)$ over the first half of this path will be $[U(x^*) - U(x)]\cdot\mu([\underline{s}, (\underline{s}+\overline{s})/2])$, whereas its response over the second half of the path will be the generally *distinct* value $[U(x^*) - U(x)]\cdot\mu(((\underline{s}+\overline{s})/2, \overline{s}])$.[21]

In fact, in our framework of purely subjective acts, where the key variables are the *events* $\{E_x \,|\, x \in \mathcal{X}\}$ and their change sets $\{(\Delta E^+_x, \Delta E^-_x) \,|\, x \in \mathcal{X}\}$, constant-direction paths *do not exist,* for the simple reason that no family of event changes $\{(\Delta E^+_x, \Delta E^-_x) \,|\, x \in \mathcal{X}\}$ can ever be applied more than once in succession. In other words, once an event E_x has expanded by ΔE^+_x to become $E_x \cup \Delta E^+_x$, it cannot expand by ΔE^+_x again, and once E_x has shrunk by ΔE^-_x to become $E_x - \Delta E^-_x$, it cannot shrink by ΔE^-_x again. Thus *no* path $\{f_\alpha(\cdot) \,|\, \alpha \in [\underline{\alpha}, \overline{\alpha}]\}$ in $\mathcal{A}$ – single sweep or otherwise – can have the property that its change sets from $f_{\underline{\alpha}}(\cdot)$ to $f_{(\underline{\alpha}+\overline{\alpha})/2}(\cdot)$ are the same as its change sets from $f_{(\underline{\alpha}+\overline{\alpha})/2}(\cdot)$ to $f_{\overline{\alpha}}(\cdot)$. Since we cannot prove global robustness results by *constant-direction paths* as in the left diagram of Figure 1, we must use *line integral approximations* along paths that come *arbitrarily close* to the constant-direction property, as in the right diagram of the figure.

The nonexistence of constant-direction paths between *subjective acts* is in distinct contrast to the case of *objective lotteries* considered in Section 3.2, where the probability mixture path $\{\mathbf{P}_\alpha = (\ldots, \alpha\cdot p^*_x + (1-\alpha)\cdot p_x, \ldots) \,|\, \alpha \in [0,1]\}$ from $\mathbf{P}$ to $\mathbf{P}^*$ *does* constitute a constant-direction path,[22] and hence allows for the use of the line integral formula (11) in establishing global robustness results. In order to obtain a path $\{f_\alpha(\cdot) \,|\, \alpha \in [\underline{\alpha}, \overline{\alpha}]\}$ from $f(\cdot)$ to $f^*(\cdot)$ that comes arbitrarily close to the constant-direction property – as in the right diagram of Figure 1 – we must work with subjective acts $f_\alpha(\cdot) \in \mathcal{A}$ that come arbitrarily close to being "objective probability mixtures" of $f^*(\cdot)$ and $f(\cdot)$. We present such "almost-objective" mixtures and paths in the following section.

[20] More generally, if a constant-direction path is divided into k equal portions, the variable changes from the beginning to the end of each portion will be identical, so any constant-sensitivity function will exhibit an equal response along each portion.

[21] The response of any constant-sensitivity form $W_{SDEU}(f(\cdot)) \equiv \int_{\mathcal{S}} U(f(s)|s)\cdot d\mu(s)$ on these two halves will be the generally distinct values $\int_{[\underline{s},(\underline{s}+\overline{s})/2]} [U(x^*|s) - U(x|s)]\cdot d\mu(s)$ and $\int_{((\underline{s}+\overline{s})/2,\overline{s}]} [U(x^*|s) - U(x|s)]\cdot d\mu(s)$.

[22] For this path, the changes from $\mathbf{P}_0$ to $\mathbf{P}_{1/2}$ are the same as from $\mathbf{P}_{1/2}$ to $\mathbf{P}_1$, namely the probability changes $\{\ldots, (p^*_{x'} - p_{x'})/2, (p^*_{x''} - p_{x''})/2, (p^*_{x'''} - p_{x'''})/2, \ldots\}$. The constant-sensitivity preference function $V_{EU}(\cdot)$ also exhibits the same response from $\mathbf{P}_0$ to $\mathbf{P}_{1/2}$ as from $\mathbf{P}_{1/2}$ to $\mathbf{P}_1$, namely $[V_{EU}(\mathbf{P}^*) - V_{EU}(\mathbf{P})]/2$.

4.3 Almost-objective events, acts and mixtures under subjective uncertainty

In the purely subjective framework of this paper, a given event E could be assigned different subjective likelihoods by different classical preference functions $W_{SEU}(f(\cdot)) \equiv \int_{\mathcal{S}} U(f(s)) \cdot d\mu(s)$ and $W^*_{SEU}(f(\cdot)) \equiv \int_{\mathcal{S}} U^*(f(s)) \cdot d\mu^*(s)$, and may not be assigned any well-defined likelihood at all by a preference function such as $W_{Choquet}(\cdot)$, $W_{maxmin}(\cdot)$, or a general event-smooth $W(\cdot)$. However, every Euclidean state space contains a special class of events – termed *almost objective events* – that approximate, and in the limit attain, the property of *unanimous, well-defined revealed likelihoods* for all event-smooth $W(\cdot)$. Examples of such events date back at least to Poincaré (1912), and as shown in Machina (2004), they will arbitrarily closely approximate virtually all of the belief and betting properties of objectively uncertain events for every event-smooth $W(\cdot)$.

Given our subjective state space $\mathcal{S} = [\underline{s}, \overline{s}] \subset R^1$, define $\lambda_{\mathcal{S}} = \lambda(\mathcal{S}) = \overline{s} - \underline{s}$. For arbitrary positive integer m, partition $\mathcal{S}$ into the m equal-length intervals

$$\left[\underline{s}\,,\underline{s}+\tfrac{\lambda_{\mathcal{S}}}{m}\right),\ldots,\left[\underline{s}+i\cdot\tfrac{\lambda_{\mathcal{S}}}{m}\,,\,\underline{s}+(i+1)\cdot\tfrac{\lambda_{\mathcal{S}}}{m}\right),\ldots,\left[\underline{s}+(m-1)\cdot\tfrac{\lambda_{\mathcal{S}}}{m}\,,\,\overline{s}\right] \quad (29)$$

and for any interval (or finite union of intervals) $\wp \subseteq [0,1]$, define the *almost-objective event*

$$\wp \underset{m}{\times} S \;=\; \bigcup_{i=0}^{m-1} \left\{\underline{s} + (i+\omega)\cdot\tfrac{\lambda_{\mathcal{S}}}{m} \,\middle|\, \omega \in \wp\right\} \quad (30)$$

that is, as the union of the linear images of $\wp$ into each of the m intervals in (29). Thus, $[0,\frac{1}{2}]\underset{m}{\times}\mathcal{S}$ would be the union of the left halves of these intervals, $[\frac{1}{3},\frac{2}{3}]\underset{m}{\times}\mathcal{S}$ would be the union of their middle thirds, etc. It is straightforward to show that such events will satisfy the limiting measure property

$$\lim_{m\to\infty} \mu\big(\wp \underset{m}{\times} \mathcal{S}\big) \;=\; \lambda(\wp) \quad (31)$$

for every continuous-density subjective probability measure $\mu(\cdot)$ on $\mathcal{S}$, where $\lambda(\cdot)$ is Lebesgue measure over [0,1]. In this sense, the events $\wp\underset{m}{\times}\mathcal{S}$ can be said to have a *limiting likelihood* of $\lambda(\wp)$ for each such subjective probability measure $\mu(\cdot)$, and hence for every probabilistically sophisticated preference function $W_{PS}(\cdot)$.

As shown in Machina (2004), this unanimous limiting likelihood property extends to every event-smooth preference function $W(\cdot)$, *whether or not* it is probabilistically sophisticated, or even based on an underlying probability measure or capacity at all. In other words, for any pair of disjoint finite interval unions $\wp, \wp' \subseteq [0,1]$ with $\lambda(\wp) > (=)\, \lambda(\wp')$, every event-smooth $W(\cdot)$ over $\mathcal{A}$ will satisfy

$$\lim_{m\to\infty} W\begin{pmatrix} x^* \text{ on } \wp\underset{m}{\times}\mathcal{S} \\ x \text{ on } \wp'\underset{m}{\times}\mathcal{S} \\ f(\cdot) \text{ elsewhere} \end{pmatrix} \underset{\substack{\text{all } x^*\succ x \\ \text{all } f(\cdot)\in\mathcal{A}}}{>(=)} \lim_{m\to\infty} W\begin{pmatrix} x \text{ on } \wp\underset{m}{\times}\mathcal{S} \\ x^* \text{ on } \wp'\underset{m}{\times}\mathcal{S} \\ f(\cdot) \text{ elsewhere} \end{pmatrix} \quad (32)$$

Besides unanimous revealed likelihoods, almost-objective events arbitrarily closely approximate another property of objective rather than subjective events – or more precisely, a property of the objective events in any Anscombe-Aumann (1963) type objective×subjective setting – namely the property that their revealed likelihoods will be *invariant to conditioning on any fixed subjective event* E. Specifically, for disjoint finite interval unions $\wp, \wp' \subseteq [0,1]$ with $\lambda(\wp) > (=)\, \lambda(\wp')$ and any fixed $E \in \mathcal{E}$, every event-smooth $W(\cdot)$ will satisfy

$$\lim_{m\to\infty} W\begin{pmatrix} x^* & \text{on } (\wp \underset{m}{\times} \mathcal{S}) \cap E \\ x & \text{on } (\wp' \underset{m}{\times} \mathcal{S}) \cap E \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \underset{\substack{\text{all } x^* \succ x \\ \text{all } f(\cdot) \in \mathcal{A}}}{> (=)} \lim_{m\to\infty} W\begin{pmatrix} x & \text{on } (\wp \underset{m}{\times} \mathcal{S}) \cap E \\ x^* & \text{on } (\wp' \underset{m}{\times} \mathcal{S}) \cap E \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \quad (33)$$

We accordingly define the *almost-objective subevents* of a subjective event E by $\wp \underset{m}{\times} E = (\wp \underset{m}{\times} \mathcal{S}) \cap E$.

Let $\{\wp_1, \ldots, \wp_n\}$ be a partition of [0,1] where each $\wp_i$ is a finite interval union. For each set of outcomes $x_1, \ldots, x_n \in \mathcal{X}$, we can define the *almost-objective act*

$$\big[x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\big] \in \mathcal{A} \quad (34)$$

and for each set of subjective acts $f_1(\cdot), \ldots, f_n(\cdot) \in \mathcal{A}$, we can define the *almost-objective mixture*

$$\big[f_1(\cdot) \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; f_n(\cdot) \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\big] \in \mathcal{A} \quad (35)$$

which yields outcome x on the event $E_x = \big(\wp_1 \underset{m}{\times} f_1^{-1}(x)\big) \cup \cdots \cup \big(\wp_n \underset{m}{\times} f_n^{-1}(x)\big)$.

Even though almost-objective acts and mixtures are elements of the subjective act space $\mathcal{A}$, preferences over them correspond more closely to preferences over objective lotteries and mixtures than to preferences over general subjective acts, in two respects. First, although a general event-smooth $W(\cdot)$ will not be probabilistically sophisticated over *general subjective acts* $f(\cdot)$, as $m \to \infty$ it will be probabilistically sophisticated over *almost-objective acts*. That is, each event-smooth $W(\cdot)$ has an associated *risk preference function* $V_W(\cdot)$ over objective lotteries $\mathbf{P} = (x_1, p_1; \ldots; x_n, p_n)$, such that

$$\lim_{m\to\infty} W\big(x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\big) \equiv V_W\big(x_1, \lambda(\wp_1); \ldots; x_n, \lambda(\wp_n)\big) \quad (36)$$

so that as $m \to \infty$, each event-smooth $W(\cdot)$ evaluates an almost-objective act solely according to its outcomes $x_1, \ldots, x_n$ and the limiting likelihood values $\lambda(\wp_1), \ldots, \lambda(\wp_n)$ of their events. The second property is that as $m \to \infty$, all event-smooth $W_{SEU}(\cdot)$ *as well as all event-smooth* $W_{SDEU}(\cdot)$[23] will be *linear* in almost-objective likelihoods and in almost-objective mixture likelihoods:

$$\begin{aligned} \lim_{m\to\infty} W_{SEU}\big(x_1 \text{ on } \wp_1 \underset{m}{\times} S; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} S\big) &\equiv \sum_{i=1}^{n} \lambda(\wp_i) \cdot U(x_i) \\ \lim_{m\to\infty} W_{SDEU}\big(x_1 \text{ on } \wp_1 \underset{m}{\times} S; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} S\big) &\equiv \sum_{i=1}^{n} \lambda(\wp_i) \cdot \int_S U(x_i|s) \cdot d\mu(s) \end{aligned} \quad (37)$$

and

[23] Recall that while state-dependent expected utility is inherently linear in *purely objective likelihoods,* and also in *objective mixtures* of subjective acts, it is generally *not* linear in the *subjective likelihoods* of a general subjective act.

$$\lim_{m\to\infty} W_{SEU}\big(f_1(\cdot) \text{ on } \wp_1 \underset{m}{\times} S;\dots;f_n(\cdot) \text{ on } \wp_n \underset{m}{\times} S\big) \equiv \sum_{i=1}^{n} \lambda(\wp_i)\cdot\int_S U\big(f_i(s)\big)\cdot d\mu(s)$$

$$\lim_{m\to\infty} W_{SDEU}\big(f_1(\cdot) \text{ on } \wp_1 \underset{m}{\times} S;\dots;f_n(\cdot) \text{ on } \wp_n \underset{m}{\times} S\big) \equiv \sum_{i=1}^{n} \lambda(\wp_i)\cdot\int_S U\big(f_i(s)\big|s\big)\cdot d\mu(s) \tag{38}$$

To summarize, almost-objective events, acts and mixtures exhibit the following properties:[24]

- as $m \to \infty$, all event-smooth $W(\cdot)$ assign *unanimous revealed likelihoods* to almost-objective events, which are *independent of conditioning on fixed subjective events*
- as $m \to \infty$, all event-smooth $W(\cdot)$ are *probabilistically sophisticated* over almost-objective acts
- as $m \to \infty$, all event-smooth $W_{SEU}(\cdot)$ and $W_{SDEU}(\cdot)$ are *linear* in almost-objective event likelihoods and in almost-objective mixtures of subjective acts

Recall that global robustness results can only be proven by line integrals along constant-direction paths (as in the left diagram of Figure 1), or by line integral approximations along paths that converge to the constant-direction property (as in the right diagram). Although we have seen that constant-direction paths do not exist in the space of subjective acts, as $m \to \infty$ the *almost-objective mixture path* between acts $f(\cdot)$ and $f^*(\cdot)$ – that is, the path $\{f_\alpha^m(\cdot)\,|\,\alpha \in [0,1]\}$ defined by

$$f_\alpha^m(\cdot) \;\equiv\; \big[f^*(\cdot) \text{ on } [0,\alpha] \underset{m}{\times} \mathcal{S};\; f(\cdot) \text{ on } (\alpha,1] \underset{m}{\times} \mathcal{S}\big] \tag{39}$$

will converge to the constant-direction property, and hence allow for global robustness proofs.

To see this, consider the almost-objective mixture path from the constant act $f(\cdot) = [x \text{ on } \mathcal{S}]$ to $f^*(\cdot) = [x^* \text{ on } \mathcal{S}]$, that is, the path defined by $f_\alpha^m(\cdot) \equiv [x^* \text{ on } [0,\alpha] \underset{m}{\times} \mathcal{S}\,;\, x \text{ on } (\alpha,1] \underset{m}{\times} \mathcal{S}]$ and illustrated in Figure 4. Although the change sets in going from the starting point of this path to its midpoint (namely $\varDelta E_{x^*}^+ = \varDelta E_x^- = [0,\tfrac{1}{2}] \underset{m}{\times} \mathcal{S}$) are distinct (and even disjoint) from the change sets in going from its midpoint to its final point (namely $\varDelta E_{x^*}^+ = \varDelta E_x^- = (\tfrac{1}{2},1] \underset{m}{\times} \mathcal{S}$), the above results imply that as $m \to \infty$ they will be viewed as being virtually equivalent to each other by every continuous-density probability measure $\mu(\cdot)$ and every event-smooth $W(\cdot)$.[25] Since (38) implies $\lim_{m\to\infty} W_{SEU}(f_\alpha^m(\cdot)) \equiv \alpha\cdot U(x^*) + (1-\alpha)\cdot U(x)$ and $\lim_{m\to\infty} W_{SDEU}(f_\alpha^m(\cdot)) \equiv \alpha\cdot\int_{\mathcal{S}} U(x^*|s)\cdot d\mu(s) + (1-\alpha)\cdot\int_{\mathcal{S}} U(x|s)\cdot d\mu(s)$, as $m \to \infty$ all preference functions that exhibit constant sensitivity in the events (namely, all $W_{SEU}(\cdot)$ and $W_{SDEU}(\cdot)$) are seen to respond at a constant rate along this path.

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24 See Machina (2004) for proofs of these properties and other aspects of almost-objective uncertainty.

25 More generally, as $m \to \infty$ the change sets from the act $f_\alpha^m(\cdot)$ to $f_{\alpha+\varDelta\alpha}^m(\cdot)$ will be viewed as being equivalent to the change sets from $f_{\alpha+\varDelta\alpha}^m(\cdot)$ to $f_{\alpha+2\cdot\varDelta\alpha}^m(\cdot)$, and similarly for all equally-spaced acts along such a path.

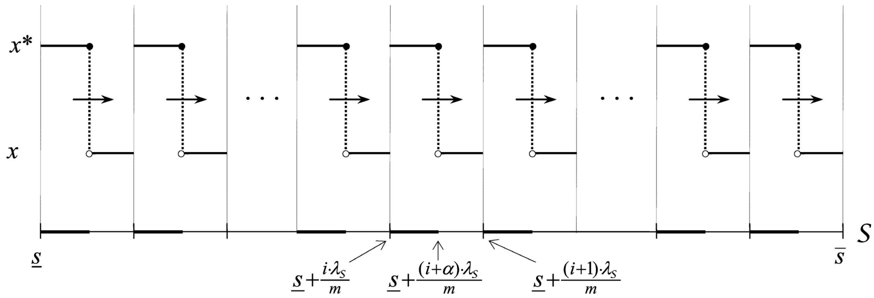

Figure 4. Almost-objective mixture path from $f(\cdot) = [x \text{ on } \mathcal{S}]$ to $f^*(\cdot) = [x^* \text{ on } \mathcal{S}]$

Since the above properties will also hold for the almost-objective mixture path (39) between any pair of *nonconstant* acts $f(\cdot), f^*(\cdot) \in \mathcal{A}$, almost-objective mixture paths can be said to arbitrarily closely approximate the property of being "constant-direction in the events." The following result from Machina (2004) shows that line integral approximations along such paths – that is, integrals that evaluate (in the same manner as (18)/(18)′) the effect of the full global change sets $\{(\Delta E_x^+, \Delta E_x^-)\,|\,x \in \mathcal{X}\}$ at each point along the path – will indeed converge to $W(\cdot)$'s exact global evaluation $W(f^*(\cdot)) - W(f(\cdot))$. Accordingly, such integrals can serve as the engine for establishing the global robustness of the classical expected utility/subjective probability model:[26]

Line Integral Approximation Theorem (Machina, 2004). If $W(\cdot)$ is event-smooth, then for any acts $f(\cdot), f^*(\cdot) \in \mathcal{A}$ and any $\varepsilon > 0$ there exists m_ε such that for each $m \geq m_\varepsilon$, $W(\cdot)$'s path derivative along the almost-objective mixture path $\{f_\alpha^m(\cdot)\,|\,\alpha \in [0,1]\} = \{[f^*(\cdot) \text{ on } [0,\alpha] \underset{m}{\times} \mathcal{S}; f(\cdot) \text{ on } (\alpha,1] \underset{m}{\times} \mathcal{S}]\,|\,\alpha \in [0,1]\}$ from $f(\cdot)$ to $f^*(\cdot)$ exists and satisfies

$$\left| \frac{dW\big(f_\alpha^m(\cdot)\big)}{d\alpha} - \sum_{x \in \mathcal{X}} \Big[\int_{\Delta E_x^+} \phi\big(x, s; f_\alpha^m\big) \cdot ds - \int_{\Delta E_x^-} \phi\big(x, s; f_\alpha^m\big) \cdot ds \Big] \right| < \varepsilon \qquad (40)$$

at all but a finite set of values of α. This implies the line integral approximation formula

$$W\big(f^*(\cdot)\big) - W\big(f(\cdot)\big)$$

$$= \lim_{m \to \infty} \int_0^1 \sum_{x \in \mathcal{X}} \Big[\int_{\Delta E_x^+} \phi\big(x, s; f_\alpha^m\big) \cdot ds - \int_{\Delta E_x^-} \phi\big(x, s; f_\alpha^m\big) \cdot ds \Big] \cdot d\alpha \qquad (41)$$

$$= \lim_{m \to \infty} \int_0^1 \sum_{x \in \mathcal{X}} \Big[\int_{E_x^*} \phi\big(x, s; f_\alpha^m\big) \cdot ds - \int_{E_x} \phi\big(x, s; f_\alpha^m\big) \cdot ds \Big] \cdot d\alpha$$

[26] In Machina (2004), the bracketed terms in (40)/(41) are equivalently expressed as $[\Phi_x(f^{*-1}(x); f_\alpha^m) - \Phi_x(f^{-1}(x); f_\alpha^m)]$.

5 Global robustness of the classical analytics

In this section we apply the robustness approach of Section 3.1 and the Line Integral Approximation Theorem to show that the fundamental analytical results of the classical expected utility/subjective probability model are globally robust to general event-smooth departures from both the expected utility hypothesis (event-separability) and the hypothesis of probabilistic sophistication. Section 5.1 establishes the property of outcome-monotonicity and robustifies the classical formula for outcome derivatives. Section 5.2 robustifies the classical characterization of global probabilistic sophistication. Section 5.3 robustifies the classical characterizations of comparative subjective likelihood and relative subjective likelihood to individuals who are not necessarily either expected utility maximizers or probabilistically sophisticated. Section 5.4 robustifies the classical characterization of comparative risk aversion, again to individuals who are not necessarily expected utility maximizers or probabilistically sophisticated.

5.1 Outcome-monotonicity and outcome derivatives

A preference function $W(\cdot)$ is said to be *outcome-monotonic* if the outcome ranking $x^* \succ x$ implies $W(x^* \text{ on } E; f(\cdot) \text{ on } {\sim}E) > W(x \text{ on } E; f(\cdot) \text{ on } {\sim}E)$ for all $f(\cdot)$ and all nonnull E.[27] For the classical form $W_{SEU}(f(\cdot)) \equiv \int_S U(f(s))\cdot d\mu(s)$ this property follows automatically, since $x^* \succ x$ implies $U(x^*) > U(x)$, so that $\int_E U(x^*)\cdot d\mu(s) + \int_{\sim E} U(f(s))\cdot d\mu(s) > \int_E U(x)\cdot d\mu(s) + \int_{\sim E} U(f(s))\cdot d\mu(s)$ for all $f(\cdot)$ and all nonnull E. For the state-dependent form $W_{SDEU}(f(\cdot)) \equiv \int_S U(f(s)|s)\cdot d\mu(s)$, although the condition "$x^* \succ x$ implies $U(x^*|s) > U(x|s)$ for all s" is *sufficient* to imply outcome-monotonicity, it is not necessary for this property, since an absolutely event-continuous $W_{SDEU}(\cdot)$ can be outcome-monotonic yet still satisfy $U(x^*|s) = U(x|s)$ at isolated states s, even isolated states s with positive subjective density $\nu(s)$. For a general event-smooth $W(\cdot)$, the third of the event-smoothness conditions $(21)/(21)'$ – namely that for each $x^* \succ x$ and nonnull E the local evaluation term $\int_E[\phi(x^*, s; f) - \phi(x, s; f)]\cdot ds$ exceeds some positive $\underline{\Phi}_{x^*,x,E}$ for all $f(\cdot)$ – is likewise sufficient to ensure that $W(\cdot)$ is outcome-monotonic, though again it is not necessary. To prove sufficiency, consider arbitrary $x^* \succ x$, $f(\cdot)$, and nonnull E, and let $\{f_\alpha^m(\cdot)\,|\,\alpha \in [0,1]\}_{m=1}^{\infty}$ be the almost-objective mixture paths from the act $[x \text{ on } E; f(\cdot) \text{ on } {\sim}E]$ to $[x^* \text{ on } E; f(\cdot) \text{ on } {\sim}E]$. The Line Integral Approximation Theorem and the third condition of $(21)/(21)'$ then imply

$$W\big(x^* \text{ on } E; f(\cdot) \text{ on } {\sim}E\big) - W\big(x \text{ on } E; f(\cdot) \text{ on } {\sim}E\big) \;=\; \lim_{m\to\infty} \int_0^1 \int_E \big[\phi(x^*, s; f_\alpha^m) - \phi(x, s; f_\alpha^m)\big]\cdot ds\cdot d\alpha \;>\; \underline{\Phi}_{x^*,x,E} \;>\; 0 \qquad (42)$$

[27] We adopt this strict preference/strict inequality definition of outcome monotonicity, rather than a weak preference/weak inequality version, in order to obtain a tighter connection between betting preferences and beliefs.

In the case of a real-valued outcome space $\mathcal{X} \subseteq R^1$, a preference function $W(\cdot)$ is said to be *outcome-differentiable* at an act $f(\cdot)$ if it possesses a *variational derivative* – that is, an absolutely continuous signed measure $\Psi(\cdot\,; f)$ such that

$$\begin{aligned} W\big(f^*(\cdot)\big) - W\big(f(\cdot)\big) &= \int_S \big[f^*(s) - f(s)\big]\cdot d\Psi(s; f) + o\big(\|f^*(\cdot) - f(\cdot)\|\big) \\ &= \int_S \big[f^*(s) - f(s)\big]\cdot \psi(s; f)\cdot ds + o\big(\|f^*(\cdot) - f(\cdot)\|\big) \end{aligned} \tag{43}$$

where $\|f^*(\cdot) - f(\cdot)\| = \sup_{s\in\mathcal{S}}|f^*(s) - f(s)|$, and $\psi(\cdot\,; f)$ is the density of $\Psi(\cdot\,; f)$. When an absolutely event-continuous $W_{SEU}(\cdot)$ or $W_{SDEU}(\cdot)$ is outcome-differentiable, its variational derivative is linked to its evaluation function $\phi(\cdot,\cdot)$ by the following formulas, which hold at each continuity point $\hat{s}$ of $f(\cdot)$:

$$\begin{aligned} &\text{for } W_{SEU}(\cdot): \;\; \psi\big(\hat{s}; f\big) = U'\big(f(\hat{s})\big)\cdot\nu(\hat{s}) = \phi_x\big(f(\hat{s}), \hat{s}\big) \\ &\text{for } W_{SDEU}(\cdot): \;\; \psi\big(\hat{s}; f\big) = U'\big(f(\hat{s})|\hat{s}\big)\cdot\nu(\hat{s}) = \phi_x\big(f(\hat{s}), \hat{s}\big) \end{aligned} \tag{44}$$

where $\phi_x(x, s) \equiv \partial\phi(x, s)/\partial x$. This generates the outcome-derivative formulas

$$\left.\begin{aligned} \left.\frac{\partial W_{SEU}\begin{pmatrix} x & \text{on} & E \\ f(\cdot) & \text{on} & \sim E\end{pmatrix}}{\partial x}\right|_{x=\hat{x}} &\equiv \int_E U'(\hat{x})\cdot\nu(s)\cdot ds \\ \left.\frac{\partial W_{SDEU}\begin{pmatrix} x & \text{on} & E \\ f(\cdot) & \text{on} & \sim E\end{pmatrix}}{\partial x}\right|_{x=\hat{x}} &\equiv \int_E U'(\hat{x}|s)\cdot\nu(s)\cdot ds \end{aligned}\right\} \equiv \int_E \phi_x\big(\hat{x}, s\big)\cdot ds \tag{45}$$

An event-smooth, outcome-differentiable $W(\cdot)$ is said to be *jointly outcome-event smooth* at $f(\cdot)$ if the expression $W(f(\hat{s})+\gamma$ on $[\hat{s}-\varepsilon_a, \hat{s}+\varepsilon_b]$; $f(\cdot)$ elsewhere) is twice continuously differentiable in $(\gamma, \varepsilon_a, \varepsilon_b)$ about (0,0,0) at each continuity point $\hat{s}$ of $f(\cdot)$. The following result globally robustifies the classical outcome-derivative formulas (44) and (45), by showing that they extend to the *local* evaluation function $\phi(\cdot,\cdot; f)$ of any jointly outcome-event smooth $W(\cdot)$.

Theorem 2 (Outcome derivatives). If an event-smooth $W(\cdot)$ is jointly outcome-event smooth at an act $f(\cdot) \in \mathcal{A}$, then its variational derivative and outcome derivatives at $f(\cdot)$ are linked to its local evaluation function by the formulas

$$\psi\big(\hat{s}; f\big) = \phi_x\big(f(\hat{s}), \hat{s}; f\big) \qquad \text{at each continuity point } \hat{s} \text{ of } f(\cdot) \tag{44$'$}$$

$$\left.\frac{\partial W\begin{pmatrix} x & \text{on} & E \\ f(\cdot) & \text{on} & \sim E\end{pmatrix}}{\partial x}\right|_{x=\hat{x}} = \int_E \phi_x\big(\hat{x}, s; f\big)\cdot ds \qquad \begin{aligned} &\text{all } \hat{x}\in\mathcal{X} \\ &\text{all } E\subseteq f^{-1}(\hat{x}) \end{aligned} \tag{45$'$}$$

where $\phi_x(x, s; f) \equiv \partial\phi(x, s; f)/\partial x$.

5.2 Characterization of probabilistic sophistication

An absolutely event-continuous expected utility preference function $W_{SEU}(\cdot)$ or $W_{SDEU}(\cdot)$ will take the probabilistically sophisticated form $W_{PS}(x_1 \text{ on } E_1; \ldots ; x_n \text{ on } E_n) \equiv V(x_1, \mu(E_1); \ldots ; x_n, \mu(E_n))$ for some subjective probability measure $\mu(\cdot)$ with density $\nu(\cdot)$ if and only if it is state-independent – that is, if and only if its $\underline{x}$-normalized evaluation function takes the multiplicatively separable form[28]

$$\phi(x,s) \;\equiv\; U(x)\cdot\nu(s) \qquad\qquad \text{all } x\in\mathcal{X},\, s\in\mathcal{S} \qquad (46)$$

That (46) *implies* probabilistic sophistication is clear. To see that it is in turn implied by it, observe that for each $x \succ \underline{x}$, $\underline{x}$-normalization, probabilistic sophistication and outcome-monotonicity[29] imply

$$\int_{E'}\phi(x,s)\cdot ds > \int_{E''}\phi(x,s)\cdot ds \Leftrightarrow \begin{bmatrix} x \text{ on } & E' \\ \underline{x} \text{ on } & \sim E' \end{bmatrix} \succ \begin{bmatrix} x \text{ on } & E'' \\ \underline{x} \text{ on } & \sim E'' \end{bmatrix} \Leftrightarrow \mu(E') > \mu(E'') \quad (47)$$

Thus for each $x \succ \underline{x}$, $\int_E \phi(x,s)\cdot ds$ is an increasing transformation of $\mu(E)$, which by additivity implies $\int_E \phi(x,s)\cdot ds = U(x)\cdot\mu(E)$ for some $U(x)$, yielding (46). A similar argument holds for each $x \prec \underline{x}$ and each $x \sim \underline{x}$.

As seen by (23), the classical characterization of global probabilistic sophistication by a multiplicatively separable evaluation function will extend to a general event-smooth preference function $W(\cdot)$ in the following manner:

- $W(\cdot)$'s local evaluation function $\phi(x,s;f) \equiv U(x;\mathbf{P}_{f,\mu})\cdot\nu(s)$ can now depend upon the act $f(\cdot)$, though only through its local utility term $U(x;\mathbf{P}_{f,\mu})$, and not its state-density term $\nu(s)$
- the local utility term $U(x;\mathbf{P}_{f,\mu})$ can only depend upon $f(\cdot)$ through that act's *implied outcome lottery* $\mathbf{P}_{f,\mu} \equiv (\ldots ; x, \mu(f^{-1}(x)); \ldots)$

Formally, we have

Theorem 3 (Characterization of probabilistic sophistication). An event-smooth preference function $W(\cdot)$ takes the probabilistically sophisticated form $W_{PS}(x_1 \text{ on } E_1; \ldots ; x_n \text{ on } E_n) \equiv V(x_1, \mu(E_1); \ldots ; x_n, \mu(E_n))$ for some subjective probability measure $\mu(\cdot)$ with density $\nu(\cdot)$ if and only if its $\underline{x}$-normalized local evaluation function takes the outcome-state separable form

$$\phi\big(x,s;f\big) \;\equiv\; U(x;\mathbf{P}_{f,\mu})\cdot\nu(s) \qquad\qquad \text{all } x\in\mathcal{X}, s\in\mathcal{S}, f(\cdot)\in\mathcal{A} \qquad (48)$$

for some function $U(x;\mathbf{P})$, where $\mathbf{P}_{f,\mu} = (x_1, \mu(E_1); \ldots ; x_n, \mu(E_n))$. In such a case, $U(x;\mathbf{P})$ will be the $\underline{x}$-normalized local utility function of $V(\cdot)$ at $\mathbf{P}$.[30]

[28] If the evaluation function $\phi(x,s)$ is multiplicatively separable for *some* choice of normalization outcome $\underline{x}$, it will be so for *any* choice of normalization outcome, and similarly for any local evaluation function of the form $\phi(x,s;f) \equiv U(x;\mathbf{P}_{f,\mu})\cdot\nu(s)$ as studied below.

[29] See Grant (1995) for a treatment of probabilistic sophistication without the assumption of outcome-monotonicity.

[30] For standard "integrability" reasons, (48) should not be taken to imply that an *arbitrary* function $U^*(x;\mathbf{P})$ can necessarily serve as the local utility function of some $V^*(\cdot)$, or as part of

Note that when an event-smooth $W(\cdot)$ is probabilistically sophisticated, its subjective density $\nu(\cdot)$ can be recovered from its local evaluation function at any $f(\cdot)$ by the formula $\nu(s) \equiv [\phi(x^*, s; f) - \phi(x, s; f)] / \int_{\mathcal{S}} [\phi(x^*, \omega; f) - \phi(x, \omega; f)] \cdot d\omega$ for any pair of outcomes $x^* \nsim x$.

5.3 Characterization of comparative and relative subjective likelihood

In the classic objective approach of von Neumann-Morgenstern, additive numerical likelihoods are specified as part of the objects of choice, and the individual is assumed to adopt these beliefs. In the subjective approaches of Savage and Machina-Schmeidler, uncertainty is represented by events or states of nature, but we impose enough conditions on subjective betting preferences to imply the existence of an additive numerical *subjective* likelihood for each event. In an Ellsberg urn, *some* events (namely "red") possess numerical likelihoods, but betting preferences are inconsistent with the existence of numerical likelihoods over *other* events ("black" or "yellow").

Although Ellsberg urns illustrate the fact that an individual can possess numerical likelihood beliefs for some events but not others, they fall short of displaying this in a completely subjective setting, since the numerical likelihoods that *do* exist are exogenously specified ("30 of the 90 balls are red") rather than inferred from betting preferences. It is thus worth behaviorally characterizing the phenomenon of "partial probabilistic sophistication," by obtaining a completely subjective characterization of classical (i.e. probabilistic) likelihood beliefs, likelihood comparisons, and likelihood ratios over *some* events or pairs of events, in a setting where additive revealed likelihoods over *other events* (i.e. complete probabilistic sophistication) need not exist.

Given an outcome-monotonic preference function $W(\cdot)$, we define its revealed *comparative likelihood relation* $A \succsim_\ell (\succ_\ell)\ B$ over pairs of disjoint events by the betting property

$$W\begin{pmatrix} x^* & \text{on } A \\ x & \text{on } B \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \underset{\substack{\text{all } x^* \succ x \\ \text{all } f(\cdot) \in \mathcal{A}}}{\geq (>)} W\begin{pmatrix} x & \text{on } A \\ x^* & \text{on } B \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \tag{49}$$

that is, if for any pair of non-indifferent outcomes, $W(\cdot)$ always prefers staking the more preferred outcome on A and the less preferred outcome on B, rather than the other way around.

Under probabilistic sophistication and event-continuity, the comparative likelihood relation will be complete and transitive, and $A \succsim_\ell B$ will be equivalent to each of the following betting properties over subjective partitions, or almost-objective subevents, of A and B:

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the local evaluation function of some $W^*(\cdot)$, probabilistically sophisticated or otherwise. By analogy, a function $G(z_1, \ldots, z_n)$ on R^n is non-decreasing if and only its partial derivative functions $g_1(z_1, \ldots, z_n), \ldots, g_n(z_1, \ldots, z_n)$ are nonnegative, although nonnegativity of an *arbitrary* set of functions $g_1^*(z_1, \ldots, z_n), \ldots, g_n^*(z_1, \ldots, z_n)$ does not necessarily imply they are the partials of some $G^*(z_1, \ldots, z_n)$.

Existence of Dominating Partitions: For each n, there exist partitions $\{A_1, \ldots, A_n\}$ of A and $\{B_1, \ldots, B_n\}$ of B such that $A_i \succcurlyeq_\ell B_j$ for each pair A_i, B_j

Nonexistence of Reverse-Dominating Partitions: All partitions $\{A_1, \ldots, A_n\}$ of A and $\{B_1, \ldots, B_n\}$ of B satisfy $A_{i'} \succcurlyeq_\ell B_{j'}$ for at least one pair $A_{i'}, B_{j'}$

Comparative Likelihood over Almost-Objective Subevents: For each finite interval union $\wp \subseteq [0,1]$, as $m \to \infty$ the almost-objective subevents $\wp \underset{m}{\times} A$ and $\wp \underset{m}{\times} B$ satisfy $\wp \underset{m}{\times} A \succcurlyeq_\ell \wp \underset{m}{\times} B$

We can extend the comparative likelihood relation to nondisjoint events by defining $A \succcurlyeq_\ell (\succ_\ell)\, B \Leftrightarrow (A - B) \succcurlyeq_\ell (\succ_\ell)\, (B - A)$, in which case it will exhibit the *monotonicity property* $A \supseteq B \Rightarrow A \succcurlyeq_\ell B$ and the *disjoint additivity property* $A \succcurlyeq_\ell (\succ_\ell)\, B \Rightarrow (A \cup C) \succcurlyeq_\ell (\succ_\ell)\, (B \cup C)$ whenever $A \cap C = B \cap C = \emptyset$.

Under probabilistic sophistication and event-continuity, we can also define the *relative likelihood* $\mathcal{L}_{A,B}$ of a pair of disjoint events A and B by the following equivalent betting properties, the last of which compares bets on subjective versus almost-objective partitions of a common event:

Existence of Dominating Partitions: For each rational value $\mathcal{L} < \mathcal{L}_{A,B}$, there exist partitions $\{A_1, \ldots, A_{n_a}\}$ of A and $\{B_1, \ldots, B_{n_b}\}$ of B with $n_a/n_b = \mathcal{L}$, such that $A_i \succ_\ell B_j$ for each pair A_i, B_j

Nonexistence of Reverse-Dominating Partitions: For each rational value $\mathcal{L} < \mathcal{L}_{A,B}$, all partitions $\{A_1, \ldots, A_{n_a}\}$ of A and $\{B_1, \ldots, B_{n_b}\}$ of B with $n_a/n_b = \mathcal{L}$ satisfy $A_{i'} \succ_\ell B_{j'}$ for some pair $A_{i'}, B_{j'}$

Relative Likelihood of Almost-Objective Subevents: For each pair of finite interval unions $\wp_a, \wp_b \subseteq [0,1]$ with the *reciprocal* likelihood ratio $\lambda(\wp_a)/\lambda(\wp_b) = 1/\mathcal{L}_{A,B}$, as $m \to \infty$ the almost-objective subevents $\wp_a \underset{m}{\times} A$ and $\wp_b \underset{m}{\times} B$ satisfy $\wp_a \underset{m}{\times} A \sim_\ell \wp_b \underset{m}{\times} B$

Comparison of Subjective versus Almost-Objective Partitions: For all disjoint $\wp_a, \wp_b \subset [0,1]$ with the *same* likelihood ratio $\lambda(\wp_a)/\lambda(\wp_b) = \mathcal{L}_{A,B}$, as $m \to \infty$ the individual is indifferent between betting on the events in the *subjective partition* $\{(\wp_a \cup \wp_b) \underset{m}{\times} A, (\wp_a \cup \wp_b) \underset{m}{\times} B\}$ versus the events in the *almost-objective partition* $\{\wp_a \underset{m}{\times} (A \cup B), \wp_b \underset{m}{\times} (A \cup B)\}$ of the event $(\wp_a \cup \wp_b) \underset{m}{\times} (A \cup B)$

Of course under probabilistic sophistication, the comparative likelihood relation $A \succcurlyeq_\ell B$ is equivalent to $\mu(A) \geq \mu(B)$, and relative likelihood $\mathcal{L}_{A,B}$ is given by $\mu(A)/\mu(B)$. Since the local evaluation function of a probabilistically sophisticated preference function takes the form $\phi(x, s; f) \equiv U(x; \mathbf{P}_{f,\mu}) \cdot \nu(s)$, the conditions $\mu(A) \geq \mu(B)$ and $\mu(A)/\mu(B) = \mathcal{L}_{A,B}$ are respectively equivalent to

$$\int_A \big[\phi(x^*, s; f) - \phi(x, s; f)\big] \cdot ds \underset{\substack{\text{all } x^* \succ x \\ \text{all } f(\cdot) \in \mathcal{A}}}{\geq} \int_B \big[\phi(x^*, s; f) - \phi(x, s; f)\big] \cdot ds \quad (50)$$

$$\int_A \big[\phi(x^*, s; f) - \phi(x, s; f)\big] \cdot ds \Big/ \int_B \big[\phi(x^*, s; f) - \phi(x, s; f)\big] \cdot ds \underset{\substack{\text{all } x^* \succ x \\ \text{all } f(\cdot) \in \mathcal{A}}}{=} \mathcal{L}_{A,B} \quad (51)$$

Intuitively, (50) states that the individual is always at least as sensitive to replacing an outcome x by a preferred outcome x^* over the event A than to making the same replacement over B, and (51) states that the ratio of these sensitivities takes the same value $\mathcal{L}_{A,B}$ for all such outcomes pairs.

However in the *absence* of probabilistic sophistication, the comparative likelihood relation $\succcurlyeq_\ell$ need not be complete, subjective likelihoods or likelihood ratios need not exist for all events, and even when two events *do* satisfy the betting property (49), neither disjoint additivity nor the associated partition properties necessarily follow from this fact.[31] Nevertheless the above characterizations are robust, in the sense that a general event-smooth $W(\cdot)$ – even though it may not be additive over *general* subjective partitions – will exhibit the comparative likelihood condition (50) or the relative likelihood condition (51) for a pair of subjective events A and B if and only if it satisfies the above betting properties over all partitions and subevents for which event-smooth preferences *are* inherently additive, namely all *almost-objective subevents*[32] and all *small-event partitions*.

The concepts of comparative likelihood ("A is at least as likely as B") and relative likelihood ("A is $\mathcal{L}_{A,B}$ times as likely as B") are both special cases of the more general comparative condition "A is at least $\mathcal{L}$ times as likely as B," which may be meaningful even when A and B are both ambiguous and neither possesses a subjective likelihood on its own. The local evaluation function conditions (50) and (51) are also both special cases of a common condition, namely condition (52) below. The following result thus serves to jointly robustify the classical characterizations of comparative subjective likelihood and relative subjective likelihood:

Theorem 4 (Characterization of comparative and relative subjective likelihood). For any event-smooth preference function $W(\cdot)$ over $\mathcal{A}$, the following conditions on a pair of disjoint nonnull events $A, B \in \mathcal{E}$ and value $\mathcal{L} \in (0, \infty)$ are equivalent:

(a) *Local likelihood ratios:* $W(\cdot)$'s local evaluation function satisfies

$$\int_A [\phi(x^*, s; f) - \phi(x, s; f)] \cdot ds \Big/ \int_B [\phi(x^*, s; f) - \phi(x, s; f)] \cdot ds \underset{\substack{\text{all } x^* \succ x \\ \text{all } f(\cdot) \in \mathcal{A}}}{\geq} \mathcal{L} \qquad (52)$$

(b) *Existence of dominating small-event partitions:* For each pair $x^* \succ x$, act $f(\cdot) \in \mathcal{A}$, rational value $\mathcal{L}^* < \mathcal{L}$ and $\varepsilon > 0$, there exist ε-partitions[33] $\{A_1, \ldots, A_{n_a}\}$ of A and $\{B_1, \ldots, B_{n_b}\}$ of B, with $n_a/n_b = \mathcal{L}^*$, such that

[31] Thus, in an Ellsberg urn with 10 red balls, 10 black balls, and 22 yellow or purple balls in unknown proportion, an ambiguity averse individual may exhibit the likelihood ranking (yellow $\cup$ purple) $\succ_\ell$ (red $\cup$ black), but also yellow $\prec_\ell$ red, yellow $\prec_\ell$ black, purple $\prec_\ell$ red and purple $\prec_\ell$ black, which violates nonexistence of reverse-dominating partitions.

[32] By way of analogy, whereas most individuals would assign the well-defined likelihood ranking $A \succ_\ell B$ to the events $A =$ (yellow $\cup$ purple) versus $B =$ (red $\cup$ black) of the previous note, most would probably *not* satisfy the associated betting properties for the *subjective partitions* $\{A_1, A_2\}$ = {yellow, purple} and $\{B_1, B_2\}$ = {red, black}, but most probably *would* satisfy them for the *objective partitions* $\{A_H, A_T\} = \{A \cap \text{heads}, A \cap \text{tails}\}$ and $\{B_H, B_T\} = \{B \cap \text{heads}, B \cap \text{tails}\}$ generated by some fair (i.e. 50:50) coin.

[33] A partition $\{E_1, \ldots, E_n\}$ is said to be an *ε-partition* if $\lambda(E_i) < \varepsilon$ for each i.

$$W\begin{pmatrix} x^* & \text{on } A_i \\ x & \text{on } B_j \\ f(\cdot) & \text{elsewhere} \end{pmatrix} > W\begin{pmatrix} x & \text{on } A_i \\ x^* & \text{on } B_j \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \quad \text{for all } A_i, B_j \tag{53}$$

(c) *Nonexistence of reverse-dominating small-event partitions:* For each pair $x^* \succ x$, act $f(\cdot) \in \mathcal{A}$ and rational value $\mathcal{L}^* < \mathcal{L}$, there exists some $\varepsilon > 0$ such that all ε-partitions $\{A_1, \ldots, A_{n_a}\}$ of A and $\{B_1, \ldots, B_{n_b}\}$ of B with $n_a/n_b = \mathcal{L}^*$ satisfy

$$W\begin{pmatrix} x^* & \text{on } A_{i'} \\ x & \text{on } B_{j'} \\ f(\cdot) & \text{elsewhere} \end{pmatrix} > W\begin{pmatrix} x & \text{on } A_{i'} \\ x^* & \text{on } B_{j'} \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \quad \text{for some } A_{i'}, B_{j'} \tag{54}$$

(d) *Comparison of almost-objective subevents:* For all finite interval unions $\wp_a, \wp_b \subseteq [0, 1]$ with $\lambda(\wp_a)/\lambda(\wp_b) = (>) 1/\mathcal{L}$, the almost-objective subevents $\wp_a \underset{m}{\times} A$ and $\wp_b \underset{m}{\times} B$ satisfy[34]

$$\lim_{m\to\infty} W\begin{pmatrix} x^* & \text{on } \wp_a \underset{m}{\times} A \\ x & \text{on } \wp_b \underset{m}{\times} B \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \underset{\substack{\text{all } x^* \succ x \\ \text{all } f(\cdot) \in \mathcal{A}}}{\geq (>)} \lim_{m\to\infty} W\begin{pmatrix} x & \text{on } \wp_a \underset{m}{\times} A \\ x^* & \text{on } \wp_b \underset{m}{\times} B \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \tag{55}$$

(e) *Comparison of subjective versus almost-objective partitions:* For all disjoint finite interval unions $\wp_a, \wp_b \subset [0, 1]$ with $\lambda(\wp_a)/\lambda(\wp_b) = (<) \mathcal{L}$

$$\lim_{m\to\infty} W\begin{pmatrix} x^* & \text{on } (\wp_a \cup \wp_b) \underset{m}{\times} A \\ x & \text{on } (\wp_a \cup \wp_b) \underset{m}{\times} B \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \underset{\substack{\text{all } x^* \succ x \\ \text{all } f(\cdot) \in \mathcal{A}}}{\geq (>)} \lim_{m\to\infty} W\begin{pmatrix} x^* & \text{on } \wp_a \underset{m}{\times} (A \cup B) \\ x & \text{on } \wp_b \underset{m}{\times} (A \cup B) \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \tag{56}$$

In such a case, we say that $W(\cdot)$ *reveals A to be at least $\mathcal{L}$ times as likely as B*, and *reveals B to be no more than $1/\mathcal{L}$ times as likely as A*.

Theorem 4 can be used to generate the following additional characterizations of subjective beliefs in the absence of probabilistic sophistication:

Conditional likelihood: Given nonnull events $A \subset B$ where neither may have a well-defined likelihood, we say that the *conditional likelihood of A given B is at least $\mathcal{L}$* if $W(\cdot)$ reveals A to be at least $\mathcal{L}/(1 - \mathcal{L})$ times as likely as $B - A$. This will be equivalent to the local evaluation function condition $\int_A [\phi(x^*,s;f) - \phi(x,s;f)] \cdot ds / \int_B [\phi(x^*,s;f) - \phi(x,s;f)] \cdot ds \geq \mathcal{L}$ for all $x^* \succ x$ and all $f(\cdot) \in \mathcal{A}$, equivalent to Theorem 4's almost-objective subevent property (d) provided $\wp_a$ and $\wp_b$ are disjoint (so the bets in (55) will be well-defined), and equivalent to Theorem 4's partition properties (b) and (c) for partitions $\{A_1, \ldots, A_{n_a}\}$ of A and $\{A_1, \ldots, A_{n_a}, B_1, \ldots, B_{n_b}\}$ of B with $n_a/(n_a + n_b) = \mathcal{L}^* < \mathcal{L}$.

[34] When $\mathcal{L} \geq 1$, setting $\wp_a = \wp_b = [0, 1]$ in (55) yields the comparative likelihood betting property (49).

Comparative numerical likelihood: Given an event E which is ambiguous and hence without a well-defined likelihood, we can still define it as having *a likelihood of at least* $\mathcal{L}$ if its conditional likelihood given the event $\mathcal{S}$ is at least $\mathcal{L}$. This will be equivalent to local evaluation function condition $\int_E [\phi(x^*, s; f) - \phi(x, s; f)] \cdot ds / \int_{\mathcal{S}} [\phi(x^*, s; f) - \phi(x, s; f)] \cdot ds \geq \mathcal{L}$ for all $x^* \succ x$ and all $f(\cdot) \in \mathcal{A}$, and to the corresponding versions of the subevent and partition properties of the previous paragraph.

Interpersonal comparative likelihood: Even if neither $W(\cdot)$ nor $W^*(\cdot)$ assigns a well-defined likelihood to an event E, we can still say that $W^*(\cdot)$ *reveals E to be at least as likely as does* $W(\cdot)$ if their local evaluation functions satisfy $\int_E [\phi^*(\hat{x}, s; f) - \phi^*(x, s; f)] \cdot ds \,/\, \int_{\mathcal{S}} [\phi^*(\hat{x}, s; f) - \phi^*(x, s; f)] \cdot ds \geq \int_E [\phi(\hat{x}, s; f) - \phi(x, s; f)] \cdot ds \,/\, \int_{\mathcal{S}} [\phi(\hat{x}, s; f) - \phi(x, s; f)] \cdot ds$ for all $\hat{x} \succ x$ and all $f(\cdot) \in \mathcal{A}$, which will be equivalent to the interpersonal versions of the above betting and partition properties.

5.4 Characterization of comparative risk aversion under subjective uncertainty[35]

We motivate our characterization of comparative risk aversion under subjective uncertainty by recalling that notion in the objective case: Given three outcomes $x'' \succ x' \succ x$, an objective lottery $\hat{\mathbf{P}}$ is said to differ from $\mathbf{P}$ by an $x \leftarrow x' \rightarrow x''$ *probability spread* if $\hat{\mathbf{P}}$ is obtained from $\mathbf{P}$ by reducing the probability assigned to x', and increasing the probabilities assigned to both x and x''. A preference function $V^*(\cdot)$ is then said to be *at least as risk averse* as $V(\cdot)$ if $V(\mathbf{P}) \geq V(\hat{\mathbf{P}}) \Rightarrow V^*(\mathbf{P}) \geq V^*(\hat{\mathbf{P}})$ whenever $\hat{\mathbf{P}}$ differs from $\mathbf{P}$ by such a spread, or equivalently, if $V^*(\hat{\mathbf{P}}) > V^*(\mathbf{P}) \Rightarrow V(\hat{\mathbf{P}}) > V(\mathbf{P})$ whenever $\hat{\mathbf{P}}$ differs from $\mathbf{P}$ by such a spread. The notion of comparative risk aversion as comparative tolerance of probability spreads (both three-point and more general) underlies the expected utility and non-expected utility characterizations of Arrow (1965), Pratt (1964), Machina (1982, Thm. 4) and others.

For a pair of classical preference functions $W_{SEU}(f(\cdot)) \equiv \int_{\mathcal{S}} U(f(s)) \cdot d\mu(s)$ and $W^*_{SEU}(\cdot) \equiv \int_{\mathcal{S}} U^*(f(s)) \cdot d\mu^*(s)$ or state-dependent functions $W_{SDEU}(f(\cdot)) \equiv \int_{\mathcal{S}} U(f(s)|s) \cdot d\mu(s)$ and $W^*_{SDEU}(\cdot) \equiv \int_{\mathcal{S}} U^*(f(s)|s) \cdot d\mu^*(s)$ in a setting of objective$\times$subjective uncertainty, comparative risk aversion over *objective lotteries* is characterized by the condition that $U^*(x)$ is at least as concave a function of x as $U(x)$, or that $\int_{\mathcal{S}} U^*(x|s) \cdot d\mu^*(s)$ is at least as concave in x as $\int_{\mathcal{S}} U(x|s) \cdot d\mu(s)$. Thus in each case, comparative risk aversion is determined by the comparative concavity (as a function of x) of the *integrated evaluation function* $\int_{\mathcal{S}} \phi(x, s) \cdot ds$, which reduces to $U(x)$ for $W_{SEU}(\cdot)$, and to $\int_{\mathcal{S}} U(x|s) \cdot d\mu(s)$ for $W_{SDEU}(\cdot)$.

Since the lotteries described in the previous paragraphs were all objective, it follows that the individuals whose risk attitudes were being compared both *agreed*

[35] Although we continue to assume a completely arbitrary outcome space $\mathcal{X}$, the discussion and results of this section will be restricted to pairs of preference functions $W(\cdot)$ and $W^*(\cdot)$ that share a common outcome ordering $\succcurlyeq$. Other characterizations of risk aversion and comparative risk aversion under subjective uncertainty include those of Yaari (1969), Montesano (1994a,b, 1999a,b), Grant and Quiggin (2001) and Nau (2003).

on the likelihoods involved in any $x \leftarrow x' \rightarrow x''$ spread they were offered. On the other hand, while a subjective act $\hat{f}(\cdot) = [x''$ on $A; x$ on $B; f_0(\cdot)$ elsewhere] can be said to differ from $f(\cdot) = [x'$ on $A \cup B; f_0(\cdot)$ elsewhere] by an $x \leftarrow x' \rightarrow x''$ *subjective spread,* there is no guarantee that the more risk averse individual will always be more averse to such a spread, since the two individuals' attitudes toward such spreads can also be affected by their respective *likelihood beliefs* over the events A and B. Thus, in extending the standard characterization of comparative risk aversion from objective to subjective uncertainty, it will be necessary to work with subjective spreads whose evaluations *only* reflect interpersonal differences in risk attitudes, and *do not* reflect any interpersonal differences in event likelihoods – including interpersonal differences in the *existence* of such event likelihoods.

Under event-smoothness, there are two types of subjective spreads whose evaluations reflect features of risk attitudes but not beliefs. The first are those in which A and B are taken from the class of events for which all event-smooth preference functions exhibit *identical* limiting likelihoods, namely the class of almost-objective events. Recall that as $m \rightarrow \infty$ all event-smooth preference functions will exhibit limiting likelihoods for the events $\wp_a \underset{m}{\times} \mathcal{S}$ and $\wp_b \underset{m}{\times} \mathcal{S}$, corresponding to the values $\lambda(\wp_a)$ and $\lambda(\wp_b)$. Given outcomes $x'' \succ x' \succ x$ and disjoint finite interval unions $\wp_a, \wp_b \subset [0,1]$, we thus say that the act $\hat{f}_m(\cdot) = [x''$ on $\wp_a \underset{m}{\times} \mathcal{S}; x$ on $\wp_b \underset{m}{\times} \mathcal{S}$; $f_0(\cdot)$ elsewhere] differs from $f_m(\cdot) = [x'$ on $(\wp_a \cup \wp_b) \underset{m}{\times} \mathcal{S}; f_0(\cdot)$ elsewhere] by an $x \leftarrow x' \rightarrow x''$ *almost-objective spread.* Since all event-smooth preference functions will agree on the limiting likelihoods involved in such spread, we would expect that if a more risk averse function $W^*(\cdot)$ had a limiting preference for such a spread, so too would any less risk averse $W(\cdot)$.

The second category of spreads whose evaluations reflect features of risk attitudes but not beliefs are those in which A and B are created out of *small subjective events* $\{E_1, \ldots, E_n\}$ which an individual considers to be "locally-exchangeable" for the outcomes in question. Given outcomes $x'' \succ x' \succ x$, act $f(\cdot)$, and real value $\mathcal{L} \in (0, \infty)$, we say that $W(\cdot)$ is *willing to accept small-event $x \leftarrow x' \rightarrow x''$ spreads about $f(\cdot)$ at any odds ratio greater than $\mathcal{L}$*, if for any n_a, n_b with $n_a/n_b > \mathcal{L}$ and any $\varepsilon > 0$, there exists an ε-partition $\{E_1, \ldots, E_n\}$ of $\mathcal{S}$ such that if A is the union of any n_a of these events and B is the union of any n_b others, then $W(x''$ on $A; x$ on $B; f(\cdot)$ elsewhere$) > W(x'$ on $A \cup B; f(\cdot)$ elsewhere$)$. Although a different preference function $W^*(\cdot)$ would generally require a different locally-exchangeable partition $\{E_1^*, \ldots, E_{n^*}^*\}$ to exhibit such a property, we would expect that, relative to their respective partitions, if a more risk averse $W^*(\cdot)$ is willing to accept small-event $x \leftarrow x' \rightarrow x''$ spreads about $f(\cdot)$ at any odds ratio greater than some value $\mathcal{L}$, so too would any less risk averse $W(\cdot)$.

Under event-smoothness, these three notions of comparative risk aversion – comparative concavity of the integrated *local* evaluation functions $\int_{\mathcal{S}} \phi(x,s;f) \cdot ds$ and $\int_{\mathcal{S}} \phi^*(x,s;f) \cdot ds$, comparative attitudes toward almost-objective spreads, and comparative attitudes toward small-event spreads – will turn out to be equivalent, and will have two additional implications.

The first additional implication involves the individuals' risk preference functions $V_{W^*}(\cdot)$ and $V_W(\cdot)$, which from (36) represent their limiting preferences over almost-objective acts $[x_1$ on $\wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n$ on $\wp_n \underset{m}{\times} \mathcal{S}]$ whether or not the individuals are probabilistically sophisticated. If $W^*(\cdot)$ is at least as risk averse as $W(\cdot)$ in any of the above three equivalent senses, it will follow that $V_{W^*}(\cdot)$ will be at least as risk averse as $V_W(\cdot)$, in the standard sense of comparative risk aversion for lottery preference functions as stated in the first paragraph of this section. But while this property is *implied* by the above three notions of comparative risk aversion, is it not strong to *imply* them. The reason is that the family of almost-objective acts is too small (and non-dense) a subset of the family $\mathcal{A}$ of subjective acts, so comparative risk aversion over almost-objective acts is not strong enough to imply comparative risk aversion over general subjective acts in the above three senses.

The second implication involves spreads in which A and B are general subjective events, but where $W^*(\cdot)$ and $W(\cdot)$ happen to *agree* on their likelihoods – even though they may disagree on the likelihoods (or have no likelihoods) for other subjective events. From Theorem 4 and its follow-up definitions, we can say that $W^*(\cdot)$ and $W(\cdot)$ each assign a likelihood p_a to A and p_b to B if $\int_A[\phi^*(\hat{x},s;f)-\phi^*(x,s;f)]\cdot ds/\int_{\mathcal{S}}[\phi^*(\hat{x},s;f)-\phi^*(x,s;f)]\cdot ds$ and $\int_A[\phi(\hat{x},s;f)-\phi(x,s;f)]\cdot ds/\int_{\mathcal{S}}[\phi(\hat{x},s;f)-\phi(x,s;f)]\cdot ds$ both equal p_a, and if $\int_B[\phi^*(\hat{x},s;f)-\phi^*(x,s;f)]\cdot ds/\int_{\mathcal{S}}[\phi^*(\hat{x},s;f)-\phi^*(x,s;f)]\cdot ds$ and $\int_B[\phi(\hat{x},s;f)-\phi(x,s;f)]\cdot ds/\int_{\mathcal{S}}[\phi(\hat{x},s;f)-\phi(x,s;f)]\cdot ds$ both equal p_b, for all $\hat{x}\succ x$ and all $f(\cdot)\in\mathcal{A}$. Given such events A and B and outcomes $x''\succ x'\succ x$, if the more risk averse $W^*(\cdot)$ prefers the subjective spread from $f(\cdot)$ $=[x'$ on $A\cup B$; $f_0(\cdot)$ elsewhere$]$ to $\hat{f}(\cdot)=[x''$ on A; x on B; $f_0(\cdot)$ elsewhere$]$, then so will the less risk averse $W(\cdot)$. We can generalize this notion by replacing the identical likelihood requirement by the condition that $W^*(\cdot)$ assigns A and B likelihoods of *at most* p_a and *at least* p_b, and $W(\cdot)$ assigns them likelihoods of *at least* p_a and *at most* p_b. Once again, while this property will be *implied* by the above three equivalent notions of comparative risk aversion, it is not strong enough to imply them.

We formalize the above discussion by:

Theorem 5 (Characterization and implications of comparative risk aversion under subjective uncertainty). The following conditions on a pair of event-smooth preference functions $W^*(\cdot)$ and $W(\cdot)$ over $\mathcal{A}$ with a common outcome ordering $\succcurlyeq$ are equivalent:

(a) *Comparative concavity of integrated local evaluation functions:* At each $f(\cdot)\in\mathcal{A}$, $W^*(\cdot)$'s and $W(\cdot)$'s integrated local evaluation functions $\int_{\mathcal{S}}\phi^*(x,s;f)\cdot ds$ and $\int_{\mathcal{S}}\phi(x,s;f)\cdot ds$ satisfy

$$\int_{\mathcal{S}}\phi^*(x,s;f)\cdot ds \underset{\text{all } x\in\mathcal{X}}{\equiv} \rho_f\Big(\int_{\mathcal{S}}\phi(x,s;f)\cdot ds\Big) \tag{57}$$

for some increasing concave function $\rho_f(\cdot)$.[36]

(b) *Comparative risk aversion over almost-objective spreads:* For all $x''\succ x'\succ x$, all $f(\cdot)\in\mathcal{A}$ and all disjoint nondegenerate finite interval unions $\wp_a,\wp_b\subset[0,1]$

[36] As in Pratt (1964, Thm. 1), this is equivalent to the condition that $[\int_{\mathcal{S}}\phi^*(x'',s;f)\cdot ds - \int_{\mathcal{S}}\phi^*(x',s;f)\cdot ds]/[\int_{\mathcal{S}}\phi^*(x',s;f)\cdot ds-\int_{\mathcal{S}}\phi^*(x,s;f)\cdot ds] \le [\int_{\mathcal{S}}\phi(x'',s;f)\cdot ds - \int_{\mathcal{S}}\phi(x',s;f)\cdot ds]/[\int_{\mathcal{S}}\phi(x',s;f)\cdot ds-\int_{\mathcal{S}}\phi(x,s;f)\cdot ds]$ for all $x''\succ x'\succ x$ and all $f(\cdot)$.

$$\lim_{m\to\infty} W^*\begin{pmatrix} x'' & \text{on } \wp_a \underset{m}{\times} \mathcal{S} \\ x & \text{on } \wp_b \underset{m}{\times} \mathcal{S} \\ f(\cdot) & \text{elsewhere} \end{pmatrix} > \lim_{m\to\infty} W^*\begin{pmatrix} x' & \text{on } (\wp_a \cup \wp_b) \underset{m}{\times} \mathcal{S} \\ f(\cdot) & \text{elsewhere} \end{pmatrix}$$
$$\Rightarrow \qquad (58)$$
$$\lim_{m\to\infty} W\begin{pmatrix} x'' & \text{on } \wp_a \underset{m}{\times} \mathcal{S} \\ x & \text{on } \wp_b \underset{m}{\times} \mathcal{S} \\ f(\cdot) & \text{elsewhere} \end{pmatrix} > \lim_{m\to\infty} W\begin{pmatrix} x' & \text{on } (\wp_a \cup \wp_b) \underset{m}{\times} \mathcal{S} \\ f(\cdot) & \text{elsewhere} \end{pmatrix}$$

(c) *Comparative risk aversion over small-event spreads:* For any $x'' \succ x' \succ x$, $f(\cdot) \in \mathcal{A}$ and $\mathcal{L} \in (0,\infty)$, if $W^*(\cdot)$ is willing to accept small-event $x \leftarrow x' \rightarrow x''$ spreads about $f(\cdot)$ at any odds ratio greater than $\mathcal{L}$, then so is $W(\cdot)$.

In such a case, we say $W^*(\cdot)$ is *at least as risk averse* as $W(\cdot)$. These conditions in turn imply:

(d) *Comparative risk aversion over almost-objective acts:* If $V_{W^*}(\cdot)$ and $V_W(\cdot)$ are $W^*(\cdot)$'s and $W(\cdot)$'s associated preference functions over lotteries as defined in (36), then $V_{W^*}(\hat{\mathbf{P}}) > V_{W^*}(\mathbf{P}) \Rightarrow V_W(\hat{\mathbf{P}}) > V_W(\mathbf{P})$ whenever $\hat{\mathbf{P}}$ differs from $\mathbf{P}$ by an $x \leftarrow x' \rightarrow x''$ probability spread for some $x'' \succ x' \succ x$.

(e) *Comparative risk aversion over likelihood-adjusted spreads:* For each pair of disjoint events A, B, if $W^*(\cdot)$ assigns A and B respective likelihoods of at most p_a and at least p_b, and $W(\cdot)$ assigns them respective likelihoods of at least p_a and at most p_b, for some $p_a, p_b \in (0,1)$, then

$$W^*\begin{pmatrix} x'' & \text{on } A \\ x & \text{on } B \\ f(\cdot) & \text{elsewhere} \end{pmatrix} > W^*\begin{pmatrix} x' & \text{on } A \cup B \\ f(\cdot) & \text{elsewhere} \end{pmatrix}$$
$$\Rightarrow \qquad (59)$$
$$W\begin{pmatrix} x'' & \text{on } A \\ x & \text{on } B \\ f(\cdot) & \text{elsewhere} \end{pmatrix} > W\begin{pmatrix} x' & \text{on } A \cup B \\ f(\cdot) & \text{elsewhere} \end{pmatrix}$$

for all $x'' \succ x' \succ x$ and all $f(\cdot) \in \mathcal{A}$.

In the case of a real-valued outcome space $\mathcal{X} = [a,b] \subseteq R^1$, we can define a preference function $W(\cdot)$ – whether or not it is probabilistically sophisticated – to be *weakly risk averse over subjective acts* if it is at least as risk averse as any *risk neutral preference function over acts,* that is, any preference function of the form $W_{RN}(f(\cdot)) \equiv \int_{\mathcal{S}} f(s)\cdot d\mu(s)$ for some absolutely continuous subjective probability measure $\mu(\cdot)$. Since conditions (a) – (e) in Theorem 5 make no reference to the respective individuals' beliefs, it turns out that if $W(\cdot)$ is at least as risk averse as *some* risk neutral preference function $W_{RN}(f(\cdot)) \equiv \int_{\mathcal{S}} f(s)\cdot d\mu(s)$, it will be at least as risk averse as *every* risk neutral $\hat{W}_{RN}(f(\cdot)) \equiv \int_{\mathcal{S}} f(s)\cdot d\hat{\mu}(s)$.[37]

[37] To see this, observe that for any such $W_{RN}(\cdot)$, its integrated local evaluation function will take the form $\int_{\mathcal{S}} \phi_{RN}(x,s;f)\cdot ds \equiv \int_{\mathcal{S}} x\cdot\nu(s)\cdot ds \equiv x$, independently of its subjective density $\nu(\cdot)$.

This definition of risk aversion over subjective acts is thus “belief-independent” – both in the sense of not requiring $W(\cdot)$ to be probabilistically sophisticated, and in the sense of being independent of particular risk neutral $W_{RN}(\cdot)$ (i.e., its particular beliefs $\mu(\cdot)$) used for comparison. By Theorem 5, any such subjectively risk averse $W(\cdot)$ will exhibit the following properties:

- $W(\cdot)$'s integrated local evaluation function $\int_{\mathcal{S}} \phi(x, s; f) \cdot ds$ is concave in x at each $f(\cdot)$
- As $m \to \infty$, $W(\cdot)$ is weakly averse to every almost-objective spread with nonpositive mean
- If $W(\cdot)$ is willing to accept small-event $x \leftarrow x' \to x''$ spreads at any odds ratio greater than $\mathcal{L}$, it must be that $\mathcal{L} \geq (x' - x)/(x'' - x')$[38]
- $W(\cdot)$'s associated risk preference function $V_W(\cdot)$ is weakly risk averse
- If $W(\cdot)$ assigns likelihoods of p_a to A and p_b to B, then it is weakly averse to any subjective spread $[x'$ on $A \cup B$; $f_0(\cdot)$ elsewhere$] \to [x''$ on A; x on B; $f_0(\cdot)$ elsewhere$]$ for which $p_a \cdot (x'' - x') \leq p_b \cdot (x' - x)$

6 Connections and extensions

6.1 Related work of Epstein

As mentioned in the introductory Note, Epstein (1999) has also proposed a notion of event-differentiability for a subjective preference function $W(\cdot)$, which can be described as follows:[39]

As in Section 4.1, view each act $f(\cdot) \in \mathcal{A}$ as a mapping $e(\cdot) : \mathcal{X} \to \mathcal{E}$ from outcomes to events that satisfies (i) $e(x) = f^{-1}(x) = \emptyset$ at all but a finite number of x; and (ii) $\{e(x) | x \in \mathcal{X}\}$ is a partition of $\mathcal{S}$, so that $\cup_{x \in \mathcal{X}} e(x) = \mathcal{S}$, and $x \neq x^*$ implies $e(x) \cap e(x^*) = \emptyset$. Let $\hat{\mathcal{A}}$ denote the larger family of all event-valued mappings $e(\cdot) : \mathcal{X} \to \mathcal{E}$ that satisfy property (i) but not necessarily property (ii). Epstein defines event-differentiability for functions $\hat{W}(\cdot)$ on $\hat{\mathcal{A}}$, so it will apply naturally to any subjective preference function $W(\cdot)$ defined over $\mathcal{A} \subset \hat{\mathcal{A}}$. Given a general event-valued mapping $e(\cdot) : \mathcal{X} \to \mathcal{E}$ that satisfies (i) but not necessarily (ii), $e^+(\cdot)$ is said to satisfy $e^+(\cdot) \cap e(\cdot) = \emptyset$ if $e^+(x) \cap e(x) = \emptyset$ for each x, in which case define $e(\cdot) + e^+(\cdot)$ as the mapping that takes x to $e(x) \cup e^+(x)$. Similarly, $e^-(\cdot)$ is said to satisfy $e^-(\cdot) \subset e(\cdot)$ if $e^-(x) \subset e(x)$ for each x, in which case define $e(\cdot) - e^-(\cdot)$ as the mapping that takes x to $e(x) - e^-(x)$. A finite collection $\{e_1(\cdot), \ldots, e_n(\cdot)\}$ is said to *partition* $e(\cdot)$ if $\{e_1(x), \ldots, e_n(x)\}$ is a partition of $e(x)$ for each $x \in \mathcal{X}$. Given the family $\{\{e_{1,\kappa}(\cdot), \ldots, e_{n_\kappa,\kappa}(\cdot)\} | \kappa \in \mathcal{K}\}$ of all such finite partitions of $e(\cdot)$, we can define a partial order $\succ_{\mathcal{K}}$ on this family by defining $\{e_{1,\kappa'}(\cdot), \ldots, e_{n_{\kappa'},\kappa'}(\cdot)\} \succ_{\mathcal{K}} \{e_{1,\kappa}(\cdot), \ldots, e_{n_\kappa,\kappa}(\cdot)\}$ if the partition of $\mathcal{S}$ implied by the former is a refinement of the partition implied by the latter.

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[38] As in the objective case, $\mathcal{L} \geq (x' - x)/(x'' - x')$ is only a *necessary* condition for a risk averse $W(\cdot)$ to be willing to accept such spreads, and it needn't be *sufficient.*

[39] In the following, Epstein's notion has been adapted to maintain consistency with the notation used in this paper.

A general function $\hat{W}(\cdot)$ over $\hat{\mathcal{A}}$ is then said to be *eventwise-differentiable* at $e_0(\cdot)$ if there exists a family of bounded and convex-ranged event-additive functions $\{d\hat{W}_x(\cdot;e_0)|x \in \mathcal{X}\}$ such that, for each $\varepsilon > 0$, each $e^+(\cdot)$ such that $e^+(\cdot) \cap e_0(\cdot) = \emptyset$, and each $e^-(\cdot)$ such that $e^-(\cdot) \subset e_0(\cdot)$, there exists partitions $\{\hat{e}_1^+(\cdot), \ldots, \hat{e}_{n^+}^+(\cdot)\}$ of $e^+(\cdot)$ and $\{\hat{e}_1^-(\cdot), \ldots, \hat{e}_{n^-}^-(\cdot)\}$ of $e^-(\cdot)$ such that

$$\sum_{j=1}^{n_\kappa} \Big| \hat{W}\big(e_0(\cdot)+e_j^+(\cdot)-e_j^-(\cdot)\big) - \hat{W}\big(e_0(\cdot)\big) - \big[d\hat{W}\big(e_j^+;e_0\big) - d\hat{W}\big(e_j^-;e_0\big)\big] \Big| < \varepsilon \quad (60)$$

for each $\{e_1^+(\cdot), \ldots, e_{n_\kappa}^+(\cdot)\} \succ_\mathcal{K} \{\hat{e}_1^+(\cdot), \ldots, \hat{e}_{n^+}^+(\cdot)\}$ and $\{e_1^-(\cdot), \ldots, e_{n_\kappa}^-(\cdot)\} \succ_\mathcal{K} \{\hat{e}_1^-(\cdot), \ldots, \hat{e}_{n^-}^-(\cdot)\}$. A preference function $W(\cdot)$ over subjective acts is then said to be eventwise-differentiable if it satisfies the above definition over its domain $\mathcal{A} \subset \hat{\mathcal{A}}$, in which case its derivative will be unique and satisfy the chain rule. Epstein applies this notion of eventwise-differentiability in his characterization of the property of uncertainty aversion, and it has been also been adapted to analyze the core of nonatomic transferable-utility games by Epstein and Marinacci (2001).

For each j, the expression inside absolute values (60) is analogous to the error-term expression $W(f(\cdot)) - W(f_0(\cdot)) - [\sum_{x\in\mathcal{X}} \Phi_x(\Delta E_x^+; f_0) - \sum_{x\in\mathcal{X}} \Phi_x(\Delta E_x^-; f_0)]$ implied by the event-differentiability formula (19). In this sense, the two notions of event-differentiability have the same structure for their first-order approximation. Epstein (1999, Appendix C) also compares his definition of event-differentiability with a definition similar to (19), proves a version of the Fundamental Theorem of Calculus, and shows that when a preference function is differentiable in both senses, the derivatives coincide. The primary *difference* between the two notions of differentiability lies in their treatment of convergence. In particular, by replacing the error-term expression $\delta(f^*(\cdot), f(\cdot))$ of (19) with the sum of n_κ such terms in (60), Epstein's definition eliminates the need for any underlying reference measure $\lambda(\cdot)$ on the state space $\mathcal{S}$, as well as the need for any distance function $\delta(\cdot,\cdot)$ between acts.

6.2 Extension to more general state spaces and preferences

Given this paper's theme of robustness, it is worth noting how its assumptions of a uniform reference measure $\lambda(\cdot)$, a univariate state space $\mathcal{S} = [\underline{s}, \overline{s}]$, and event-smoothness can each be relaxed.

Although many state spaces possess a natural uniform reference measure, others do not: For example, if the state of nature is the exchange rate, then a uniform reference measure on the Dollar/Euro rate is not equivalent to a uniform measure on the Euro/Dollar rate. However, it is straightforward to show that if two alternative choices of reference measure $\lambda(\cdot)$ and $\lambda^*(\cdot)$ are mutually absolutely continuous with respect to each other, with Radon-Nikodym derivatives that are bounded away from both 0 and ∞, then event-differentiability with respect to one reference measure will be equivalent to event-differentiability with respect to another, and more importantly, $W(\cdot)$'s local evaluations of the growth and shrinkage sets $\{(\Delta E_x^+, \Delta E_x^-)|x \in \mathcal{X}\}$ between any pair of acts $f(\cdot)$ and $f^*(\cdot)$ will be invariant to the choice of reference measure.

As with the almost-objective uncertainty analysis of Machina (2004), the analysis of this paper can also be extended to more general state spaces – in particular, to multivariate Euclidean states spaces as well as smooth univariate or multivariate manifolds. The key requirement is that the state space admit of almost-objective events, which will be the case if it can be "tiled" by arbitrarily small but suitable measure spaces. Compared with the approach of Savage (1954), the analysis of this paper is seen to require *more structure* on the state space $\mathcal{S}$, and hence on the choice space $\mathcal{A}$ of subjective acts, but *less structure* (neither expected utility nor probabilistic sophistication) on *beliefs* or *preferences* over this choice space. Of course, from a scientific and observational point of view, verifying structural assumptions on a *state space* or a *choice space* is – if anything – much easier than verifying such assumptions on an agent's (or a collection of agents') *beliefs* or *preferences* over such spaces, under either certainty or uncertainty.

Finally, the assumption of event-smoothness can also be somewhat relaxed. Recall that in standard analysis, the Fundamental Theorem of Calculus continues to hold for functions that are not everywhere differentiable, so long as they are absolutely continuous: we simply "integrate over the kinks." In a similar manner, the analysis and results of this paper can be extended to subjective preference functions $W(\cdot)$ that exhibit kinks in the events, so long as these kinks are sufficiently isolated.

The results of this paper have shown that the analytics of the classical expected utility/subjective probability model of risk preferences and beliefs – which continues to dominate research in choice under uncertainty – are in fact quite robust to smooth departures from its basic underlying assumptions of the Sure-Thing Principle and Probabilistic Sophistication. Or as Sir William Gilbert might have said: It is very very general, for a modern major model.

Appendix – Proofs of Theorems[40]

Proof of Theorem 1. (a) ⇒ (b): Given $U(\cdot|\cdot)$ and $\mu(\cdot)$, select arbitrary $\hat{x} \in \mathcal{X}$ and for each $x \in \mathcal{X}$, define the signed measure $\Phi_x(\cdot)$ by

$$\Phi_x(E) \;\equiv\; \int_E \Big[U(x|s) - U(\hat{x}|s) + \int_{\mathcal{S}} U(\hat{x}|\omega)\cdot d\mu(\omega)\Big]\cdot d\mu(s) \tag{A.1}$$

For any $f(\cdot) \in \mathcal{A}$, we can thus write $W(f(\cdot)) \equiv \int_S U(f(s)|s)\cdot d\mu(s)$ as

$$\begin{aligned} W\big(f(\cdot)\big) &\equiv \int_S \Big[U\big(f(s)|s\big) - U(\hat{x}|s) + \int_S U(\hat{x}|\omega)\cdot d\mu(\omega)\Big]\cdot d\mu(s) \;\equiv \\ \sum_{x\in\mathcal{X}} \int_{f^{-1}(x)} &\Big[U(x|s) - U(\hat{x}|s) + \int_S U(\hat{x}|\omega)\cdot d\mu(\omega)\Big]\cdot d\mu(s) \equiv \sum_{x\in\mathcal{X}} \Phi_x\big(f^{-1}(x)\big) \end{aligned} \tag{A.2}$$

Absolute continuity of each $\Phi_x(\cdot)$ follows since $\Phi_x([\underline{s},s]) \;\equiv\; W(x \text{ on } [\underline{s},s];\ \hat{x} \text{ on } (s,\overline{s}]) - \Phi_{\hat{x}}((s,\overline{s}\,]) \;\equiv\; W(x \text{ on } [\underline{s},s];\hat{x} \text{ on } (s,\overline{s}]) - [\int_{\mathcal{S}} U(\hat{x}|\omega)\cdot d\mu(\omega)]\cdot \mu((s,\overline{s}\,])$ is the difference of two absolutely continuous functions of the event boundary s.

(b) ⇒ (a): Absolute continuity of the signed measures $\{\Phi_x(\cdot)\,|\,x \in \mathcal{X}\}$ implies they can be represented by a family of signed densities $\{\phi(x,\cdot)\,|\,x \in \mathcal{X}\}$ on $\mathcal{S}$. Define $\mu(\cdot) = \lambda(\cdot)/\lambda(\mathcal{S})$ (where $\lambda(\cdot)$ is uniform Lebesgue measure), and define $U(x|s) \underset{x,s}{\equiv} \phi(x,s)\cdot\lambda(\mathcal{S})$. For any $f(\cdot) \in \mathcal{A}$, we then have

$$\begin{aligned} W\big(f(\cdot)\big) &= \sum_{x\in\mathcal{X}} \Phi_x\big(f^{-1}(x)\big) = \sum_{x\in\mathcal{X}} \int_{f^{-1}(x)} \phi(x,s)\cdot ds \\ &= \int_S \phi\big(f(s),s\big)\cdot ds = \int_S U\big(f(s)|s\big)\cdot d\mu(s) \end{aligned} \tag{A.3}$$

(b) ⇒ (c): This follows since $W(f^*(\cdot)) - W(f(\cdot)) = \sum_{x\in\mathcal{X}}[\Phi_x(E_x^*) - \Phi_x(E_x)] = \sum_{x\in\mathcal{X}}[\Phi_x(\Delta E_x^+) - \Phi_x(\Delta E_x^-)]$ for all $f(\cdot), f^*(\cdot) \in \mathcal{A}$.

(c) ⇒ (b): Select arbitrary $\hat{x} \in \mathcal{X}$ and for each $x \in \mathcal{X}$ and $E \in \mathcal{E}$, define $\Phi_x(E) = W(x \text{ on } E; \hat{x} \text{ on } \mathcal{S}-E) - (1-(\lambda(E)/\lambda(\mathcal{S})))\cdot W(\hat{x} \text{ on } \mathcal{S})$. To see that each $\Phi_x(\cdot)$ is additive and hence a signed measure, pick arbitrary disjoint E, E' and observe that constant sensitivity in the events implies

$$\begin{aligned} \Phi_x(E\cup E') &= W\big(x \text{ on } E\cup E'; \hat{x} \text{ on } \mathcal{S}-(E\cup E')\big) - \Big(1-\frac{\lambda(E\cup E')}{\lambda(\mathcal{S})}\Big)\cdot W\big(\hat{x} \text{ on } \mathcal{S}\big) \\ &= \Big[W\big(x \text{ on } E\cup E'; \hat{x} \text{ on } \mathcal{S}-(E\cup E')\big) - W\big(x \text{ on } E'; \hat{x} \text{ on } \mathcal{S}-E'\big) + \\ &\quad \frac{\lambda(E)}{\lambda(\mathcal{S})}\cdot W\big(\hat{x} \text{ on } \mathcal{S}\big)\Big] + \Big[W\big(x \text{ on } E'; \hat{x} \text{ on } \mathcal{S}-E'\big) - \Big(1-\frac{\lambda(E')}{\lambda(\mathcal{S})}\Big)\cdot W\big(\hat{x} \text{ on } \mathcal{S}\big)\Big] \\ &= \Big[W\big(x \text{ on } E; \hat{x} \text{ on } \mathcal{S}-E\big) - W\big(\hat{x} \text{ on } \mathcal{S}\big) + \frac{\lambda(E)}{\lambda(\mathcal{S})}\cdot W\big(\hat{x} \text{ on } \mathcal{S}\big)\Big] + \Phi_x(E') \\ &= \Phi_x(E) + \Phi_x(E') \end{aligned} \tag{A.4}$$

[40] In the following proofs, it will often be notationally more convenient to work with a preference function's *local evaluation measures* $\Phi_x(E;f) \equiv \int_E \phi(x,s;f)\cdot ds$ as defined in (19), rather than its local evaluation function $\phi(x,s;f)$.

Absolute continuity of each $\Phi_x(\cdot)$ follows since $\Phi_x([\underline{s}, s]) \equiv W(x \text{ on } [\underline{s}, s]; \hat{x} \text{ on } (s, \overline{s}]) - (1 - ((s - \underline{s})/\lambda(\mathcal{S}))) \cdot W(\hat{x} \text{ on } \mathcal{S})$ is the difference of two absolutely continuous functions of the event boundary s. Given arbitrary $f(\cdot) = [x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n]$, constant sensitivity in the events thus implies

$$\begin{aligned}
W\big(f(\cdot)\big) &= W\big(f(\cdot)\big) - W\big(\hat{x} \text{ on } E_1; f(\cdot) \text{ elsewhere}\big) \\
&+ W\big(\hat{x} \text{ on } E_1; f(\cdot) \text{ elsewhere}\big) - W\big(\hat{x} \text{ on } E_1 \cup E_2; f(\cdot) \text{ elsewhere}\big) \\
&+ W\big(\hat{x} \text{ on } E_1 \cup E_2; f(\cdot) \text{ elsewhere}\big) - W\big(\hat{x} \text{ on } E_1 \cup E_2 \cup E_3; f(\cdot) \text{ elsewhere}\big) \\
&\;\vdots \\
&+ W\big(\hat{x} \text{ on } E_1 \cup \ldots \cup E_{n-1}; f(\cdot) \text{ elsewhere}\big) - W\big(\hat{x} \text{ on } \mathcal{S}\big) + W\big(\hat{x} \text{ on } \mathcal{S}\big) \\
&= \sum_{i=1}^{n} \big[W\big(x_i \text{ on } E_i; \hat{x} \text{ on } \mathcal{S} - E_i\big) - W\big(\hat{x} \text{ on } \mathcal{S}\big)\big] + \sum_{i=1}^{n} \frac{\lambda(E_i)}{\lambda(\mathcal{S})} \cdot W\big(\hat{x} \text{ on } \mathcal{S}\big) \\
&= \textstyle\sum_{i=1}^{n} \Phi_{x_i}(E_i)
\end{aligned} \tag{A.5}$$

Proof of Theorem 2. Let $\hat{s}$ be an arbitrary continuity point of $f(\cdot)$, let $\varepsilon_a \geq 0$ and $\varepsilon_b \geq 0$ be small enough so that $[\hat{s} - \varepsilon_a, \hat{s} + \varepsilon_b] \subseteq f^{-1}(f(\hat{s}))$, and define $f_{\gamma,\varepsilon_a,\varepsilon_b}(\cdot) \equiv [f(\hat{s}) + \gamma \text{ on } [\hat{s} - \varepsilon_a, \hat{s} + \varepsilon_b]; f(\cdot) \text{ elsewhere}]$. By (43) we have that $W(f_{\gamma,\varepsilon_a,\varepsilon_b}(\cdot)) - W(f(\cdot)) = \gamma \cdot \Psi([\hat{s} - \varepsilon_a, \hat{s} + \varepsilon_b]; f) + o(|\gamma|)$. By joint outcome-event smoothness, this expression is twice continuously differentiable in $(\gamma, \varepsilon_a, \varepsilon_b)$ about (0,0,0), which implies that $\psi(s; f)$ is continuous in s about $\hat{s}$, and also that $\partial^2 W(f_{\gamma,\varepsilon_a,\varepsilon_b}(\cdot))/\partial\gamma\,\partial\varepsilon_b|_{\gamma=\varepsilon_a=\varepsilon_b=0} = \psi(\hat{s}; f)$. By (19), we have $W(f_{\gamma,\varepsilon_a,\varepsilon_b}(\cdot)) - W(f(\cdot)) = \Phi_{f(\hat{s})+\gamma}([\hat{s} - \varepsilon_a, \hat{s} + \varepsilon_b]; f) - \Phi_{f(\hat{s})}([\hat{s} - \varepsilon_a, \hat{s} + \varepsilon_b]; f) + o(|\varepsilon_a + \varepsilon_b|)$, so that $\partial W(f_{\gamma,\varepsilon_a,\varepsilon_b}(\cdot))/\partial\varepsilon_b|_{\varepsilon_a=\varepsilon_b=0} = \phi(f(\hat{s}) + \gamma, \hat{s}; f) - \phi(f(\hat{s}), \hat{s}; f)$. By joint outcome-event smoothness, we thus have $\psi(\hat{s}; f) = \partial^2 W(f_{\gamma,\varepsilon_a,\varepsilon_b}(\cdot))/\partial\gamma\,\partial\varepsilon_b|_{\gamma=\varepsilon_a=\varepsilon_b=0} = \phi_x(f(\hat{s}), \hat{s}; f)$, which is (44)′. Equations (43) and (44)′ then yield (45)′.

Proof of Theorem 3. (48) ⇒ probabilistic sophistication: We first show that each finite-outcome lottery **P** can be associated with a unique "basic reference act" $f_{\mathbf{P}}(\cdot) \in \mathcal{A}$, and that $W(\cdot)$ is probabilistically sophisticated over the family of such acts: Well-order the elements of each indifference class of the outcome preference relation $\succcurlyeq$, and construct the strict order $\succ^+$ over outcomes defined by $x^* \succ^+ x$ if and only if (i) $x^* \succ x$ or (ii) $x^* \sim x$ and x^* is ordered above x within their common indifference class. Express each lottery in the form $\mathbf{P} = (\hat{x}_1, \hat{p}_1; \ldots; \hat{x}_n, \hat{p}_n)$ with $\hat{x}_1 \prec^+ \hat{x}_2 \prec^+ \ldots \prec^+ \hat{x}_n$ and $\hat{p}_1, \ldots, \hat{p}_n > 0$, and define its associated *basic reference act* $f_{\mathbf{P}}(\cdot) = [\hat{x}_1 \text{ on } [\underline{s}, s_1]; \hat{x}_2 \text{ on } (s_1, s_2]; \ldots; \hat{x}_n \text{ on } (s_{n-1}, \overline{s}]]$, where s_1 is the smallest state in $\mathcal{S}$ such that $\mu([\underline{s}, s_1]) = \hat{p}_1$, s_2 is the smallest state such that $\mu((s_1, s_2]) = \hat{p}_2$, etc. For each **P**, define $V(\mathbf{P}) = W(f_{\mathbf{P}}(\cdot))$. Since this implies $W(f_{\mathbf{P}}(\cdot)) = V(\hat{x}_1, \hat{p}_1; \ldots; \hat{x}_n, \hat{p}_n) = V(\hat{x}_1, \mu([\underline{s}, s_1]); \ldots; \hat{x}_n, \mu((s_{n-1}, \overline{s}])) = V(\hat{x}_1, \mu(f_{\mathbf{P}}^{-1}(\hat{x}_1)); \ldots; \hat{x}_n, \mu(f_{\mathbf{P}}^{-1}(\hat{x}_n)))$, $W(\cdot)$ is probabilistically sophisticated over basic reference acts, with lottery preference function $V(\cdot)$ and subjective probability measure $\mu(\cdot)$.

We now show that every act $f(\cdot) \in \mathcal{A}$ with an implied outcome lottery **P** will be indifferent to $f_{\mathbf{P}}(\cdot)$, so that $W(\cdot)$ is probabilistically sophisticated over $\mathcal{A}$: Since

$f(\cdot)$ implies the lottery **P**, we have $\mu(f^{-1}(x)) = \mu(f_{\mathbf{P}}^{-1}(x))$ for each $x \in \mathcal{X}$. Defining $\{f_\alpha^m(\cdot)|\alpha \in [0,1]\}_{m=1}^\infty$ as the almost-objective mixture paths from $f_{\mathbf{P}}(\cdot)$ to $f(\cdot)$, the Line Integral Approximation Theorem and (48) imply

$$W\big(f(\cdot)\big) - W\big(f_{\mathbf{P}}(\cdot)\big) = \lim_{m\to\infty}\int_0^1 \sum_{x\in\mathcal{X}} \big[\Phi_x\big(f^{-1}(x); f_\alpha^m\big) - \Phi_x\big(f_{\mathbf{P}}^{-1}(x); f_\alpha^m\big)\big]\cdot d\alpha \tag{A.6}$$

$$= \lim_{m\to\infty}\int_0^1 \sum_{x\in\mathcal{X}} \big[U\big(x; \mathbf{P}_{f_\alpha^m,\mu}\big)\cdot\mu\big(f^{-1}(x)\big) - U\big(x; \mathbf{P}_{f_\alpha^m,\mu}\big)\cdot\mu\big(f_{\mathbf{P}}^{-1}(x)\big)\big]\cdot d\alpha = 0$$

Probabilistic sophistication ⇒ (48): Event-smoothness of $W(\cdot)$, probabilistic sophistication, and absolute continuity of $\mu(\cdot)$ jointly ensure that $V(\cdot)$ is differentiable in the probabilities. For arbitrary $x \in \mathcal{X}$, interior state $s \in \mathcal{S}$ and act $f(\cdot) = [x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n] \in \mathcal{A}$, formulas (22) and (10) imply[41]

$$\phi(x,s;f) \;=\; \lim_{\varepsilon\to 0}\frac{W\begin{pmatrix} x & \text{on} & B_{s,\varepsilon} \\ f(\cdot) & \text{on} & \mathcal{S}-B_{s,\varepsilon}\end{pmatrix} - W\begin{pmatrix} \underline{x} & \text{on} & B_{s,\varepsilon} \\ f(\cdot) & \text{on} & \mathcal{S}-B_{s,\varepsilon}\end{pmatrix}}{2\cdot\varepsilon} \;= \tag{A.7}$$

$$\lim_{\varepsilon\to 0}\frac{\begin{bmatrix} V\big(x,\mu(B_{s,\varepsilon}); x_1,\mu(E_1-B_{s,\varepsilon}); \ldots; x_n,\mu(E_n-B_{s,\varepsilon})\big) \\ -\, V\big(\underline{x},\mu(B_{s,\varepsilon}); x_1,\mu(E_1-B_{s,\varepsilon}); \ldots; x_n,\mu(E_n-B_{s,\varepsilon})\big)\end{bmatrix}}{\mu(B_{s,\varepsilon})}\cdot\frac{\mu(B_{s,\varepsilon})}{2\cdot\varepsilon}$$

$$=\; U(x;\mathbf{P}_{f,\mu})\cdot\nu(s)$$

where $U(x;\mathbf{P})$ is the $\underline{x}$-normalized local utility function of $V(\cdot)$ at **P**. When $\mu(B_{s,\varepsilon}) = 0$ for all small $B_{s,\varepsilon}$ about s, so that the left fraction in the second line of (A.7) is undefined, the numerator in the first line will be zero, as will be $\nu(s)$, so we again obtain $\phi(x,s;f) = 0 = U(x;\mathbf{P}_{f,\mu})\cdot\nu(s)$.

Proof of Theorem 4. (a) ⇒ (b): Given arbitrary $x^* \succ x$, $f(\cdot) \in \mathcal{A}$, rational $\mathcal{L}^* < \mathcal{L}$ and $\varepsilon > 0$, define $\gamma = [\Phi_{x^*}(A;f) - \Phi_x(A;f)] - \mathcal{L}^*\cdot[\Phi_{x^*}(B;f) - \Phi_x(B;f)] > 0$. By (19) there exists $\delta^* > 0$ such that $\delta(\hat{f}(\cdot), f(\cdot)) < \delta^*$ implies

$$\Big|W\big(\hat{f}(\cdot)\big) - W\big(f(\cdot)\big) - \Big[\sum_{\hat{x}\in\mathcal{X}}\Phi_{\hat{x}}\big(\hat{f}^{-1}(\hat{x}); f\big) - \sum_{\hat{x}\in\mathcal{X}}\Phi_{\hat{x}}\big(f^{-1}(\hat{x}); f\big)\Big]\Big| < \frac{\gamma/2}{\lambda(A) + \mathcal{L}^*\cdot\lambda(B)}\cdot\delta\big(\hat{f}(\cdot), f(\cdot)\big) \tag{A.8}$$

Select integers n_a, n_b such that $n_a/n_b = \mathcal{L}^*$ and $\lambda(A)/n_a + \lambda(B)/n_b < \min\{\varepsilon, \delta^*\}$. By Stromquist and Woodall (1985, Thm. 1) and $(21)'$, there exist $\mathcal{E}$-measurable partitions $\{A_1, \ldots, A_{n_a}\}$ of A and $\{B_1, \ldots, B_{n_b}\}$ of B with $\lambda(A_i) = \lambda(A)/n_a < \varepsilon$ and $\Phi_{x^*}(A_i;f) - \Phi_x(A_i;f) = [\Phi_{x^*}(A;f) - \Phi_x(A;f)]/n_a$ for each i, and $\lambda(B_j) = \lambda(B)/n_b < \varepsilon$ and $\Phi_{x^*}(B_j;f) - \Phi_x(B_j;f) = [\Phi_{x^*}(B;f) - \Phi_x(B;f)]/n_b$ for each j. For each i,j we thus have

$$\Phi_{x^*}(A_i;f) - \Phi_x(A_i;f) - \Phi_{x^*}(B_j;f) + \Phi_x(B_j;f) \;= \tag{A.9}$$

[41] For states on the boundary of $\mathcal{S}$, replace $B_{s,\varepsilon} = [s-\varepsilon, s+\varepsilon]$ by the appropriate half-ball, and replace $2\cdot\varepsilon$ by ε.

$$\big[\Phi_{x^*}(A;f) - \Phi_x(A;f)\big]/n_a \;-\; \mathcal{L}^*\cdot\big[\Phi_{x^*}(B;f) - \Phi_x(B;f)\big]/n_a \;=\; \gamma/n_a$$

For each i,j, define the acts $f^L_{i,j}(\cdot) \;=\; [x^*$ on $A_i; x$ on $B_j; f(\cdot)$ elsewhere$]$ and $f^R_{i,j}(\cdot) = [x$ on $A_i; x^*$ on $B_j; f(\cdot)$ elsewhere$]$ from (53), so that we have both

$$\begin{aligned}\delta\big(f^L_{i,j},f\big) \;\le\; \lambda(A_i)+\lambda(B_j) \;=\; \frac{\lambda(A)}{n_a}+\frac{\lambda(B)}{n_b} \;=\; \frac{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)}{n_a} \;<\; \delta^*\\ \delta\big(f^R_{i,j},f\big) \;\le\; \lambda(A_i)+\lambda(B_j) \;=\; \frac{\lambda(A)}{n_a}+\frac{\lambda(B)}{n_b} \;=\; \frac{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)}{n_a} \;<\; \delta^*\end{aligned} \tag{A.10}$$

For each i,j, (A.8) and (A.9) thus imply

$$\begin{aligned} &W\big(f^L_{i,j}(\cdot)\big) - W\big(f^R_{i,j}(\cdot)\big) \;> \\ &\sum_{\hat{x}\in\mathcal{X}}\Phi_{\hat{x}}\big(f^{L^{-1}}_{i,j}(\hat{x});f\big) - \sum_{\hat{x}\in\mathcal{X}}\Phi_{\hat{x}}\big(f^{R^{-1}}_{i,j}(\hat{x});f\big) - \frac{\frac{1}{2}\cdot\gamma\cdot\big(\delta(f^L_{i,j},f)+\delta(f^R_{i,j},f)\big)}{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)}\\ &\ge\; \Phi_{x^*}(A_i;f) - \Phi_x(A_i;f) - \Phi_{x^*}(B_j;f) + \Phi_x(B_j;f) - \gamma/n_a \;=\; 0\end{aligned} \tag{A.11}$$

(b) ⇒ (a): Say (a) failed, so that $\gamma = \Phi_{x^*}(A;f) - \Phi_x(A;f) - \mathcal{L}^*\cdot[\Phi_{x^*}(B;f) - \Phi_x(B;f)] < 0$ for some $x^* \succ x$, $f(\cdot)\in\mathcal{A}$ and rational $\mathcal{L}^*\in(0,\mathcal{L})$. By (19) there exists some $\varepsilon>0$ such that $\delta(\hat{f},f) < 2\cdot\varepsilon$ implies

$$\begin{aligned}&\Big|W\big(\hat{f}(\cdot)\big) - W\big(f(\cdot)\big) - \Big[\sum_{\hat{x}\in\mathcal{X}}\Phi_{\hat{x}}\big(\hat{f}^{-1}(\hat{x});f\big) - \sum_{\hat{x}\in\mathcal{X}}\Phi_{\hat{x}}\big(f^{-1}(\hat{x});f\big)\Big]\Big|\\ &<\; \frac{|\gamma|/2}{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)}\cdot\delta\big(\hat{f}(\cdot),f(\cdot)\big)\end{aligned} \tag{A.12}$$

Given arbitrary ε-partitions $\{A_1,\ldots,A_{n_a}\}$ of A and $\{B_1,\ldots,B_{n_b}\}$ of B with $n_a/n_b = \mathcal{L}^*$, additivity of local evaluation measures and the definition of γ imply

$$\begin{aligned}&\sum\nolimits_{i=1}^{n_a}\Big[\Phi_{x^*}(A_i;f) - \Phi_x(A_i;f)\Big] - \mathcal{L}^*\cdot\sum\nolimits_{j=1}^{n_b}\Big[\Phi_{x^*}(B_j;f) - \Phi_x(B_j;f)\Big]\\ &=\; \frac{\gamma}{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)}\cdot\Big[\sum\nolimits_{i=1}^{n_a}\lambda(A_i) + \mathcal{L}^*\cdot\sum\nolimits_{j=1}^{n_b}\lambda(B_j)\Big]\end{aligned} \tag{A.13}$$

Substituting n_a/n_b for the first and third occurrence of $\mathcal{L}^*$ and rearranging yields

$$\begin{aligned}&\sum\nolimits_{i=1}^{n_a}\Big[\Phi_{x^*}(A_i;f) - \Phi_x(A_i;f) - \frac{\gamma}{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)}\cdot\lambda(A_i)\Big] \;=\\ &\frac{n_a}{n_b}\cdot\sum\nolimits_{j=1}^{n_b}\Big[\Phi_{x^*}(B_j;f) - \Phi_x(B_j;f) + \frac{\gamma}{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)}\cdot\lambda(B_j)\Big]\end{aligned} \tag{A.14}$$

This implies there exists some $A_{i'}$ and $B_{j'}$ such that

$$\begin{aligned}&\Phi_{x^*}(A_{i'};f) \;-\; \Phi_x(A_{i'};f) \;-\; \frac{\gamma}{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)}\cdot\lambda(A_{i'})\\ &\le\; \Phi_{x^*}(B_{j'};f) \;-\; \Phi_x(B_{j'};f) \;+\; \frac{\gamma}{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)}\cdot\lambda(B_{j'})\end{aligned} \tag{A.15}$$

Define $f_L(\cdot) = [x^*\text{ on }A_{i'}; x\text{ on }B_{j'}; f(\cdot)\text{ elsewhere}]$ and $f_R(\cdot) = [x\text{ on }A_{i'}; x^*\text{ on }B_{j'}; f(\cdot)\text{ elsewhere}]$, so $\delta(f_L, f) \le \lambda(A_{i'}) + \lambda(B_{j'}) < 2\cdot\varepsilon$ and $\delta(f_R, f) \le \lambda(A_{i'}) + \lambda(B_{j'}) < 2\cdot\varepsilon$. By (A.12) and (A.15),

$$W\big(f_L(\cdot)\big) - W\big(f_R(\cdot)\big) \;<\; \tag{A.16}$$
$$\sum_{\hat{x}\in\mathcal{X}} \varPhi_{\hat{x}}\big(f_L^{-1}(\hat{x}); f\big) \;-\; \sum_{\hat{x}\in\mathcal{X}} \varPhi_{\hat{x}}\big(f_R^{-1}(\hat{x}); f\big) \;+\; \frac{|\gamma|}{2}\cdot\frac{\delta(f_L, f) + \delta(f_R, f)}{\lambda(A) + \mathcal{L}^*\cdot\lambda(B)} \;\le$$
$$\varPhi_{x^*}(A_{i'};f) + \varPhi_x(B_{j'};f) - \varPhi_x(A_{i'};f) - \varPhi_{x^*}(B_{j'};f) + |\gamma|\cdot\frac{\lambda(A_{i'}) + \lambda(B_{j'})}{\lambda(A) + \mathcal{L}^*\cdot\lambda(B)} \le 0$$

But since $\{A_1, \ldots, A_{n_a}\}$ and $\{B_1, \ldots, B_{n_b}\}$ were arbitrary ε-partitions with $n_a/n_b = \mathcal{L}^*$, this contradicts (b).

(a) $\Rightarrow$ (c): Given arbitrary $x^* \succ x$, $f(\cdot) \in \mathcal{A}$ and rational $\mathcal{L}^* < \mathcal{L}$, (52) implies $\gamma = \varPhi_{x^*}(A; f) - \varPhi_x(A; f) - \mathcal{L}^*\cdot[\varPhi_{x^*}(B; f) - \varPhi_x(B; f)] > 0$. By (19) there exists some $\varepsilon > 0$ such that $\delta(\hat{f}, f) < 2\cdot\varepsilon$ implies

$$\Big|W\big(\hat{f}(\cdot)\big) - W\big(f(\cdot)\big) - \Big[\sum_{\hat{x}\in\mathcal{X}} \varPhi_{\hat{x}}\big(\hat{f}^{-1}(\hat{x}); f\big) - \sum_{\hat{x}\in\mathcal{X}} \varPhi_{\hat{x}}\big(f^{-1}(\hat{x}); f\big)\Big]\Big|$$
$$< \;\frac{\gamma/2}{\lambda(A) + \mathcal{L}^*\cdot\lambda(B)} \cdot \delta\big(\hat{f}(\cdot), f(\cdot)\big) \tag{A.17}$$

Given arbitrary ε-partitions $\{A_1, \ldots, A_{n_a}\}$ of A and $\{B_1, \ldots, B_{n_b}\}$ of B with $n_a/n_b = \mathcal{L}^*$, we have

$$\sum\nolimits_{i=1}^{n_a} \Big[\varPhi_{x^*}(A_i; f) - \varPhi_x(A_i; f)\Big] \;-\; \mathcal{L}^*\cdot\sum\nolimits_{j=1}^{n_b} \Big[\varPhi_{x^*}(B_j; f) - \varPhi_x(B_j; f)\Big]$$
$$= \;\frac{\gamma}{\lambda(A) + \mathcal{L}^*\cdot\lambda(B)} \cdot \Big[\sum\nolimits_{i=1}^{n_a} \lambda(A_i) + \mathcal{L}^*\cdot\sum\nolimits_{j=1}^{n_b} \lambda(B_j)\Big] \tag{A.18}$$

Substituting n_a/n_b for the first and third occurrence of $\mathcal{L}^*$ and rearranging yields

$$\sum\nolimits_{i=1}^{n_a} \Big[\varPhi_{x^*}(A_i; f) - \varPhi_x(A_i; f) - \frac{\gamma}{\lambda(A) + \mathcal{L}^*\cdot\lambda(B)}\cdot\lambda(A_i)\Big] \;=$$
$$\frac{n_a}{n_b}\cdot\sum\nolimits_{j=1}^{n_b} \Big[\varPhi_{x^*}(B_j; f) - \varPhi_x(B_j; f) + \frac{\gamma}{\lambda(A) + \mathcal{L}^*\cdot\lambda(B)}\cdot\lambda(B_j)\Big] \tag{A.19}$$

This implies there exists some $A_{i'}$ and $B_{j'}$ such that

$$\varPhi_{x^*}(A_{i'}; f) - \varPhi_x(A_{i'}; f) - \frac{\gamma}{\lambda(A) + \mathcal{L}^*\cdot\lambda(B)}\cdot\lambda(A_{i'})$$
$$\ge\; \varPhi_{x^*}(B_{j'}; f) - \varPhi_x(B_{j'}; f) + \frac{\gamma}{\lambda(A) + \mathcal{L}^*\cdot\lambda(B)}\cdot\lambda(B_{j'}) \tag{A.20}$$

Define $f_L(\cdot) = [x^*\text{ on }A_{i'}; x\text{ on }B_{j'}; f(\cdot)\text{ elsewhere}]$ and $f_R(\cdot) = [x\text{ on }A_{i'}; x^*\text{ on }B_{j'}; f(\cdot)\text{ elsewhere}]$, so $\delta(f_L, f) \le \lambda(A_{i'}) + \lambda(B_{j'}) < 2\cdot\varepsilon$ and $\delta(f_R, f) \le \lambda(A_{i'}) + \lambda(B_{j'}) < 2\cdot\varepsilon$. By (A.17) and (A.20), we have

$$W\big(f_L(\cdot)\big) - W\big(f_R(\cdot)\big) \;> \tag{A.21}$$

$$\sum_{\hat{x}\in\mathcal{X}} \Phi_{\hat{x}}\big(f_L^{-1}(\hat{x});f\big) - \sum_{\hat{x}\in\mathcal{X}} \Phi_{\hat{x}}\big(f_R^{-1}(\hat{x});f\big) - \frac{\frac{1}{2}\cdot\gamma\cdot\big(\delta(f_L,f)+\delta(f_R,f)\big)}{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)} \geq$$

$$\Phi_{x^*}(A_{i'};f)+\Phi_x(B_{j'};f)-\Phi_x(A_{i'};f)-\Phi_{x^*}(B_{j'};f)-\frac{\gamma\cdot\big(\lambda(A_{i'})+\lambda(B_{j'})\big)}{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)} \geq 0$$

(c) ⇒ (a): Say (a) failed, so that $\gamma = \Phi_{x^*}(A;f) - \Phi_x(A;f) - \mathcal{L}^*\cdot[\Phi_{x^*}(B;f) - \Phi_x(B;f)] < 0$ for some $x^* \succ x$, $f(\cdot)\in\mathcal{A}$, and rational $\mathcal{L}^* \in (0,\mathcal{L})$. By (19) there exists $\varepsilon^* > 0$ such that $\delta(\hat{f},f) < 2\cdot\varepsilon^*$ implies

$$\Big|W\big(\hat{f}(\cdot)\big) - W\big(f(\cdot)\big) - \Big[\sum_{\hat{x}\in\mathcal{X}} \Phi_{\hat{x}}\big(\hat{f}^{-1}(\hat{x});f\big) - \sum_{\hat{x}\in\mathcal{X}} \Phi_{\hat{x}}\big(f^{-1}(\hat{x});f\big)\Big]\Big| < \frac{\frac{1}{2}\cdot|\gamma|}{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)}\cdot\delta\big(\hat{f}(\cdot),f(\cdot)\big) \tag{A.22}$$

Select arbitrary $\varepsilon \in (0,\varepsilon^*)$ and integers $n_a > \lambda(A)/\varepsilon$ and $n_b > \lambda(B)/\varepsilon$ such that $n_a/n_b = \mathcal{L}^*$. By Stromquist and Woodall (1985, Thm.1), there exists an $\mathcal{E}$-measurable partition $\{A_1,\ldots,A_{n_a}\}$ of A with $\lambda(A_i) = \lambda(A)/n_a < \varepsilon$ and $\Phi_{x^*}(A_i;f) - \Phi_x(A_i;f) = [\Phi_{x^*}(A;f) - \Phi_x(A;f)]/n_a$ for each i, and an $\mathcal{E}$-measurable partition $\{B_1,\ldots,B_{n_b}\}$ of B with $\lambda(B_j) = \lambda(B)/n_b < \varepsilon$ and $\Phi_{x^*}(B_j;f) - \Phi_x(B_j;f) = [\Phi_{x^*}(B;f) - \Phi_x(B;f)]/n_b$ for each j. This implies that for all i,j we have

$$\begin{aligned}\Phi_{x^*}(A_i;f) - \Phi_x(A_i;f) - \Phi_{x^*}(B_j;f) + \Phi_x(B_j;f) \;&=\\ \frac{\Phi_{x^*}(A;f)-\Phi_x(A;f)}{n_a} - \mathcal{L}^*\cdot\frac{\Phi_{x^*}(B;f)-\Phi_x(B;f)}{n_a} \;&=\; \gamma/n_a\end{aligned} \tag{A.23}$$

Define $f_{i,j}^L(\cdot) = [x^*$ on A_i; x on B_j; $f(\cdot)$ elsewhere$]$ and $f_{i,j}^R(\cdot) = [x$ on A_i; x^* on B_j; $f(\cdot)$ elsewhere$]$, so that we have both

$$\begin{aligned}\delta\big(f_{i,j}^L,f\big) \leq \lambda(A_i)+\lambda(B_j) = \frac{\lambda(A)}{n_a}+\frac{\lambda(B)}{n_b} = \frac{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)}{n_a} \;<\; 2\cdot\varepsilon^*\\ \delta\big(f_{i,j}^R,f\big) \leq \lambda(A_i)+\lambda(B_j) = \frac{\lambda(A)}{n_a}+\frac{\lambda(B)}{n_b} = \frac{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)}{n_a} \;<\; 2\cdot\varepsilon^*\end{aligned} \tag{A.24}$$

For each i,j, (A.22) and (A.23) thus imply

$$W\big(f_{i,j}^L(\cdot)\big) - W\big(f_{i,j}^R(\cdot)\big) \;< \tag{A.25}$$

$$\sum_{\hat{x}\in\mathcal{X}} \Phi_{\hat{x}}\big(f_{i,j}^{L^{-1}}(\hat{x});f\big) - \sum_{\hat{x}\in\mathcal{X}} \Phi_{\hat{x}}\big(f_{i,j}^{R^{-1}}(\hat{x});f\big) + \frac{|\gamma|}{2}\cdot\frac{\delta(f_{i,j}^L,f)+\delta(f_{i,j}^R,f)}{\lambda(A)+\mathcal{L}^*\cdot\lambda(B)}$$

$$\leq \Phi_{x^*}(A_i;f) - \Phi_x(A_i;f) - \Phi_{x^*}(B_j;f) + \Phi_x(B_j;f) + |\gamma|/n_a \;=\; 0$$

But since $\{A_1,\ldots,A_{n_a}\}$ and $\{B_1,\ldots,B_{n_b}\}$ were ε-partitions with $n_a/n_b = \mathcal{L}^*$ for arbitrarily small $\varepsilon > 0$, this contradicts (c).

(a) ⇒ (d): Consider arbitrary $\wp_a, \wp_b \subseteq [0,1]$ with $\lambda(\wp_a)/\lambda(\wp_b) = (>) 1/\mathcal{L}$, $x^* \succ x$, and $f(\cdot) \in \mathcal{A}$. By Step 1 of the proof of Theorem 4 in Machina (2004), both limits in (55) exist. For arbitrary $\varepsilon > 0$, event-smoothness (boundedness and uniform continuity of the families $\{\phi(x^*,\cdot,\hat{f}) | \hat{f}(\cdot) \in \mathcal{A}\}$ and $\{\phi(x,\cdot,\hat{f}) | \hat{f}(\cdot) \in \mathcal{A}\}$) and Theorem 0 of Machina (2004) imply some m_ε such that both

$$\begin{aligned} \left|\Phi_{x^*}\big(\wp_a \underset{m}{\times} A;\hat{f}\big) - \Phi_x\big(\wp_a \underset{m}{\times} A;\hat{f}\big) - \lambda(\wp_a)\cdot\big(\Phi_{x^*}(A;\hat{f}) - \Phi_x(A;\hat{f})\big)\right| < \varepsilon/2 \\ \left|\Phi_{x^*}\big(\wp_b \underset{m}{\times} B;\hat{f}\big) - \Phi_x\big(\wp_b \underset{m}{\times} B;\hat{f}\big) - \lambda(\wp_b)\cdot\big(\Phi_{x^*}(B;\hat{f}) - \Phi_x(B;\hat{f})\big)\right| < \varepsilon/2 \end{aligned} \tag{A.26}$$

for all $m > m_\varepsilon$ and all $\hat{f}(\cdot) \in \mathcal{A}$. Event-smoothness (the bound $\Phi_{x^*}(B;\hat{f}) - \Phi_x(B;\hat{f}) > \underline{\Phi}_{x^*,x,B} > 0$ for all $\hat{f}(\cdot)$) and (52) then imply

$$\begin{aligned} &\Phi_{x^*}\big(\wp_a \underset{m}{\times} A;\hat{f}\big) - \Phi_x\big(\wp_a \underset{m}{\times} A;\hat{f}\big) - \Phi_{x^*}\big(\wp_b \underset{m}{\times} B;\hat{f}\big) + \Phi_x\big(\wp_b \underset{m}{\times} B;\hat{f}\big) > \\ &\lambda(\wp_a)\cdot\big(\Phi_{x^*}(A;\hat{f}) - \Phi_x(A;\hat{f})\big) - \lambda(\wp_b)\cdot\big(\Phi_{x^*}(B;\hat{f}) - \Phi_x(B;\hat{f})\big) - \varepsilon \geq \\ &\big[\lambda(\wp_a)\cdot\mathcal{L} - \lambda(\wp_b)\big]\cdot\big(\Phi_{x^*}(B;\hat{f}) - \Phi_x(B;\hat{f})\big) - \varepsilon \geq \big[\lambda(\wp_a)\cdot\mathcal{L} - \lambda(\wp_b)\big]\cdot\underline{\Phi}_{x^*,x,B} - \varepsilon \end{aligned} \tag{A.27}$$

for all $m > m_\varepsilon$ and all $\hat{f}(\cdot)$. The change sets from $f_R^m(\cdot) = [x \text{ on } \wp_a \underset{m}{\times} A;\ x^* \text{ on } \wp_b \underset{m}{\times} B;\ f(\cdot) \text{ elsewhere}]$ to $f_L^m(\cdot) = [x^* \text{ on } \wp_a \underset{m}{\times} A;\ x \text{ on } \wp_b \underset{m}{\times} B;\ f(\cdot) \text{ elsewhere}]$ in (55) are $\Delta E_{x^*}^+ = \Delta E_x^- = \wp_a \underset{m}{\times} A$ and $\Delta E_{x^*}^- = \Delta E_x^+ = \wp_b \underset{m}{\times} B$. For the almost-objective mixture paths $f_\alpha^{k,m}(\cdot) \equiv [f_L^m(\cdot) \text{ on } [0,\alpha] \underset{k}{\times} \mathcal{S};\ f_R^m(\cdot) \text{ on } (\alpha,1] \underset{k}{\times} \mathcal{S}]$ from $f_R^m(\cdot)$ to $f_L^m(\cdot)$, the Line Integral Approximation Theorem and (A.27) imply

$$\begin{aligned} &W\big(f_L^m(\cdot)\big) - W\big(f_R^m(\cdot)\big) = \lim_{k\to\infty} \int_0^1 \Big[\Phi_{x^*}\big(\wp_a \underset{m}{\times} A; f_\alpha^{k,m}\big) + \Phi_x\big(\wp_b \underset{m}{\times} B; f_\alpha^{k,m}\big) - \\ &\Phi_{x^*}\big(\wp_b \underset{m}{\times} B; f_\alpha^{k,m}\big) - \Phi_x\big(\wp_a \underset{m}{\times} A; f_\alpha^{k,m}\big)\Big]\cdot d\alpha \geq \big[\lambda(\wp_a)\cdot\mathcal{L} - \lambda(\wp_b)\big]\cdot\underline{\Phi}_{x^*,x,B} - \varepsilon \end{aligned} \tag{A.28}$$

for all $m > m_\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\lim_{m\to\infty} W(f_L^m(\cdot)) - \lim_{m\to\infty} W(f_R^m(\cdot)) \geq [\lambda(\wp_a)\cdot\mathcal{L} - \lambda(\wp_b)]\cdot\underline{\Phi}_{x^*,x,B} = (>)\, 0$.

(d) ⇒ (a): Say (a) failed, so that $\gamma = \Phi_{x^*}(A;f) - \Phi_x(A;f) - \mathcal{L}\cdot[\Phi_{x^*}(B;f) - \Phi_x(B;f)] < 0$ for some $x^* \succ x$ and $f(\cdot)$. By (19) there exists some $\delta^* > 0$ such that $\delta(\hat{f}, f) < \delta^*$ implies

$$\begin{aligned} &\left|W\big(\hat{f}(\cdot)\big) - W\big(f(\cdot)\big) - \Big[\sum_{\hat{x}\in\mathcal{X}} \Phi_{\hat{x}}\big(\hat{f}^{-1}(\hat{x}); f\big) - \sum_{\hat{x}\in\mathcal{X}} \Phi_{\hat{x}}\big(f^{-1}(\hat{x}); f\big)\Big]\right| \\ &\qquad < \frac{\frac{1}{16}\cdot|\gamma|}{\lambda(A) + \mathcal{L}\cdot\lambda(B)}\cdot\delta\big(\hat{f}(\cdot), f(\cdot)\big) \end{aligned} \tag{A.29}$$

Select $\wp_a, \wp_b \subset [0,1]$ such that $\lambda(\wp_a)/\lambda(\wp_b) = 1/\mathcal{L}$ and $\lambda(\wp_a)\cdot\lambda(A) + \lambda(\wp_b)\cdot\lambda(B) < \delta^*/2$. By event-smoothness and Theorem 0 of Machina (2004) there exists m^* such that for all $m > m^*$ we have the following three relationships

$$\lambda\big(\wp_a \underset{m}{\times} A\big) + \lambda\big(\wp_b \underset{m}{\times} B\big) < 2\cdot\big(\lambda(\wp_a)\cdot\lambda(A) + \lambda(\wp_b)\cdot\lambda(B)\big) < \delta^* \tag{A.30}$$

$$\left|\Phi_{x^*}\big(\wp_a \underset{m}{\times} A;f\big) - \Phi_x\big(\wp_a \underset{m}{\times} A;f\big) - \lambda(\wp_a)\cdot\big(\Phi_{x^*}(A;f) - \Phi_x(A;f)\big)\right| < \tfrac{1}{4}\cdot|\gamma|\cdot\lambda(\wp_a)$$

$$\left|\Phi_{x^*}\big(\wp_b \underset{m}{\times} B;f\big) - \Phi_x\big(\wp_b \underset{m}{\times} B;f\big) - \lambda(\wp_b)\cdot\big(\Phi_{x^*}(B;f) - \Phi_x(B;f)\big)\right| < \tfrac{1}{4}\cdot|\gamma|\cdot\lambda(\wp_a)$$

For all $m > m^*$ we thus have

$$\Phi_{x^*}\big(\wp_a \underset{m}{\times} A;f\big) - \Phi_x\big(\wp_a \underset{m}{\times} A;f\big) - \Phi_{x^*}\big(\wp_b \underset{m}{\times} B;f\big) + \Phi_x\big(\wp_b \underset{m}{\times} B;f\big) \;<\; \tag{A.31}$$
$$\lambda(\wp_a)\cdot\big(\Phi_{x^*}(A;f) - \Phi_x(A;f)\big) - \lambda(\wp_b)\cdot\big(\Phi_{x^*}(B;f) - \Phi_x(B;f)\big) + \tfrac{1}{2}\cdot|\gamma|\cdot\lambda(\wp_a)$$
$$= \lambda(\wp_a)\cdot\big[\big(\Phi_{x^*}(A;f) - \Phi_x(A;f)\big) - \mathcal{L}\cdot\big(\Phi_{x^*}(B;f) - \Phi_x(B;f)\big)\big] + \tfrac{1}{2}\cdot|\gamma|\cdot\lambda(\wp_a)$$
$$= \lambda(\wp_a)\cdot\gamma + \tfrac{1}{2}\cdot|\gamma|\cdot\lambda(\wp_a) \;=\; -\tfrac{1}{2}\cdot|\gamma|\cdot\lambda(\wp_a)$$

For each m, define $f_L^m(\cdot) = [x^* \text{ on } \wp_a \underset{m}{\times} A; x \text{ on } \wp_b \underset{m}{\times} B; f(\cdot) \text{ elsewhere}]$ and $f_R^m(\cdot) = [x \text{ on } \wp_a \underset{m}{\times} A; x^* \text{ on } \wp_b \underset{m}{\times} B; f(\cdot) \text{ elsewhere}]$, so the change sets from $f_R^m(\cdot)$ to $f_L^m(\cdot)$ are $\Delta E_{x^*}^+ = \Delta E_x^- = \wp_a \underset{m}{\times} A$ and $\Delta E_{x^*}^- = \Delta E_x^+ = \wp_b \underset{m}{\times} B$. For each $m > m^*$, (A.30) implies $\delta(f_L^m, f)$ and $\delta(f_R^m, f)$ are each less than or equal to $\lambda(\wp_a \underset{m}{\times} A) + \lambda(\wp_b \underset{m}{\times} B) < 2\cdot(\lambda(\wp_a)\cdot\lambda(A) + \lambda(\wp_b)\cdot\lambda(B)) = 2\cdot\lambda(\wp_a)\cdot(\lambda(A) + \mathcal{L}\cdot\lambda(B)) < \delta^*$. This yields a contradiction of (d), since $\lambda(\wp_a)/\lambda(\wp_b) = 1/\mathcal{L}$, yet for each $m > m^*$, (A.29) and (A.31) imply

$$W\big(f_L^m(\cdot)\big) - W\big(f_R^m(\cdot)\big) \;<\; \tag{A.32}$$
$$\sum_{\hat{x}\in\mathcal{X}} \Phi_{\hat{x}}\big(f_L^{m^{-1}}(\hat{x});f\big) - \sum_{\hat{x}\in\mathcal{X}} \Phi_{\hat{x}}\big(f_R^{m^{-1}}(\hat{x});f\big) + \frac{\frac{1}{16}\cdot|\gamma|\cdot\big(\delta(f_L^m,f)+\delta(f_R^m,f)\big)}{\lambda(A) + \mathcal{L}\cdot\lambda(B)}$$
$$< \Phi_{x^*}\big(\wp_a \underset{m}{\times} A;f\big) + \Phi_x\big(\wp_b \underset{m}{\times} B;f\big) - \Phi_{x^*}\big(\wp_b \underset{m}{\times} B;f\big) - \Phi_x\big(\wp_a \underset{m}{\times} A;f\big) + \tfrac{1}{4}\cdot|\gamma|\cdot\lambda(\wp_a)$$
$$< -\tfrac{1}{2}\cdot|\gamma|\cdot\lambda(\wp_a) + \tfrac{1}{4}\cdot|\gamma|\cdot\lambda(\wp_a) \;=\; -\tfrac{1}{4}\cdot|\gamma|\cdot\lambda(\wp_a) \;<\; 0$$

(a) $\Rightarrow$ (e): Consider arbitrary disjoint $\wp_a, \wp_b \subset [0,1]$ with $\lambda(\wp_a)/\lambda(\wp_b) = (<)\,\mathcal{L}$, $x^* \succ x$, and $f(\cdot) \in \mathcal{A}$. By Step 1 of the proof of Theorem 4 in Machina (2004), both limits in (56) exist. For arbitrary $\varepsilon > 0$, event-smoothness and Theorem 0 of Machina (2004) imply some m_ε such that both

$$\begin{aligned}\big|\Phi_{x^*}\big(\wp_b \underset{m}{\times} A;\hat{f}\big) - \Phi_x\big(\wp_b \underset{m}{\times} A;\hat{f}\big) - \lambda(\wp_b)\cdot\big(\Phi_{x^*}(A;\hat{f}) - \Phi_x(A;\hat{f})\big)\big| &< \varepsilon/2\\ \big|\Phi_{x^*}\big(\wp_a \underset{m}{\times} B;\hat{f}\big) - \Phi_x\big(\wp_a \underset{m}{\times} B;\hat{f}\big) - \lambda(\wp_a)\cdot\big(\Phi_{x^*}(B;\hat{f}) - \Phi_x(B;\hat{f})\big)\big| &< \varepsilon/2\end{aligned} \tag{A.33}$$

for all $m > m_\varepsilon$ and all $\hat{f}(\cdot) \in \mathcal{A}$. Event-smoothness (the bound $\Phi_{x^*}(B;\hat{f}) - \Phi_x(B;\hat{f}) > \underline{\Phi}_{x^*,x,B} > 0$ for all $\hat{f}(\cdot)$) and (52) then imply

$$\Phi_{x^*}\big(\wp_b \underset{m}{\times} A;\hat{f}\big) - \Phi_x\big(\wp_b \underset{m}{\times} A;\hat{f}\big) - \Phi_{x^*}\big(\wp_a \underset{m}{\times} B;\hat{f}\big) + \Phi_x\big(\wp_a \underset{m}{\times} B;\hat{f}\big) \;>$$
$$\lambda(\wp_b)\cdot\big(\Phi_{x^*}(A;\hat{f}) - \Phi_x(A;\hat{f})\big) - \lambda(\wp_a)\cdot\big(\Phi_{x^*}(B;\hat{f}) - \Phi_x(B;\hat{f})\big) - \varepsilon \;\geq \tag{A.34}$$
$$\big[\lambda(\wp_b)\cdot\mathcal{L} - \lambda(\wp_a)\big]\cdot\big(\Phi_{x^*}(B;\hat{f}) - \Phi_x(B;\hat{f})\big) - \varepsilon \;\geq\; \big[\lambda(\wp_b)\cdot\mathcal{L} - \lambda(\wp_a)\big]\cdot\underline{\Phi}_{x^*,x,B} - \varepsilon$$

The change sets from $f_R^m(\cdot) = [x^* \text{ on } \wp_a \underset{m}{\times} (A \cup B); x \text{ on } \wp_b \underset{m}{\times} (A \cup B); f(\cdot)$ elsewhere$]$ to $f_L^m(\cdot) = [x^* \text{ on } (\wp_a \cup \wp_b) \underset{m}{\times} A; x \text{ on } (\wp_a \cup \wp_b) \underset{m}{\times} B; f(\cdot) \text{ elsewhere}]$

in (56) are $\Delta E^+_{x^*} = \Delta E^-_x = \wp_b \underset{m}{\times} A$ and $\Delta E^-_{x^*} = \Delta E^+_x = \wp_a \underset{m}{\times} B$. For the almost-objective mixture paths $f^{k,m}_\alpha(\cdot) \equiv [f^m_L(\cdot)$ on $[0,\alpha]\underset{k}{\times}\mathcal{S}$; $f^m_R(\cdot)$ on $(\alpha,1]\underset{k}{\times}\mathcal{S}]$ from $f^m_R(\cdot)$ to $f^m_L(\cdot)$, the Line Integral Approximation Theorem and (A.34) yield

$$W\big(f^m_L(\cdot)\big) - W\big(f^m_R(\cdot)\big) = \lim_{k\to\infty}\int_0^1 \big[\Phi_{x^*}\big(\wp_b \underset{m}{\times} A; f^{k,m}_\alpha\big) + \Phi_x\big(\wp_a \underset{m}{\times} B; f^{k,m}_\alpha\big) \tag{A.35}$$

$$-\Phi_{x^*}\big(\wp_a \underset{m}{\times} B; f^{k,m}_\alpha\big) - \Phi_x\big(\wp_b \underset{m}{\times} A; f^{k,m}_\alpha\big)\big]\cdot d\alpha \ge \big[\lambda(\wp_b)\cdot\mathcal{L} - \lambda(\wp_a)\big]\cdot \underline{\Phi}_{x^*,x,B} - \varepsilon$$

for all $m > m_\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\lim_{m\to\infty} W(f^m_L(\cdot)) - \lim_{m\to\infty} W(f^m_R(\cdot)) \ge [\lambda(\wp_b)\cdot\mathcal{L} - \lambda(\wp_a)]\cdot\underline{\Phi}_{x^*,x,B} = (>)\, 0$.

(e) ⇒ (a): Say (a) failed, so that $\gamma = \Phi_{x^*}(A;f) - \Phi_x(A;f) - \mathcal{L}\cdot[\Phi_{x^*}(B;f) - \Phi_x(B;f)] < 0$ for some $x^* \succ x$ and $f(\cdot)$. By (19) there exists some $\delta^* > 0$ such that $\delta(\hat{f}, f) < \delta^*$ implies

$$\Big| W\big(\hat{f}(\cdot)\big) - W\big(f(\cdot)\big) - \Big[\sum_{\hat{x}\in\mathcal{X}} \Phi_{\hat{x}}\big(\hat{f}^{-1}(\hat{x}); f\big) - \sum_{\hat{x}\in\mathcal{X}} \Phi_{\hat{x}}\big(f^{-1}(\hat{x}); f\big)\Big]\Big| < \frac{\frac{1}{16}\cdot|\gamma|}{(\mathcal{L}+1)\cdot\lambda(A\cup B)}\cdot\delta\big(\hat{f}(\cdot), f(\cdot)\big) \tag{A.36}$$

Select disjoint $\wp_a, \wp_b \subset [0,1]$ such that $\lambda(\wp_a)/\lambda(\wp_b) = \mathcal{L}$ and $\lambda(\wp_b)\cdot(\mathcal{L}+1)\cdot\lambda(A\cup B) < \delta^*/2$. By event-smoothness and Theorem 0 of Machina (2004) there exists m^* such that for all $m > m^*$ we have each of the following three relationships

$$\lambda\big((\wp_a\cup\wp_b)\underset{m}{\times}(A\cup B)\big) < 2\cdot\lambda(\wp_a\cup\wp_b)\cdot\lambda(A\cup B) = 2\cdot\lambda(\wp_b)\cdot(\mathcal{L}+1)\cdot\lambda(A\cup B) < \delta^* \tag{A.37}$$

$$\big|\Phi_{x^*}\big(\wp_b \underset{m}{\times} A; f\big) - \Phi_x\big(\wp_b \underset{m}{\times} A; f\big) - \lambda(\wp_b)\cdot\big(\Phi_{x^*}(A;f) - \Phi_x(A;f)\big)\big| < \tfrac{1}{4}\cdot|\gamma|\cdot\lambda(\wp_b)$$

$$\big|\Phi_{x^*}\big(\wp_a \underset{m}{\times} B; f\big) - \Phi_x\big(\wp_a \underset{m}{\times} B; f\big) - \lambda(\wp_a)\cdot\big(\Phi_{x^*}(B;f) - \Phi_x(B;f)\big)\big| < \tfrac{1}{4}\cdot|\gamma|\cdot\lambda(\wp_b)$$

For all $m > m^*$, the second and third relationships in (A.37) imply

$$\Phi_{x^*}\big(\wp_b \underset{m}{\times} A; f\big) - \Phi_x\big(\wp_b \underset{m}{\times} A; f\big) - \Phi_{x^*}\big(\wp_a \underset{m}{\times} B; f\big) + \Phi_x\big(\wp_a \underset{m}{\times} B; f\big) < \tag{A.38}$$

$$\lambda(\wp_b)\cdot\big(\Phi_{x^*}(A;f) - \Phi_x(A;f)\big) - \lambda(\wp_a)\cdot\big(\Phi_{x^*}(B;f) - \Phi_x(B;f)\big) + \tfrac{1}{2}\cdot|\gamma|\cdot\lambda(\wp_b)$$

$$= \lambda(\wp_b)\cdot\big[\big(\Phi_{x^*}(A;f) - \Phi_x(A;f)\big) - \mathcal{L}\cdot\big(\Phi_{x^*}(B;f) - \Phi_x(B;f)\big)\big] + \tfrac{1}{2}\cdot|\gamma|\cdot\lambda(\wp_b)$$

$$= \lambda(\wp_b)\cdot\gamma + \tfrac{1}{2}\cdot|\gamma|\cdot\lambda(\wp_b) = -\tfrac{1}{2}\cdot|\gamma|\cdot\lambda(\wp_b)$$

For each m, define $f^m_L(\cdot) = [x^*$ on $(\wp_a\cup\wp_b)\underset{m}{\times} A$; x on $(\wp_a\cup\wp_b)\underset{m}{\times} B$; $f(\cdot)$ elsewhere] and $f^m_R(\cdot) = [x^*$ on $\wp_a \underset{m}{\times}(A\cup B)$; x on $\wp_b\underset{m}{\times}(A\cup B)$; $f(\cdot)$ elsewhere], so the change sets from $f^m_R(\cdot)$ to $f^m_L(\cdot)$ are $\Delta E^+_{x^*} = \Delta E^-_x = \wp_b\underset{m}{\times} A$ and $\Delta E^-_{x^*} = \Delta E^+_x = \wp_a \underset{m}{\times} B$. For each $m > m^*$, the first relationship in (A.37) implies $\delta(f^m_L, f)$ and $\delta(f^m_R, f)$ are each less than or equal to $\lambda((\wp_a\cup\wp_b)\underset{m}{\times}(A\cup B)) < 2\cdot\lambda(\wp_b)\cdot(\mathcal{L}+1)\cdot\lambda(A\cup B) < \delta^*$. This yields a contradiction of (e), since $\lambda(\wp_a)/\lambda(\wp_b) = \mathcal{L}$, yet for each $m > m^*$, (A.36) and (A.38) imply

$$W\big(f^m_L(\cdot)\big) - W\big(f^m_R(\cdot)\big) <$$

$$\sum_{\hat{x}\in\mathcal{X}}\Phi_{\hat{x}}\big(f_L^{m^{-1}}(\hat{x});f\big) - \sum_{\hat{x}\in\mathcal{X}}\Phi_{\hat{x}}\big(f_R^{m^{-1}}(\hat{x});f\big) + \frac{\frac{1}{16}\cdot|\gamma|\cdot\big(\delta(f_L^m,f)+\delta(f_R^m,f)\big)}{(\mathcal{L}+1)\cdot\lambda(A\cup B)} \tag{A.39}$$
$$< \Phi_{x^*}\big(\wp_b \underset{m}{\times} A;f\big) + \Phi_x\big(\wp_a \underset{m}{\times} B;f\big) - \Phi_{x^*}\big(\wp_a \underset{m}{\times} B;f\big) - \Phi_x\big(\wp_b \underset{m}{\times} A;f\big) + \tfrac{1}{4}\cdot|\gamma|\cdot\lambda(\wp_b)$$
$$< -\tfrac{1}{2}\cdot|\gamma|\cdot\lambda(\wp_b) + \tfrac{1}{4}\cdot|\gamma|\cdot\lambda(\wp_b) = -\tfrac{1}{4}\cdot|\gamma|\cdot\lambda(\wp_b) < 0$$

Proof of Theorem 5. (a) ⇒ (b): Consider arbitrary $x'' \succ x' \succ x,\ f(\cdot) \in \mathcal{A}$ and disjoint nondegenerate $\wp_a, \wp_b \subset [0,1]$ that satisfy the upper inequality of (58). For each m define $A_m = \wp_a \underset{m}{\times} \mathcal{S},\ B_m = \wp_b \underset{m}{\times} \mathcal{S},\ f_L^m(\cdot) = [x''$ on A_m; x on B_m; $f(\cdot)$ elsewhere$]$ and $f_R^m(\cdot) = [x'$ on $A_m \cup B_m$; $f(\cdot)$ elsewhere$]$, so there exist m^* and $\gamma > 0$ such that $2\cdot\gamma > W^*(f_L^m(\cdot)) - W^*(f_R^m(\cdot)) > \gamma$ for all $m > m^*$. Select $\varepsilon \in (0,1)$ less than $\min\{\lambda(\wp_a), \lambda(\wp_b)\}$, and small enough so that both

$$\left(1 - \frac{1-\varepsilon}{1+\varepsilon}\cdot\frac{\lambda(\wp_a)-\varepsilon}{\lambda(\wp_a)+\varepsilon}\right)\cdot\left[2\cdot\gamma + (1+\varepsilon)\cdot\big(\lambda(\wp_b)+\varepsilon\big)\cdot\overline{\Phi}^*_{x',x,\mathcal{S}}\right] < \gamma/4$$
$$\left(\frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{\lambda(\wp_b)+\varepsilon}{\lambda(\wp_b)-\varepsilon} - 1\right)\cdot(1+\varepsilon)\cdot\big(\lambda(\wp_b)+\varepsilon\big)\cdot\overline{\Phi}^*_{x',x,\mathcal{S}} < \gamma/4 \tag{A.40}$$

Event-smoothness and Machina (2004, Thm. 0) imply the four relationships

$$\big|\Phi_{x''}(A_m;\hat{f}) - \Phi_{x'}(A_m;\hat{f}) - \lambda(\wp_a)\cdot\big[\Phi_{x''}(\mathcal{S};\hat{f}) - \Phi_{x'}(\mathcal{S};\hat{f})\big]\big|$$
$$< \varepsilon\cdot\underline{\Phi}_{x'',x',\mathcal{S}} < \varepsilon\cdot\big[\Phi_{x''}(\mathcal{S};\hat{f}) - \Phi_{x'}(\mathcal{S};\hat{f})\big]$$
$$\big|\Phi_{x'}(B_m;\hat{f}) - \Phi_x(B_m;\hat{f}) - \lambda(\wp_b)\cdot\big[\Phi_{x'}(\mathcal{S};\hat{f}) - \Phi_x(\mathcal{S};\hat{f})\big]\big|$$
$$< \varepsilon\cdot\underline{\Phi}_{x',x,\mathcal{S}} < \varepsilon\cdot\big[\Phi_{x'}(\mathcal{S};\hat{f}) - \Phi_x(\mathcal{S};\hat{f})\big] \tag{A.41}$$
$$\big|\Phi^*_{x''}(A_m;\hat{f}) - \Phi^*_{x'}(A_m;\hat{f}) - \lambda(\wp_a)\cdot\big[\Phi^*_{x''}(\mathcal{S};\hat{f}) - \Phi^*_{x'}(\mathcal{S};\hat{f})\big]\big|$$
$$< \varepsilon\cdot\underline{\Phi}^*_{x'',x',\mathcal{S}} < \varepsilon\cdot\big[\Phi^*_{x''}(\mathcal{S};\hat{f}) - \Phi^*_{x'}(\mathcal{S};\hat{f})\big]$$
$$\big|\Phi^*_{x'}(B_m;\hat{f}) - \Phi^*_x(B_m;\hat{f}) - \lambda(\wp_b)\cdot\big[\Phi^*_{x'}(\mathcal{S};\hat{f}) - \Phi^*_x(\mathcal{S};\hat{f})\big]\big|$$
$$< \varepsilon\cdot\underline{\Phi}^*_{x',x,\mathcal{S}} < \varepsilon\cdot\big[\Phi^*_{x'}(\mathcal{S};\hat{f}) - \Phi^*_x(\mathcal{S};\hat{f})\big]$$

for all m greater than some m^{**} and all $\hat{f}(\cdot) \in \mathcal{A}$, and thus the three relationships

$$\frac{\Phi_{x''}(A_m;\hat{f}) - \Phi_{x'}(A_m;\hat{f})}{\Phi_{x'}(B_m;\hat{f}) - \Phi_x(B_m;\hat{f})} > \frac{\lambda(\wp_a)-\varepsilon}{\lambda(\wp_b)+\varepsilon}\cdot\frac{\Phi_{x''}(\mathcal{S};\hat{f}) - \Phi_{x'}(\mathcal{S};\hat{f})}{\Phi_{x'}(\mathcal{S};\hat{f}) - \Phi_x(\mathcal{S};\hat{f})}$$
$$\frac{\Phi^*_{x''}(A_m;\hat{f}) - \Phi^*_{x'}(A_m;\hat{f})}{\Phi^*_{x'}(B_m;\hat{f}) - \Phi^*_x(B_m;\hat{f})} < \frac{\lambda(\wp_a)+\varepsilon}{\lambda(\wp_b)-\varepsilon}\cdot\frac{\Phi^*_{x''}(\mathcal{S};\hat{f}) - \Phi^*_{x'}(\mathcal{S};\hat{f})}{\Phi^*_{x'}(\mathcal{S};\hat{f}) - \Phi^*_x(\mathcal{S};\hat{f})} \tag{A.42}$$
$$\frac{\Phi^*_{x'}(B_m;\hat{f}) - \Phi^*_x(B_m;\hat{f})}{\Phi_{x'}(B_m;\hat{f}) - \Phi_x(B_m;\hat{f})} < \frac{\lambda(\wp_b)+\varepsilon}{\lambda(\wp_b)-\varepsilon}\cdot\frac{\Phi^*_{x'}(\mathcal{S};\hat{f}) - \Phi^*_x(\mathcal{S};\hat{f})}{\Phi_{x'}(\mathcal{S};\hat{f}) - \Phi_x(\mathcal{S};\hat{f})} < \frac{\lambda(\wp_b)+\varepsilon}{\lambda(\wp_b)-\varepsilon}\cdot\frac{\overline{\Phi}^*_{x',x,\mathcal{S}}}{\underline{\Phi}_{x',x,\mathcal{S}}}$$

for all $m > m^{**}$ and all $\hat{f}(\cdot)$. Select arbitrary $m > \max\{m^*, m^{**}\}$ and define $\gamma_m = W^*(f_L^m(\cdot)) - W^*(f_R^m(\cdot))$, so $2\cdot\gamma > \gamma_m > \gamma$. For each k, define the parametrized family of acts $\{f^k_{\alpha,\beta}(\cdot)|\alpha,\beta \in [0,1]\}$ by

$$f^k_{\alpha,\beta}(\cdot) = \begin{bmatrix} x'' & \text{on } [0,\alpha] \underset{k}{\times} A_m \\ x' & \text{on } (\alpha,1] \underset{k}{\times} A_m \\ x & \text{on } [0,\beta] \underset{k}{\times} B_m \\ x' & \text{on } (\beta,1] \underset{k}{\times} B_m \\ f(\cdot) & \text{elsewhere} \end{bmatrix} \tag{A.43}$$

so that for each k we have $f^k_{1,1}(\cdot) = f^m_L(\cdot)$ and $\delta(f^k_{0,0}, f^m_R) = 0$.
By an argument similar to that in Machina (2004, pp. 39-41), we can select large enough k such that the partial derivatives $\partial W(f^k_{\alpha,\beta}(\cdot))/\partial\alpha$ and $\partial W^*(f^k_{\alpha,\beta}(\cdot))/\partial\alpha$ exist and satisfy

$$\begin{aligned} &\left|\partial W\big(f^k_{\alpha,\beta}(\cdot)\big)/\partial\alpha - \big[\Phi_{x''}\big(A_m; f^k_{\alpha,\beta}\big) - \Phi_{x'}\big(A_m; f^k_{\alpha,\beta}\big)\big]\right| \\ &< \quad \varepsilon\cdot\underline{\Phi}_{x'',x',A_m} \quad < \quad \varepsilon\cdot\big[\Phi_{x''}\big(A_m; f^k_{\alpha,\beta}\big) - \Phi_{x'}\big(A_m; f^k_{\alpha,\beta}\big)\big] \\ &\left|\partial W^*\big(f^k_{\alpha,\beta}(\cdot)\big)/\partial\alpha - \big[\Phi^*_{x''}\big(A_m; f^k_{\alpha,\beta}\big) - \Phi^*_{x'}\big(A_m; f^k_{\alpha,\beta}\big)\big]\right| \\ &< \quad \varepsilon\cdot\underline{\Phi}^*_{x'',x',A_m} \quad < \quad \varepsilon\cdot\big[\Phi^*_{x''}\big(A_m; f^k_{\alpha,\beta}\big) - \Phi^*_{x'}\big(A_m; f^k_{\alpha,\beta}\big)\big] \end{aligned} \tag{A.44}$$

at all but a finite set of values of $\alpha \in [0,1]$ (where these values are independent of β), and the partials $\partial W(f^k_{\alpha,\beta}(\cdot))/\partial\beta$ and $W^*(f^k_{\alpha,\beta}(\cdot))/\partial\beta$ exist and satisfy

$$\begin{aligned} &\left|\partial W\big(f^k_{\alpha,\beta}(\cdot)\big)/\partial\beta - \big[\Phi_{x}\big(B_m; f^k_{\alpha,\beta}\big) - \Phi_{x'}\big(B_m; f^k_{\alpha,\beta}\big)\big]\right| \\ &< \quad \varepsilon\cdot\underline{\Phi}_{x',x,B_m} \quad < \quad \varepsilon\cdot\big[\Phi_{x'}\big(B_m; f^k_{\alpha,\beta}\big) - \Phi_{x}\big(B_m; f^k_{\alpha,\beta}\big)\big] \\ &\left|\partial W^*\big(f^k_{\alpha,\beta}(\cdot)\big)/\partial\beta - \big[\Phi^*_{x}\big(B_m; f^k_{\alpha,\beta}\big) - \Phi^*_{x'}\big(B_m; f^k_{\alpha,\beta}\big)\big]\right| \\ &< \quad \varepsilon\cdot\underline{\Phi}^*_{x',x,B_m} \quad < \quad \varepsilon\cdot\big[\Phi^*_{x'}\big(B_m; f^k_{\alpha,\beta}\big) - \Phi^*_{x}\big(B_m; f^k_{\alpha,\beta}\big)\big] \end{aligned} \tag{A.45}$$

at all but a finite set of values of β (where these values are independent of α). Since $\varepsilon < 1$ this implies $\partial W(f^k_{\alpha,\beta}(\cdot))/\partial\alpha$, $\partial W^*(f^k_{\alpha,\beta}(\cdot))/\partial\alpha > 0$ and $\partial W(f^k_{\alpha,\beta}(\cdot))/\partial\beta$, $\partial W^*(f^k_{\alpha,\beta}(\cdot))/\partial\beta < 0$ except at these values, so $W^*(f^k_{\alpha,\beta}(\cdot))$ and $W(f^k_{\alpha,\beta}(\cdot))$ are strictly increasing in α and strictly decreasing in β.

For each value of $\beta \in [0,1]$, let $\alpha(\beta)$ be the unique solution to $W^*(f^k_{\alpha(\beta),\beta}(\cdot)) = W^*(f^k_{0,0}(\cdot)) + \gamma_m\cdot\beta = W^*(f^m_R(\cdot)) + \gamma_m\cdot\beta$, so the mapping $\alpha(\cdot) : [0,1] \to [0,1]$ is continuous, strictly increasing, onto, and $\alpha'(\beta)$ exists with

$$\alpha'(\beta)\cdot\partial W^*\big(f^k_{\alpha(\beta),\beta}(\cdot)\big)/\partial\alpha + \partial W^*\big(f^k_{\alpha(\beta),\beta}(\cdot)\big)/\partial\beta \;=\; \gamma_m > 0 \tag{A.46}$$

at all but a finite number of values of β. By (A.45) and (A.41) we have

$$\begin{aligned} \partial W^*\big(f^k_{\alpha(\beta),\beta}(\cdot)\big)/\partial\beta \;&>\; -(1+\varepsilon)\cdot\big[\Phi^*_{x'}\big(B_m; f^k_{\alpha(\beta),\beta}\big) - \Phi^*_{x}\big(B_m; f^k_{\alpha(\beta),\beta}\big)\big] \\ &> \quad -(1+\varepsilon)\cdot\big(\lambda(\wp_b)+\varepsilon\big)\cdot\big[\Phi^*_{x'}\big(\mathcal{S}; f^k_{\alpha(\beta),\beta}\big) - \Phi^*_{x}\big(\mathcal{S}; f^k_{\alpha(\beta),\beta}\big)\big] \\ &> \quad -(1+\varepsilon)\cdot\big(\lambda(\wp_b)+\varepsilon\big)\cdot\overline{\Phi}^*_{x',x,\mathcal{S}} \end{aligned} \tag{A.47}$$

so that at all but a finite number of values of β we also have

$$\begin{aligned} \alpha'(\beta)\cdot\partial W^*\big(f^k_{\alpha(\beta),\beta}(\cdot)\big)/\partial\alpha \quad &= \quad \gamma_m - \partial W^*\big(f^k_{\alpha(\beta),\beta}(\cdot)\big)/\partial\beta \\ &< \quad \gamma_m + (1+\varepsilon)\cdot\big(\lambda(\wp_b)+\varepsilon\big)\cdot\overline{\Phi}^*_{x',x,\mathcal{S}} \end{aligned} \tag{A.48}$$

By (A.46), (A.47), (A.48), (A.40) and $2\cdot\gamma > \gamma_m > \gamma$, we have

$$\begin{aligned} &\frac{1-\varepsilon}{1+\varepsilon}\cdot\alpha'(\beta)\cdot\frac{\lambda(\wp_a)-\varepsilon}{\lambda(\wp_a)+\varepsilon}\cdot\frac{\partial W^*\big(f^k_{\alpha(\beta),\beta}\big)}{\partial\alpha} + \frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{\lambda(\wp_b)+\varepsilon}{\lambda(\wp_b)-\varepsilon}\cdot\frac{\partial W^*\big(f^k_{\alpha(\beta),\beta}\big)}{\partial\beta} \\ &= \alpha'(\beta)\cdot\frac{\partial W^*\big(f^k_{\alpha(\beta),\beta}\big)}{\partial\alpha} - \left(1-\frac{1-\varepsilon}{1+\varepsilon}\cdot\frac{\lambda(\wp_a)-\varepsilon}{\lambda(\wp_a)+\varepsilon}\right)\cdot\alpha'(\beta)\cdot\frac{\partial W^*\big(f^k_{\alpha(\beta),\beta}\big)}{\partial\alpha} \\ &\qquad + \frac{\partial W^*\big(f^k_{\alpha(\beta),\beta}\big)}{\partial\beta} + \left(\frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{\lambda(\wp_b)+\varepsilon}{\lambda(\wp_b)-\varepsilon}-1\right)\cdot\frac{\partial W^*\big(f^k_{\alpha(\beta),\beta}\big)}{\partial\beta} \\ &> \gamma_m - \left(1-\frac{1-\varepsilon}{1+\varepsilon}\cdot\frac{\lambda(\wp_a)-\varepsilon}{\lambda(\wp_a)+\varepsilon}\right)\cdot\big[\gamma_m + (1+\varepsilon)\cdot\big(\lambda(\wp_b)+\varepsilon\big)\cdot\overline{\Phi}^*_{x',x,\mathcal{S}}\big] \\ &\qquad - \left(\frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{\lambda(\wp_b)+\varepsilon}{\lambda(\wp_b)-\varepsilon}-1\right)\cdot(1+\varepsilon)\cdot\big(\lambda(\wp_b)+\varepsilon\big)\cdot\overline{\Phi}^*_{x',x,\mathcal{S}} \\ &> \gamma - \left(1-\frac{1-\varepsilon}{1+\varepsilon}\cdot\frac{\lambda(\wp_a)-\varepsilon}{\lambda(\wp_a)+\varepsilon}\right)\cdot\big[2\cdot\gamma + (1+\varepsilon)\cdot\big(\lambda(\wp_b)+\varepsilon\big)\cdot\overline{\Phi}^*_{x',x,\mathcal{S}}\big] - \frac{\gamma}{4} > \frac{\gamma}{2} > 0 \end{aligned} \tag{A.49}$$

The above inequalities and (57) imply that at all but a finite number of values of β, we have

$$\begin{aligned} &\alpha'(\beta)\cdot\partial W\big(f^k_{\alpha(\beta),\beta}(\cdot)\big)/\partial\alpha + \partial W\big(f^k_{\alpha(\beta),\beta}(\cdot)\big)/\partial\beta \\ &> (1-\varepsilon)\cdot\alpha'(\beta)\cdot\big[\Phi_{x''}(A_m;f^k_{\alpha(\beta),\beta}) - \Phi_{x'}(A_m;f^k_{\alpha(\beta),\beta})\big] \\ &\qquad - (1+\varepsilon)\cdot\big[\Phi_{x'}(B_m;f^k_{\alpha(\beta),\beta}) - \Phi_{x}(B_m;f^k_{\alpha(\beta),\beta})\big] \\ &= \frac{(1-\varepsilon)\cdot\alpha'(\beta)\cdot\dfrac{\Phi_{x''}(A_m;f^k_{\alpha(\beta),\beta}) - \Phi_{x'}(A_m;f^k_{\alpha(\beta),\beta})}{\Phi_{x'}(B_m;f^k_{\alpha(\beta),\beta}) - \Phi_{x}(B_m;f^k_{\alpha(\beta),\beta})} - (1+\varepsilon)}{1/[\Phi_{x'}(B_m;f^k_{\alpha(\beta),\beta}) - \Phi_{x}(B_m;f^k_{\alpha(\beta),\beta})]} \\ &> \frac{(1-\varepsilon)\cdot\alpha'(\beta)\cdot\dfrac{\lambda(\wp_a)-\varepsilon}{\lambda(\wp_b)+\varepsilon}\cdot\dfrac{\Phi_{x''}(\mathcal{S};f^k_{\alpha(\beta),\beta}) - \Phi_{x'}(\mathcal{S};f^k_{\alpha(\beta),\beta})}{\Phi_{x'}(\mathcal{S};f^k_{\alpha(\beta),\beta}) - \Phi_{x}(\mathcal{S};f^k_{\alpha(\beta),\beta})} - (1+\varepsilon)}{1/[\Phi_{x'}(B_m;f^k_{\alpha(\beta),\beta}) - \Phi_{x}(B_m;f^k_{\alpha(\beta),\beta})]} \\ &\geq \frac{(1-\varepsilon)\cdot\alpha'(\beta)\cdot\dfrac{\lambda(\wp_a)-\varepsilon}{\lambda(\wp_b)+\varepsilon}\cdot\dfrac{\Phi^*_{x''}(\mathcal{S};f^k_{\alpha(\beta),\beta}) - \Phi^*_{x'}(\mathcal{S};f^k_{\alpha(\beta),\beta})}{\Phi^*_{x'}(\mathcal{S};f^k_{\alpha(\beta),\beta}) - \Phi^*_{x}(\mathcal{S};f^k_{\alpha(\beta),\beta})} - (1+\varepsilon)}{1/[\Phi_{x'}(B_m;f^k_{\alpha(\beta),\beta}) - \Phi_{x}(B_m;f^k_{\alpha(\beta),\beta})]} > \\ &\frac{(1-\varepsilon)\cdot\alpha'(\beta)\cdot\dfrac{\lambda(\wp_a)-\varepsilon}{\lambda(\wp_b)+\varepsilon}\cdot\dfrac{\lambda(\wp_b)-\varepsilon}{\lambda(\wp_a)+\varepsilon}\cdot\dfrac{\Phi^*_{x''}(A_m;f^k_{\alpha(\beta),\beta}) - \Phi^*_{x'}(A_m;f^k_{\alpha(\beta),\beta})}{\Phi^*_{x'}(B_m;f^k_{\alpha(\beta),\beta}) - \Phi^*_{x}(B_m;f^k_{\alpha(\beta),\beta})} - (1+\varepsilon)}{1/[\Phi_{x'}(B_m;f^k_{\alpha(\beta),\beta}) - \Phi_{x}(B_m;f^k_{\alpha(\beta),\beta})]} \end{aligned} \tag{A.50}$$

$$= \frac{(1-\varepsilon)\cdot\alpha'(\beta)\cdot\dfrac{\lambda(\wp_a)-\varepsilon}{\lambda(\wp_b)+\varepsilon}\cdot\dfrac{\lambda(\wp_b)-\varepsilon}{\lambda(\wp_a)+\varepsilon}\cdot[\Phi^*_{x''}(A_m;f^k_{\alpha(\beta),\beta})-\Phi^*_{x'}(A_m;f^k_{\alpha(\beta),\beta})]}{[\Phi^*_{x'}(B_m;f_{\alpha(\beta),\beta})-\Phi^*_{x}(B_m;f^k_{\alpha(\beta),\beta})]/[\Phi_{x'}(B_m;f^k_{\alpha(\beta),\beta})-\Phi_{x}(B_m;f^k_{\alpha(\beta),\beta})]}$$

$$- \frac{(1+\varepsilon)\cdot[\Phi^*_{x'}(B_m;f^k_{\alpha(\beta),\beta})-\Phi^*_{x}(B_m;f^k_{\alpha(\beta),\beta})]}{[\Phi^*_{x'}(B_m;f_{\alpha(\beta),\beta})-\Phi^*_{x}(B_m;f^k_{\alpha(\beta),\beta})]/[\Phi_{x'}(B_m;f^k_{\alpha(\beta),\beta})-\Phi_{x}(B_m;f^k_{\alpha(\beta),\beta})]}$$

$$> \frac{\dfrac{1-\varepsilon}{1+\varepsilon}\cdot\alpha'(\beta)\cdot\dfrac{\lambda(\wp_a)-\varepsilon}{\lambda(\wp_b)+\varepsilon}\cdot\dfrac{\lambda(\wp_b)-\varepsilon}{\lambda(\wp_a)+\varepsilon}\cdot\dfrac{\partial W^*(f^k_{\alpha(\beta),\beta})}{\partial\alpha}+\dfrac{1+\varepsilon}{1-\varepsilon}\cdot\dfrac{\partial W^*(f^k_{\alpha(\beta),\beta})}{\partial\beta}}{[\Phi^*_{x'}(B_m;f_{\alpha(\beta),\beta})-\Phi^*_{x}(B_m;f^k_{\alpha(\beta),\beta})]/[\Phi_{x'}(B_m;f^k_{\alpha(\beta),\beta})-\Phi_{x}(B_m;f^k_{\alpha(\beta),\beta})]}$$

$$= \frac{\dfrac{1-\varepsilon}{1+\varepsilon}\cdot\alpha'(\beta)\cdot\dfrac{\lambda(\wp_a)-\varepsilon}{\lambda(\wp_a)+\varepsilon}\cdot\dfrac{\partial W^*(f^k_{\alpha(\beta),\beta})}{\partial\alpha}+\dfrac{1+\varepsilon}{1-\varepsilon}\cdot\dfrac{\lambda(\wp_b)+\varepsilon}{\lambda(\wp_b)-\varepsilon}\cdot\dfrac{\partial W^*(f^k_{\alpha(\beta),\beta})}{\partial\beta}}{\dfrac{(\lambda(\wp_b)+\varepsilon)\cdot[\Phi^*_{x'}(B_m;f^k_{\alpha(\beta),\beta})\;-\;\Phi^*_{x}(B_m;f^k_{\alpha(\beta),\beta})]}{(\lambda(\wp_b)-\varepsilon)\cdot[\Phi_{x'}(B_m;f^k_{\alpha(\beta),\beta})\;-\;\Phi_{x}(B_m;f^k_{\alpha(\beta),\beta})]}}$$

$$> \quad \frac{\gamma/2}{\big[(\lambda(\wp_b)+\varepsilon)^2\cdot\overline{\Phi}^*_{x',x,\mathcal{S}}\big]/\big[(\lambda(\wp_b)-\varepsilon)^2\cdot\underline{\Phi}_{x',x,\mathcal{S}}\big]} \quad > \quad 0$$

We thus have

$$W\big(f^m_L(\cdot)\big)-W\big(f^m_R(\cdot)\big)=W\big(f^k_{1,1}(\cdot)\big)-W\big(f^k_{0,0}(\cdot)\big)=\int_0^1\frac{dW(f^k_{\alpha(\beta),\beta})}{d\beta}\cdot d\beta=$$
$$\int_0^1\Big[\alpha'(\beta)\cdot\frac{\partial W(f^k_{\alpha(\beta),\beta})}{\partial\alpha}+\frac{\partial W(f^k_{\alpha(\beta),\beta})}{\partial\beta}\Big]\cdot d\beta>\frac{\gamma}{2}\cdot\frac{(\lambda(\wp_b)-\varepsilon)^2\cdot\underline{\Phi}_{x',x,\mathcal{S}}}{(\lambda(\wp_b)+\varepsilon)^2\cdot\overline{\Phi}^*_{x',x,\mathcal{S}}}>0 \tag{A.51}$$

Since m was an arbitrary integer greater than $\max\{m^*,m^{**}\}$, the values $\lim_{m\to\infty}W(f^m_L(\cdot))$ and $\lim_{m\to\infty}W(f^m_R(\cdot))$, which by Machina (2004, Thm.2) both exist, satisfy $\lim_{m\to\infty}W(f^m_L(\cdot))>\lim_{m\to\infty}W(f^m_R(\cdot))$.

(b) ⇒ (a): Say (a) fails, so there exist $x''\succ x'\succ x$, $f(\cdot)$ and $\gamma>0$ with $[\Phi^*_{x''}(\mathcal{S};f)-\Phi^*_{x'}(\mathcal{S};f)]/[\Phi^*_{x'}(\mathcal{S};f)-\Phi^*_{x}(\mathcal{S};f)]>\gamma>[\Phi_{x''}(\mathcal{S};f)-\Phi_{x'}(\mathcal{S};f)]/[\Phi_{x'}(\mathcal{S};f)-\Phi_{x}(\mathcal{S};f)]$. This implies some $\varepsilon>0$ such that both

$$\begin{aligned}\big[\Phi_{x''}(\mathcal{S};f)-\Phi_{x'}(\mathcal{S};f)\big]-\gamma\cdot\big[\Phi_{x'}(\mathcal{S};f)-\Phi_{x}(\mathcal{S};f)\big]&<-\varepsilon\cdot(1+\gamma)\\ \big[\Phi^*_{x''}(\mathcal{S};f)-\Phi^*_{x'}(\mathcal{S};f)\big]-\gamma\cdot\big[\Phi^*_{x'}(\mathcal{S};f)-\Phi^*_{x}(\mathcal{S};f)\big]&>\;\;\varepsilon\cdot(1+\gamma)\end{aligned} \tag{A.52}$$

By (19) there exists some $\delta^*>0$ such that $\delta(\hat f,f)<\delta^*$ implies both

$$\begin{aligned}&\Big|W\big(\hat f(\cdot)\big)-W\big(f(\cdot)\big)-\Big[\sum_{\hat x\in\mathcal{X}}\Phi_{\hat x}\big(\hat f^{-1}(\hat x);f\big)-\sum_{\hat x\in\mathcal{X}}\Phi_{\hat x}\big(f^{-1}(\hat x);f\big)\Big]\Big|\\ &\qquad<\;\big(\tfrac{1}{8}\cdot\varepsilon/\lambda(\mathcal{S})\big)\cdot\delta\big(\hat f(\cdot),f(\cdot)\big)\\ &\Big|W^*\big(\hat f(\cdot)\big)-W^*\big(f(\cdot)\big)-\Big[\sum_{\hat x\in\mathcal{X}}\Phi^*_{\hat x}\big(\hat f^{-1}(\hat x);f\big)-\sum_{\hat x\in\mathcal{X}}\Phi^*_{\hat x}\big(f^{-1}(\hat x);f\big)\Big]\Big|\\ &\qquad<\;\big(\tfrac{1}{8}\cdot\varepsilon/\lambda(\mathcal{S})\big)\cdot\delta\big(\hat f(\cdot),f(\cdot)\big)\end{aligned} \tag{A.53}$$

Define $\tau = \min\{\frac{1}{4}\cdot\delta^*/\lambda(\mathcal{S}),1\}$ and the disjoint intervals $\wp_a = [0,\tau/(1+\gamma)]$ and $\wp_b = (\tau/(1+\gamma),\tau]$, so $\lambda(\wp_b) = \tau\cdot\gamma/(1+\gamma) = \gamma\cdot\lambda(\wp_a)$ and $\lambda(\wp_a \underset{m}{\times} \mathcal{S}) + \lambda(\wp_b \underset{m}{\times} \mathcal{S}) = \lambda(\wp_a)\cdot\lambda(\mathcal{S})+\lambda(\wp_b)\cdot\lambda(\mathcal{S}) = \tau\cdot\lambda(\mathcal{S}) < \delta^*/2$. By event-smoothness and Machina (2004, Thm. 0) there exists m^* such that for all $m > m^*$ we have

$$\begin{aligned}
&\left|\Phi_{x''}\big(\wp_a\underset{m}{\times}\mathcal{S};f\big)-\Phi_{x'}\big(\wp_a\underset{m}{\times}\mathcal{S};f\big)-\lambda(\wp_a)\cdot\big[\Phi_{x''}(\mathcal{S};f)-\Phi_{x'}(\mathcal{S};f)\big]\right| < \tfrac{1}{4}\cdot\tau\cdot\varepsilon\\
&\left|\Phi_{x'}\big(\wp_b\underset{m}{\times}\mathcal{S};f\big)-\Phi_{x}\big(\wp_b\underset{m}{\times}\mathcal{S};f\big)-\lambda(\wp_b)\cdot\big[\Phi_{x'}(\mathcal{S};f)-\Phi_{x}(\mathcal{S};f)\big]\right| < \tfrac{1}{4}\cdot\tau\cdot\varepsilon\\
&\left|\Phi^*_{x''}\big(\wp_a\underset{m}{\times}\mathcal{S};f\big)-\Phi^*_{x'}\big(\wp_a\underset{m}{\times}\mathcal{S};f\big)-\lambda(\wp_a)\cdot\big[\Phi^*_{x''}(\mathcal{S};f)-\Phi^*_{x'}(\mathcal{S};f)\big]\right| < \tfrac{1}{4}\cdot\tau\cdot\varepsilon\\
&\left|\Phi^*_{x'}\big(\wp_b\underset{m}{\times}\mathcal{S};f\big)-\Phi^*_{x}\big(\wp_b\underset{m}{\times}\mathcal{S};f\big)-\lambda(\wp_b)\cdot\big[\Phi^*_{x'}(\mathcal{S};f)-\Phi^*_{x}(\mathcal{S};f)\big]\right| < \tfrac{1}{4}\cdot\tau\cdot\varepsilon
\end{aligned}\tag{A.54}$$

For all $m > m^*$, (A.54) and (A.52) thus imply

$$\begin{aligned}
&\Phi_{x''}\big(\wp_a\underset{m}{\times}\mathcal{S};f\big) - \Phi_{x'}\big(\wp_a\underset{m}{\times}\mathcal{S};f\big) - \Phi_{x'}\big(\wp_b\underset{m}{\times}\mathcal{S};f\big) + \Phi_{x}\big(\wp_b\underset{m}{\times}\mathcal{S};f\big)\\
&< \lambda(\wp_a)\cdot\big[\Phi_{x''}(\mathcal{S};f)-\Phi_{x'}(\mathcal{S};f)\big] - \lambda(\wp_b)\cdot\big[\Phi_{x'}(\mathcal{S};f)-\Phi_{x}(\mathcal{S};f)\big] + \tfrac{1}{2}\cdot\tau\cdot\varepsilon\\
&= \lambda(\wp_a)\cdot\Big[\big[\Phi_{x''}(\mathcal{S};f)-\Phi_{x'}(\mathcal{S};f)\big] - \gamma\cdot\big[\Phi_{x'}(\mathcal{S};f)-\Phi_{x}(\mathcal{S};f)\big]\Big] + \tfrac{1}{2}\cdot\tau\cdot\varepsilon\\
&< -\lambda(\wp_a)\cdot\varepsilon\cdot(1+\gamma) + \tfrac{1}{2}\cdot\tau\cdot\varepsilon = -\tau\cdot\varepsilon + \tfrac{1}{2}\cdot\tau\cdot\varepsilon = -\tfrac{1}{2}\cdot\tau\cdot\varepsilon
\end{aligned}\tag{A.55}$$

and similarly

$$\begin{aligned}
&\Phi^*_{x''}\big(\wp_a\underset{m}{\times}\mathcal{S};f\big) - \Phi^*_{x'}\big(\wp_a\underset{m}{\times}\mathcal{S};f\big) - \Phi^*_{x'}\big(\wp_b\underset{m}{\times}\mathcal{S};f\big) + \Phi^*_{x}\big(\wp_b\underset{m}{\times}\mathcal{S};f\big)\\
&> \lambda(\wp_a)\cdot\big[\Phi^*_{x''}(\mathcal{S};f)-\Phi^*_{x'}(\mathcal{S};f)\big] - \lambda(\wp_b)\cdot\big[\Phi^*_{x'}(\mathcal{S};f)-\Phi^*_{x}(\mathcal{S};f)\big] - \tfrac{1}{2}\cdot\tau\cdot\varepsilon\\
&= \lambda(\wp_a)\cdot\Big[\big[\Phi^*_{x''}(\mathcal{S};f)-\Phi^*_{x'}(\mathcal{S};f)\big] - \gamma\cdot\big[\Phi^*_{x'}(\mathcal{S};f)-\Phi^*_{x}(\mathcal{S};f)\big]\Big] - \tfrac{1}{2}\cdot\tau\cdot\varepsilon\\
&> \lambda(\wp_a)\cdot\varepsilon\cdot(1+\gamma) - \tfrac{1}{2}\cdot\tau\cdot\varepsilon = \tau\cdot\varepsilon - \tfrac{1}{2}\cdot\tau\cdot\varepsilon = \tfrac{1}{2}\cdot\tau\cdot\varepsilon
\end{aligned}\tag{A.55$'$}$$

Define $f_L^m(\cdot) = [x''$ on $\wp_a \underset{m}{\times} \mathcal{S}; x$ on $\wp_b \underset{m}{\times} \mathcal{S}; f(\cdot)$ elsewhere$]$ and $f_R^m(\cdot) = [x'$ on $(\wp_a\cup\wp_b)\underset{m}{\times}\mathcal{S}; f(\cdot)$ elsewhere$]$, so that both $\delta(f_L^m,f) \le \lambda((\wp_a\cup\wp_b)\underset{m}{\times}\mathcal{S}) = \tau\cdot\lambda(\mathcal{S}) < \delta^*/2$ and $\delta(f_R^m,f) \le \lambda((\wp_a\cup\wp_b)\underset{m}{\times}\mathcal{S}) = \tau\cdot\lambda(\mathcal{S}) < \delta^*/2$. By (A.53), (A.54) and (A.55), we have that for all $m > m^*$

$$W\big(f_L^m(\cdot)\big) - W\big(f_R^m(\cdot)\big) < \tag{A.56}$$

$$\sum_{\hat{x}\in\mathcal{X}}\Phi_{\hat{x}}\big(f_L^{m^{-1}}(\hat{x});f\big) - \sum_{\hat{x}\in\mathcal{X}}\Phi_{\hat{x}}\big(f_R^{m^{-1}}(\hat{x});f\big) + \frac{\frac{1}{8}\cdot\varepsilon}{\lambda(\mathcal{S})}\cdot\big[\delta(f_L^m,f)+\delta(f_R^m,f)\big] \le$$

$$\Phi_{x''}\big(\wp_a\underset{m}{\times}\mathcal{S};f\big) - \Phi_{x'}\big(\wp_a\underset{m}{\times}\mathcal{S};f\big) - \Phi_{x'}\big(\wp_b\underset{m}{\times}\mathcal{S};f\big) + \Phi_{x}\big(\wp_b\underset{m}{\times}\mathcal{S};f\big) + \frac{\tau\cdot\varepsilon}{4} < -\frac{\tau\cdot\varepsilon}{4} < 0$$

and by (A.53), (A.54) and (A.55)$'$ we have that for all $m > m^*$

$$W^*\big(f_L^m(\cdot)\big) - W^*\big(f_R^m(\cdot)\big) > \tag{A.56$'$}$$

$$\sum_{\hat{x}\in\mathcal{X}}\Phi^*_{\hat{x}}\big(f_L^{m^{-1}}(\hat{x});f\big) - \sum_{\hat{x}\in\mathcal{X}}\Phi^*_{\hat{x}}\big(f_R^{m^{-1}}(\hat{x});f\big) - \frac{\frac{1}{8}\cdot\varepsilon}{\lambda(\mathcal{S})}\cdot\big[\delta(f_L^m,f)+\delta(f_R^m,f)\big] \ge$$

$$\Phi^*_{x''}\big(\wp_a\underset{m}{\times}\mathcal{S};f\big) - \Phi^*_{x'}\big(\wp_a\underset{m}{\times}\mathcal{S};f\big) - \Phi^*_{x'}\big(\wp_b\underset{m}{\times}\mathcal{S};f\big) + \Phi^*_{x}\big(\wp_b\underset{m}{\times}\mathcal{S};f\big) - \frac{\tau\cdot\varepsilon}{4} > \frac{\tau\cdot\varepsilon}{4} > 0$$

Thus the following four limits – which by Machina (2004, Thm.1) must each exist – will satisfy $\lim_{m\to\infty} W^*(f_L^m(\cdot)) > \lim_{m\to\infty} W^*(f_R^m(\cdot))$, but $\lim_{m\to\infty} W(f_L^m(\cdot)) < \lim_{m\to\infty} W(f_R^m(\cdot))$, which contradicts (b).

(a) ⇔ (c): We establish this equivalence by showing that $W(\cdot)$ is willing to accept small-event $x \leftarrow x' \rightarrow x''$ spreads about $f(\cdot)$ at any odds ratio greater than some $\mathcal{L}$ if and only if $\mathcal{L} \geq [\Phi_{x'}(\mathcal{S};f) - \Phi_x(\mathcal{S};f)]/[\Phi_{x''}(\mathcal{S};f) - \Phi_{x'}(\mathcal{S};f)]$, and similarly for $W^*(\cdot)$. To prove the "if" direction, consider arbitrary $x'' \succ x' \succ x,\ f(\cdot) \in \mathcal{A}$ and $\mathcal{L} \in (0,\infty)$ that satisfy this inequality. Given arbitrary n_a, n_b with $n_a/n_b > \mathcal{L}$ and arbitrary $\varepsilon > 0$, define $\gamma = n_a\cdot[\Phi_{x''}(\mathcal{S};f) - \Phi_{x'}(\mathcal{S};f)] - n_b\cdot[\Phi_{x'}(\mathcal{S};f) - \Phi_x(\mathcal{S};f)] > 0$. By (19) there exists $\delta^* > 0$ such that $\delta(\hat{f},f) < \delta^*$ implies

$$\Big|W\big(\hat{f}(\cdot)\big) - W\big(f(\cdot)\big) - \Big[\sum_{\hat{x}\in\mathcal{X}} \Phi_{\hat{x}}\big(\hat{f}^{-1}(\hat{x});f\big) - \sum_{\hat{x}\in\mathcal{X}} \Phi_{\hat{x}}\big(f^{-1}(\hat{x});f\big)\Big]\Big| < \frac{\frac{1}{4}\cdot\gamma}{\lambda(\mathcal{S})\cdot(n_a+n_b)}\cdot\delta\big(\hat{f}(\cdot),f(\cdot)\big) \tag{A.57}$$

Select $n > \max\{n_a+n_b, \lambda(\mathcal{S})/\varepsilon, (n_a+n_b)\cdot\lambda(\mathcal{S})/\delta^*\}$. By Stromquist and Woodall (1985, Thm. 1) and (21)′, there exists an $\mathcal{E}$-measurable partition $\{E_1,\ldots,E_n\}$ of $\mathcal{S}$ such that for each i, $\lambda(E_i) = \lambda(\mathcal{S})/n < \varepsilon$, $\Phi_{x''}(E_i;f) - \Phi_{x'}(E_i;f) = [\Phi_{x''}(\mathcal{S};f) - \Phi_{x'}(\mathcal{S};f)]/n$ and $\Phi_{x'}(E_i;f) - \Phi_x(E_i;f) = [\Phi_{x'}(\mathcal{S};f) - \Phi_x(\mathcal{S};f)]/n$.

Let A be the union of any n_a of the events $E_1,\ldots,E_n$ and B be the union of any n_b others, so that $\lambda(A) + \lambda(B) = (n_a+n_b)\cdot\lambda(\mathcal{S})/n < \delta^*$, and also

$$\begin{aligned}&\big[\Phi_{x''}(A;f) - \Phi_{x'}(A;f)\big] - \big[\Phi_{x'}(B;f) - \Phi_x(B;f)\big] = \\ &n_a\cdot\big[\Phi_{x''}(\mathcal{S};f) - \Phi_{x'}(\mathcal{S};f)\big]/n - n_b\cdot\big[\Phi_{x'}(\mathcal{S};f) - \Phi_x(\mathcal{S};f)\big]/n = \gamma/n\end{aligned} \tag{A.58}$$

Define $f_L(\cdot) = [x''$ on $A; x$ on $B; f(\cdot)$ elsewhere$]$ and $f_R(\cdot) = [x'$ on $A\cup B$; $f(\cdot)$ elsewhere$]$ so $\delta(f_L,f) \leq \lambda(A)+\lambda(B) < \delta^*$ and $\delta(f_R,f) \leq \lambda(A)+\lambda(B) < \delta^*$. By (A.57) we have

$$\begin{aligned}&W\big(f_L(\cdot)\big) - W\big(f_R(\cdot)\big) > \\ &\Phi_{x''}(A;f) - \Phi_{x'}(A;f) - \Phi_{x'}(B;f) + \Phi_x(B;f) - \frac{\frac{1}{4}\cdot\gamma\cdot[\delta(f_L,f)+\delta(f_R,f)]}{\lambda(\mathcal{S})\cdot(n_a+n_b)} \\ &\geq \gamma/n - \frac{\frac{1}{2}\cdot\gamma\cdot[\lambda(A)+\lambda(B)]}{\lambda(\mathcal{S})\cdot(n_a+n_b)} = \gamma/n - \tfrac{1}{2}\cdot\gamma/n = \tfrac{1}{2}\cdot\gamma/n > 0\end{aligned} \tag{A.59}$$

To prove the "only if" direction, consider arbitrary $x'' \succ x' \succ x,\ f(\cdot)$ and value $\mathcal{L} < [\Phi_{x'}(\mathcal{S};f) - \Phi_x(\mathcal{S};f)]/[\Phi_{x''}(\mathcal{S};f) - \Phi_{x'}(\mathcal{S};f)]$. Select n_a, n_b such that $[\Phi_{x'}(\mathcal{S};f) - \Phi_x(\mathcal{S};f)]/[\Phi_{x''}(\mathcal{S};f) - \Phi_{x'}(\mathcal{S};f)] > n_a/n_b > \mathcal{L}$, and positive η such that $\tau = n_a\cdot[\Phi_{x''}(\mathcal{S};f) - \Phi_{x'}(\mathcal{S};f) + \eta\cdot\lambda(\mathcal{S})] - n_b\cdot[\Phi_{x'}(\mathcal{S};f) - \Phi_x(\mathcal{S};f) - \eta\cdot\lambda(\mathcal{S})] < 0$. By (19) there exists $\delta^{**} > 0$ such that $\delta(\hat{f},f) < \delta^{**}$ implies

$$\Big|W\big(\hat{f}(\cdot)\big)-W\big(f(\cdot)\big)-\Big[\sum_{\hat{x}\in\mathcal{X}}\Phi_{\hat{x}}\big(\hat{f}^{-1}(\hat{x});f\big)-\sum_{\hat{x}\in\mathcal{X}}\Phi_{\hat{x}}\big(f^{-1}(\hat{x});f\big)\Big]\Big| < \tfrac{1}{2}\cdot\eta\cdot\delta\big(\hat{f}(\cdot),f(\cdot)\big) \tag{A.60}$$

Select positive $\varepsilon < \min\{\delta^{**}/(n_a+n_b), \lambda(\mathcal{S})/(n_a+n_b)\}$.

Let $\{E_1,\ldots,E_n\}$ be an arbitrary ε-partition of $\mathcal{S}$, which implies $n > \lambda(\mathcal{S})/\varepsilon > n_a+n_b$. By a standard combinatoric argument, there are $K = n!/(n_a!\cdot n_b!\cdot(n-n_a-n_b)!)$ pairs (A,B) such that A is the union of n_a of the events $E_1,\ldots,E_n$ and B is the union of n_b others. Let $\mathcal{K} = \{(A_k,B_k)\,|\,k=1,\ldots,K\}$ be the family of all such pairs. Since each event E_i will be included in A_k for exactly $(n_a/n)\cdot K$ of the pairs in $\mathcal{K}$, and will be included in B_k for exactly $(n_b/n)\cdot K$ others, we have

$$\begin{aligned}&\sum\nolimits_{k=1}^{K}\big[\Phi_{x''}(A_k;f)-\Phi_{x'}(A_k;f)+\eta\cdot\lambda(A_k)\big]\\ &=\sum\nolimits_{i=1}^{n}(n_a/n)\cdot K\cdot\big[\Phi_{x''}(E_i;f)-\Phi_{x'}(E_i;f)+\eta\cdot\lambda(E_i)\big]\\ &=(n_a/n)\cdot K\cdot\big[\Phi_{x''}(\mathcal{S};f)-\Phi_{x'}(\mathcal{S};f)+\eta\cdot\lambda(\mathcal{S})\big]\end{aligned} \tag{A.61}$$

$$\begin{aligned}&\sum\nolimits_{k=1}^{K}\big[\Phi_{x'}(B_k;f)-\Phi_{x}(B_k;f)-\eta\cdot\lambda(B_k)\big]\\ &=\sum\nolimits_{i=1}^{n}(n_b/n)\cdot K\cdot\big[\Phi_{x'}(E_i;f)-\Phi_{x}(E_i;f)-\eta\cdot\lambda(E_i)\big]\\ &=(n_b/n)\cdot K\cdot\big[\Phi_{x'}(\mathcal{S};f)-\Phi_{x}(\mathcal{S};f)-\eta\cdot\lambda(\mathcal{S})\big]\end{aligned} \tag{A.61$'$}$$

so that

$$\begin{aligned}&\sum_{k=1}^{K}\frac{\Phi_{x''}(A_k;f)-\Phi_{x'}(A_k;f)-\Phi_{x'}(B_k;f)+\Phi_{x}(B_k;f)+\eta\cdot\big(\lambda(A_k)+\lambda(B_k)\big)}{K}\\ &=\quad(n_a/n)\cdot\big[\Phi_{x''}(\mathcal{S};f)-\Phi_{x'}(\mathcal{S};f)+\eta\cdot\lambda(\mathcal{S})\big]\\ &\quad-(n_b/n)\cdot\big[\Phi_{x'}(\mathcal{S};f)-\Phi_{x}(\mathcal{S};f)-\eta\cdot\lambda(\mathcal{S})\big]\quad=\quad\tau/n\end{aligned} \tag{A.62}$$

This implies that at least one pair $(A_{\hat{k}},B_{\hat{k}})$ satisfies

$$\Phi_{x''}(A_{\hat{k}};f)-\Phi_{x'}(A_{\hat{k}};f)-\Phi_{x'}(B_{\hat{k}};f)+\Phi_{x}(B_{\hat{k}};f)+\eta\cdot\big(\lambda(A_{\hat{k}})+\lambda(B_{\hat{k}})\big)\le\frac{\tau}{n} \tag{A.63}$$

Define $f_L(\cdot) = [x''$ on $A_{\hat{k}}; x$ on $B_{\hat{k}}; f(\cdot)$ elsewhere$]$ and $f_R(\cdot) = [x'$ on $A_{\hat{k}}\cup B_{\hat{k}}; f(\cdot)$ elsewhere$]$, so $\delta(f_L,f) \le \lambda(A_{\hat{k}})+\lambda(B_{\hat{k}}) < (n_a+n_b)\cdot\varepsilon < \delta^{**}$ and $\delta(f_R,f)\le\lambda(A_{\hat{k}})+\lambda(B_{\hat{k}}) < (n_a+n_b)\cdot\varepsilon<\delta^{**}$. By (A.60) and (A.63) we have that $W(\cdot)$ is *not* willing to accept the spread from $f_R(\cdot)$ to $f_L(\cdot)$, since

$$\begin{aligned}&W\big(f_L(\cdot)\big)-W\big(f_R(\cdot)\big)\;<\\ &\Phi_{x''}(A_{\hat{k}};f)-\Phi_{x'}(A_{\hat{k}};f)-\Phi_{x'}(B_{\hat{k}};f)+\Phi_{x}(B_{\hat{k}};f)+\tfrac{1}{2}\cdot\eta\cdot\big(\delta(f_L,f)+\delta(f_R,f)\big)\\ &\le\;\tau/n-\eta\cdot\big(\lambda(A_{\hat{k}})+\lambda(B_{\hat{k}})\big)+\tfrac{1}{2}\cdot\eta\cdot2\cdot\big(\lambda(A_{\hat{k}})+\lambda(B_{\hat{k}})\big)\;=\;\tau/n\;<\;0\end{aligned} \tag{A.64}$$

(a) ⇒ (d): Say $V_{W^*}(\hat{\mathbf{P}}) > V_{W^*}(\mathbf{P})$ where $\hat{\mathbf{P}}$ differs from $\mathbf{P}$ by an $x \leftarrow x' \rightarrow x''$ probability spread for $x'' \succ x' \succ x$. We can write $\mathbf{P} = (x,p\,;x',p';x'',p'';$

$x_1, p_1; \ldots; x_n, p_n)$ and $\hat{\mathbf{P}} = (x, \hat{p}; x', \hat{p}'; x'', \hat{p}''; x_1, p_1; \ldots; x_n, p_n)$ for $\hat{p} > p$, $\hat{p}' < p', \hat{p}'' > p''$ and $p + p' + p'' = \hat{p} + \hat{p}' + \hat{p}'' = \overline{p}$. Define the intervals $\wp = [0, p]$, $\wp' = (p, p + p']$, $\wp'' = (p + p', \overline{p}]$, $\hat{\wp} = [0, \hat{p}]$, $\hat{\wp}' = (\hat{p}, \hat{p} + \hat{p}']$, $\hat{\wp}'' = (\hat{p} + \hat{p}', \overline{p}]$, $\wp_1 = (\overline{p}, \overline{p} + p_1]$, $\wp_2 = (\overline{p} + p_1, \overline{p} + p_1 + p_2], \ldots, \wp_n = (\overline{p} + p_1 + \ldots + p_{n-1}, 1]$, and define the almost-objective acts

$$f_m(\cdot) = [x \text{ on } \wp \underset{m}{\times} \mathcal{S}; x' \text{ on } \wp' \underset{m}{\times} \mathcal{S}; x'' \text{ on } \wp'' \underset{m}{\times} \mathcal{S}; x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}] \tag{A.65}$$

$$\hat{f}_m(\cdot) = [x \text{ on } \hat{\wp} \underset{m}{\times} \mathcal{S}; x' \text{ on } \hat{\wp}' \underset{m}{\times} \mathcal{S}; x'' \text{ on } \hat{\wp}'' \underset{m}{\times} \mathcal{S}; x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}]$$

Since $\lim_{m\to\infty} W^*(\hat{f}_m(\cdot)) = V_{W^*}(\hat{\mathbf{P}}) > V_{W^*}(\mathbf{P}) = \lim_{m\to\infty} W^*(f_m(\cdot))$, there exists some m^* and $\gamma > 0$ such that $W^*(\hat{f}_m(\cdot)) - W^*(f_m(\cdot)) > \gamma$ for each $m \geq m^*$. Select $\varepsilon \in (0, 1)$ small enough so that both the following relationships hold

$$\begin{gathered} \left(1 - \frac{1-\varepsilon}{1+\varepsilon}\right) \cdot \left(\frac{\gamma}{\hat{p} - p} + (1+\varepsilon) \cdot \overline{\Phi}^*_{x', x, \mathcal{S}}\right) < \frac{\gamma/4}{\hat{p} - p} \\ \left(\frac{1+\varepsilon}{1-\varepsilon} - 1\right) \cdot (1+\varepsilon) \cdot \overline{\Phi}^*_{x', x, \mathcal{S}} < \frac{\gamma/4}{\hat{p} - p} \end{gathered} \tag{A.66}$$

For each m, define the parametrized family $\{f^m_{\alpha,\beta}(\cdot) \,|\, \alpha \in [p'', \hat{p}''], \beta \in [p, \hat{p}]\}$ by

$$\begin{gathered} f^m_{\alpha,\beta}(\cdot) = \big[x \text{ on } [0, \beta] \underset{m}{\times} \mathcal{S}; x' \text{ on } (\beta, \bar{p} - \alpha] \underset{m}{\times} \mathcal{S}; x'' \text{ on } (\bar{p} - \alpha, \bar{p}] \underset{m}{\times} \mathcal{S}; \\ x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\big] \end{gathered} \tag{A.67}$$

so that for each m we have $f^m_{p'', p}(\cdot) = f_m(\cdot)$ and $f^m_{\hat{p}'', \hat{p}}(\cdot) = \hat{f}_m(\cdot)$.

By an argument similar to that in Machina (2004, pp. 39-41) we can select large enough m^{**} such that for any $m \geq m^{**}$, the partial derivatives $\partial W(f^m_{\alpha,\beta}(\cdot))/\partial\alpha$ and $\partial W^*(f^m_{\alpha,\beta}(\cdot))/\partial\alpha$ exist and satisfy

$$\begin{aligned} &\left|\partial W\big(f^m_{\alpha,\beta}(\cdot)\big)/\partial\alpha - \big[\Phi_{x''}(\mathcal{S}; f^m_{\alpha,\beta}) - \Phi_{x'}(\mathcal{S}; f^m_{\alpha,\beta})\big]\right| \\ &\qquad < \varepsilon \cdot \underline{\Phi}_{x'', x', \mathcal{S}} < \varepsilon \cdot \big[\Phi_{x''}(\mathcal{S}; f^m_{\alpha,\beta}) - \Phi_{x'}(\mathcal{S}; f^m_{\alpha,\beta})\big] \\ &\left|\partial W^*\big(f^m_{\alpha,\beta}(\cdot)\big)/\partial\alpha - \big[\Phi^*_{x''}(\mathcal{S}; f^m_{\alpha,\beta}) - \Phi^*_{x'}(\mathcal{S}; f^m_{\alpha,\beta})\big]\right| \\ &\qquad < \varepsilon \cdot \underline{\Phi}^*_{x'', x', \mathcal{S}} < \varepsilon \cdot \big[\Phi^*_{x''}(\mathcal{S}; f^m_{\alpha,\beta}) - \Phi^*_{x'}(\mathcal{S}; f^m_{\alpha,\beta})\big] \end{aligned} \tag{A.68}$$

at all but a finite set of values of α (where these values are independent of β), and the partials $\partial W(f^m_{\alpha,\beta}(\cdot))/\partial\beta$ and $W^*(f^m_{\alpha,\beta}(\cdot))/\partial\beta$ exist and satisfy

$$\begin{aligned} &\left|\partial W\big(f^m_{\alpha,\beta}(\cdot)\big)/\partial\beta - \big[\Phi_{x}(\mathcal{S}; f^m_{\alpha,\beta}) - \Phi_{x'}(\mathcal{S}; f^m_{\alpha,\beta})\big]\right| \\ &\qquad < \varepsilon \cdot \underline{\Phi}_{x', x, \mathcal{S}} < \varepsilon \cdot \big[\Phi_{x'}(\mathcal{S}; f^m_{\alpha,\beta}) - \Phi_{x}(\mathcal{S}; f^m_{\alpha,\beta})\big] \\ &\left|\partial W^*\big(f^m_{\alpha,\beta}(\cdot)\big)/\partial\beta - \big[\Phi^*_{x}(\mathcal{S}; f^m_{\alpha,\beta}) - \Phi^*_{x'}(\mathcal{S}; f^m_{\alpha,\beta})\big]\right| \\ &\qquad < \varepsilon \cdot \underline{\Phi}^*_{x', x, \mathcal{S}} < \varepsilon \cdot \big[\Phi^*_{x'}(\mathcal{S}; f^m_{\alpha,\beta}) - \Phi^*_{x}(\mathcal{S}; f^m_{\alpha,\beta})\big] \end{aligned} \tag{A.69}$$

at all but a finite set of values of β (where these values are independent of α). Since $\varepsilon < 1$ this implies $\partial W(f^m_{\alpha,\beta}(\cdot))/\partial\alpha$ and $\partial W^*(f^m_{\alpha,\beta}(\cdot))/\partial\alpha$ are both positive and $\partial W(f^m_{\alpha,\beta}(\cdot))/\partial\beta$ and $\partial W^*(f^m_{\alpha,\beta}(\cdot))/\partial\beta$ are both negative except at these values, so $W^*(f^m_{\alpha,\beta}(\cdot))$ and $W(f^m_{\alpha,\beta}(\cdot))$ are each strictly increasing in α and strictly decreasing in β.

Select arbitrary $m > \max\{m^*, m^{**}\}$ and define $\gamma_m = W^*(\hat{f}_m(\cdot)) - W^*(f_m(\cdot)) > \gamma$. For each $\beta \in [p, \hat{p}]$, define $\alpha(\beta)$ as the unique solution to $W^*(f^m_{\alpha(\beta),\beta}(\cdot)) = W^*(f_m(\cdot)) + \gamma_m \cdot (\beta - p)/(\hat{p} - p)$, so the mapping $\alpha(\cdot)\colon [p, \hat{p}] \to [p'', \hat{p}'']$ is continuous, strictly increasing, onto, and $\alpha'(\beta)$ exists and satisfies

$$\alpha'(\beta)\cdot\partial W^*\big(f^m_{\alpha(\beta),\beta}(\cdot)\big)/\partial\alpha \;+\; \partial W^*\big(f^m_{\alpha(\beta),\beta}(\cdot)\big)/\partial\beta \;=\; \frac{\gamma_m}{\hat{p} - p} \;>\; 0 \qquad \text{(A.70)}$$

at all but a finite number of values of β. By (A.69) we have

$$\begin{aligned} \partial W^*\big(f^m_{\alpha(\beta),\beta}(\cdot)\big)/\partial\beta \;&>\; -(1+\varepsilon)\cdot\big[\Phi^*_{x'}(\mathcal{S}; f^m_{\alpha(\beta),\beta}) - \Phi^*_x(\mathcal{S}; f^m_{\alpha(\beta),\beta})\big] \\ &>\; -(1+\varepsilon)\cdot\overline{\Phi}^*_{x',x,\mathcal{S}} \end{aligned} \qquad \text{(A.71)}$$

and thus by (A.70), that

$$\begin{aligned} \alpha'(\beta)\cdot\partial W^*\big(f^m_{\alpha(\beta),\beta}(\cdot)\big)/\partial\alpha \;&=\; \gamma_m/(\hat{p} - p) \;-\; \partial W^*\big(f^m_{\alpha(\beta),\beta}(\cdot)\big)/\partial\beta \\ &<\; \gamma_m/(\hat{p} - p) \;+\; (1+\varepsilon)\cdot\overline{\Phi}^*_{x',x,\mathcal{S}} \end{aligned} \qquad \text{(A.72)}$$

at all but a finite number of values of β. By (A.70), (A.71), (A.72), (A.66) and $\gamma_m > \gamma$, we have

$$\begin{aligned} &\frac{1-\varepsilon}{1+\varepsilon}\cdot\alpha'(\beta)\cdot\partial W^*(f^m_{\alpha(\beta),\beta})/\partial\alpha \;+\; \frac{1+\varepsilon}{1-\varepsilon}\cdot\partial W^*(f^m_{\alpha(\beta),\beta})/\partial\beta \\ &\quad=\; \alpha'(\beta)\cdot\partial W^*(f^m_{\alpha(\beta),\beta})/\partial\alpha \;+\; \partial W^*(f^m_{\alpha(\beta),\beta})/\partial\beta \qquad \text{(A.73)} \\ &-\Big(1 - \frac{1-\varepsilon}{1+\varepsilon}\Big)\cdot\alpha'(\beta)\cdot\partial W^*(f^m_{\alpha(\beta),\beta})/\partial\alpha + \Big(\frac{1+\varepsilon}{1-\varepsilon} - 1\Big)\cdot\partial W^*(f^m_{\alpha(\beta),\beta})/\partial\beta \\ &> \frac{\gamma_m}{\hat{p} - p} - \Big(1 - \frac{1-\varepsilon}{1+\varepsilon}\Big)\cdot\Big(\frac{\gamma_m}{\hat{p} - p} + (1+\varepsilon)\cdot\overline{\Phi}^*_{x',x,\mathcal{S}}\Big) - \Big(\frac{1+\varepsilon}{1-\varepsilon} - 1\Big)\cdot(1+\varepsilon)\cdot\overline{\Phi}^*_{x',x,\mathcal{S}} \\ &> \frac{\gamma}{\hat{p} - p} - \Big(1 - \frac{1-\varepsilon}{1+\varepsilon}\Big)\cdot\Big(\frac{\gamma}{\hat{p} - p} + (1+\varepsilon)\cdot\overline{\Phi}^*_{x',x,\mathcal{S}}\Big) - \frac{\gamma/4}{\hat{p} - p} \;>\; \frac{\gamma/2}{\hat{p} - p} \;>\; 0 \end{aligned}$$

The above inequalities and (57) imply that at all but a finite number of values of β, we have:

$$\begin{aligned} &\alpha'(\beta)\cdot\partial W\big(f^m_{\alpha(\beta),\beta}(\cdot)\big)/\partial\alpha + \partial W\big(f^m_{\alpha(\beta),\beta}(\cdot)\big)/\partial\beta \\ &>\; (1-\varepsilon)\cdot\alpha'(\beta)\cdot\big[\Phi_{x''}(\mathcal{S}; f^m_{\alpha(\beta),\beta}) - \Phi_{x'}(\mathcal{S}; f^m_{\alpha(\beta),\beta})\big] \\ &\qquad -(1+\varepsilon)\cdot\big[\Phi_{x'}(\mathcal{S}; f^m_{\alpha(\beta),\beta}) - \Phi_x(\mathcal{S}; f^m_{\alpha(\beta),\beta})\big] \\ &=\; \big[\Phi_{x''}(\mathcal{S}; f^m_{\alpha(\beta),\beta}) - \Phi_{x'}(\mathcal{S}; f^m_{\alpha(\beta),\beta})\big] \\ &\qquad \cdot\Big[(1-\varepsilon)\cdot\alpha'(\beta) - (1+\varepsilon)\cdot\frac{\Phi_{x'}(\mathcal{S}; f^m_{\alpha(\beta),\beta}) - \Phi_x(\mathcal{S}; f^m_{\alpha(\beta),\beta})}{\Phi_{x''}(\mathcal{S}; f^m_{\alpha(\beta),\beta}) - \Phi_{x'}(\mathcal{S}; f^m_{\alpha(\beta),\beta})}\Big] \\ &\geq\; \big[\Phi_{x''}(\mathcal{S}; f^m_{\alpha(\beta),\beta}) - \Phi_{x'}(\mathcal{S}; f^m_{\alpha(\beta),\beta})\big] \qquad \text{(A.74)} \\ &\qquad \cdot\Big[(1-\varepsilon)\cdot\alpha'(\beta) - (1+\varepsilon)\cdot\frac{\Phi^*_{x'}(\mathcal{S}; f^m_{\alpha(\beta),\beta}) - \Phi^*_x(\mathcal{S}; f^m_{\alpha(\beta),\beta})}{\Phi^*_{x''}(\mathcal{S}; f^m_{\alpha(\beta),\beta}) - \Phi^*_{x'}(\mathcal{S}; f^m_{\alpha(\beta),\beta})}\Big] \end{aligned}$$

$$= \frac{(1-\varepsilon)\cdot\alpha'(\beta)\cdot[\Phi^*_{x''}(\mathcal{S};f^m_{\alpha(\beta),\beta}) - \Phi^*_{x'}(\mathcal{S};f^m_{\alpha(\beta),\beta})]}{\big[\Phi^*_{x''}(\mathcal{S};f^m_{\alpha(\beta),\beta}) - \Phi^*_{x'}(\mathcal{S};f^m_{\alpha(\beta),\beta})\big]/\big[\Phi_{x''}(\mathcal{S};f^m_{\alpha(\beta),\beta}) - \Phi_{x'}(\mathcal{S};f^m_{\alpha(\beta),\beta})\big]}$$
$$- \frac{(1+\varepsilon)\cdot[\Phi^*_{x'}(\mathcal{S};f^m_{\alpha(\beta),\beta}) - \Phi^*_{x}(\mathcal{S};f^m_{\alpha(\beta),\beta})]}{\big[\Phi^*_{x''}(\mathcal{S};f^m_{\alpha(\beta),\beta}) - \Phi^*_{x'}(\mathcal{S};f^m_{\alpha(\beta),\beta})\big]/\big[\Phi_{x''}(\mathcal{S};f^m_{\alpha(\beta),\beta}) - \Phi_{x'}(\mathcal{S};f^m_{\alpha(\beta),\beta})\big]}$$

$$> \frac{\dfrac{1-\varepsilon}{1+\varepsilon}\cdot\alpha'(\beta)\cdot\dfrac{\partial W^*(f^m_{\alpha(\beta),\beta})}{\partial\alpha} + \dfrac{1+\varepsilon}{1-\varepsilon}\cdot\dfrac{\partial W^*(f^m_{\alpha(\beta),\beta})}{\partial\beta}}{\dfrac{\Phi^*_{x''}(\mathcal{S};f^m_{\alpha(\beta),\beta}) - \Phi^*_{x'}(\mathcal{S};f^m_{\alpha(\beta),\beta})}{\Phi_{x''}(\mathcal{S};f^m_{\alpha(\beta),\beta}) - \Phi_{x'}(\mathcal{S};f^m_{\alpha(\beta),\beta})}} > \frac{\gamma/2}{\hat{p}-p}\cdot\frac{\underline{\Phi}_{x'',x',\mathcal{S}}}{\overline{\Phi}^*_{x'',x',\mathcal{S}}}$$

We thus have

$$W\big(\hat{f}_m(\cdot)\big) - W\big(f_m(\cdot)\big) \;=\; W\big(f^m_{\alpha(\hat{p}),\hat{p}}(\cdot)\big) - W\big(f^m_{\alpha(p),p}(\cdot)\big) \;=$$
$$\int_p^{\hat{p}} \frac{dW(f^m_{\alpha(\beta),\beta})}{d\beta}\cdot d\beta \;=\; \int_p^{\hat{p}}\Big[\alpha'(\beta)\cdot\frac{\partial W(f^m_{\alpha(\beta),\beta})}{\partial\alpha} + \frac{\partial W(f^m_{\alpha(\beta),\beta})}{\partial\beta}\Big]\cdot d\beta$$
$$> \;(\gamma/2)\cdot\big(\underline{\Phi}_{x'',x',\mathcal{S}}/\overline{\Phi}^*_{x'',x',\mathcal{S}}\big) \;>\; 0 \qquad \text{(A.75)}$$

Since m was an arbitrary integer greater than $\max\{m^*, m^{**}\}$ we have $\lim_{m\to\infty} W(\hat{f}_m(\cdot)) > \lim_{m\to\infty} W(f_m(\cdot))$, and hence that $V_W(\hat{\mathbf{P}}) > V_W(\mathbf{P})$.

(a) ⇒ (e): Given events A, B satisfying the stated likelihood properties, arbitrary $x'' \succ x' \succ x$ and arbitrary $f(\cdot)$, define $f_L(\cdot) = [x''$ on A; x on B; $f(\cdot)$ elsewhere$]$, $f_R(\cdot) = [x'$ on $A \cup B$; $f(\cdot)$ elsewhere$]$ and $\gamma = W^*(f_L(\cdot)) - W^*(f_R(\cdot)) > 0$. Select positive $\varepsilon < \min\{1/4, \gamma/(16\cdot\overline{\Phi}^*_{x',x,B})\}$, so that $2\cdot\varepsilon/(1+\varepsilon) < 1/2$ and $(4\cdot\varepsilon/(1-\varepsilon))\cdot\overline{\Phi}^*_{x',x,B} < 8\cdot\varepsilon\cdot\overline{\Phi}^*_{x',x,B} < \gamma/2$.

For each m, define the parametrized family of acts $\{f^m_{\alpha,\beta}(\cdot)\,|\,\alpha,\beta\in[0,1]\}$ by

$$f^m_{\alpha,\beta}(\cdot) \equiv \big[x'' \text{ on } [0,\alpha]\underset{m}{\times}A;\; x' \text{ on } (\alpha,1]\underset{m}{\times}A;\; x \text{ on } [0,\beta]\underset{m}{\times}B;\; x' \text{ on } (\beta,1]\underset{m}{\times}B;\; f(\cdot) \text{ elsewhere}\big] \qquad \text{(A.76)}$$

so that $f^m_{1,1}(\cdot) = f_L(\cdot)$ and $\delta(f^m_{0,0}, f_R) = 0$ for each m. The likelihood properties of A and B imply both $[\Phi^*_{x''}(A;f^m_{\alpha,\beta}) - \Phi^*_{x'}(A;f^m_{\alpha,\beta})]/[\Phi^*_{x''}(\mathcal{S};f^m_{\alpha,\beta}) - \Phi^*_{x'}(\mathcal{S};f^m_{\alpha,\beta})] \le p_a \le [\Phi_{x''}(A;f^m_{\alpha,\beta}) - \Phi_{x'}(A;f^m_{\alpha,\beta})]/[\Phi_{x''}(\mathcal{S};f^m_{\alpha,\beta}) - \Phi_{x'}(\mathcal{S};f^m_{\alpha,\beta})]$ and $[\Phi^*_{x'}(B;f^m_{\alpha,\beta}) - \Phi^*_{x}(B;f^m_{\alpha,\beta})]/[\Phi^*_{x'}(\mathcal{S};f^m_{\alpha,\beta}) - \Phi^*_{x}(\mathcal{S};f^m_{\alpha,\beta})] \ge p_b \ge [\Phi_{x'}(B;f^m_{\alpha,\beta}) - \Phi_{x}(B;f^m_{\alpha,\beta})]/[\Phi_{x'}(\mathcal{S};f^m_{\alpha,\beta}) - \Phi_{x}(\mathcal{S};f^m_{\alpha,\beta})]$ for all m and all α, β.

By an argument similar to that in Machina (2004, pp. 39-41) we can select a large enough m such that the partial derivatives $\partial W(f^m_{\alpha,\beta}(\cdot))/\partial\alpha$ and $\partial W^*(f^m_{\alpha,\beta}(\cdot))/\partial\alpha$ exist and satisfy

$$\Big|\partial W\big(f^m_{\alpha,\beta}(\cdot)\big)/\partial\alpha - \big[\Phi_{x''}(A;f^m_{\alpha,\beta}) - \Phi_{x'}(A;f^m_{\alpha,\beta})\big]\Big|$$
$$< \quad \varepsilon\cdot\underline{\Phi}_{x'',x',A} \quad < \quad \varepsilon\cdot\big[\Phi_{x''}(A;f^m_{\alpha,\beta}) - \Phi_{x'}(A;f^m_{\alpha,\beta})\big] \qquad \text{(A.77)}$$
$$\Big|\partial W^*\big(f^m_{\alpha,\beta}(\cdot)\big)/\partial\alpha - \big[\Phi^*_{x''}(A;f^m_{\alpha,\beta}) - \Phi^*_{x'}(A;f^m_{\alpha,\beta})\big]\Big|$$

$$< \quad \varepsilon \cdot \underline{\Phi}^{*}_{x'',x',A} \quad < \quad \varepsilon \cdot \big[\Phi^{*}_{x''}(A; f^{m}_{\alpha,\beta}) - \Phi^{*}_{x'}(A; f^{m}_{\alpha,\beta})\big]$$

at all but a finite number of values of α (where these values are independent of β), and the partials $\partial W(f^{m}_{\alpha,\beta}(\cdot))/\partial\beta$ and $\partial W^{*}(f^{m}_{\alpha,\beta}(\cdot))/\partial\beta$ exist and satisfy

$$\begin{aligned} &\Big|\partial W\big(f^{m}_{\alpha,\beta}(\cdot)\big)/\partial\beta - \big[\Phi_{x}(B; f^{m}_{\alpha,\beta}) - \Phi_{x'}(B; f^{m}_{\alpha,\beta})\big]\Big| \\ &\quad < \quad \varepsilon \cdot \underline{\Phi}_{x',x,B} \quad < \quad \varepsilon \cdot \big[\Phi_{x'}(B; f^{m}_{\alpha,\beta}) - \Phi_{x}(B; f^{m}_{\alpha,\beta})\big] \\ &\Big|\partial W^{*}\big(f^{m}_{\alpha,\beta}(\cdot)\big)/\partial\beta - \big[\Phi^{*}_{x}(B; f^{m}_{\alpha,\beta}) - \Phi^{*}_{x'}(B; f^{m}_{\alpha,\beta})\big]\Big| \\ &\quad < \quad \varepsilon \cdot \underline{\Phi}^{*}_{x',x,B} \quad < \quad \varepsilon \cdot \big[\Phi^{*}_{x'}(B; f^{m}_{\alpha,\beta}) - \Phi^{*}_{x}(B; f^{m}_{\alpha,\beta})\big] \end{aligned} \tag{A.78}$$

at all but a finite number of values of β (where these values are independent of α). Since $\varepsilon < 1/4$, this implies that $\partial W(f^{m}_{\alpha,\beta}(\cdot))/\partial\alpha$ and $\partial W^{*}(f^{m}_{\alpha,\beta}(\cdot))/\partial\alpha$ are each positive and $\partial W(f^{m}_{\alpha,\beta}(\cdot))/\partial\beta$ and $\partial W^{*}(f^{m}_{\alpha,\beta}(\cdot))/\partial\beta$ are each negative except at these values, so that $W^{*}(f^{m}_{\alpha,\beta}(\cdot))$ and $W(f^{m}_{\alpha,\beta}(\cdot))$ are each strictly increasing in α and strictly decreasing in β.

For each $\beta \in [0,1]$, let $\alpha(\beta)$ be the unique solution to $W^{*}(f^{m}_{\alpha(\beta),\beta}(\cdot)) = W^{*}(f^{m}_{0,0}(\cdot)) + \gamma\cdot\beta$, so the mapping $\alpha(\cdot) : [0,1] \to [0,1]$ is continuous, strictly increasing, onto, and $\alpha'(\beta)$ exists with

$$\alpha'(\beta)\cdot\partial W^{*}\big(f^{m}_{\alpha(\beta),\beta}(\cdot)\big)/\partial\alpha + \partial W^{*}\big(f^{m}_{\alpha(\beta),\beta}(\cdot)\big)/\partial\beta \;=\; \gamma \;>\; 0 \tag{A.79}$$

at all but a finite number of values of β. Since (A.78) implies $\partial W^{*}(f^{m}_{\alpha(\beta),\beta}(\cdot))/\partial\beta > -(1+\varepsilon)\cdot[\Phi^{*}_{x'}(B; f^{m}_{\alpha(\beta),\beta}) - \Phi^{*}_{x}(B; f^{m}_{\alpha(\beta),\beta})] > -(1+\varepsilon)\cdot\overline{\Phi}^{*}_{x',x,B}$, we also have $\alpha'(\beta)\cdot\partial W^{*}(f^{m}_{\alpha(\beta),\beta}(\cdot))/\partial\alpha = \gamma - \partial W^{*}(f^{m}_{\alpha(\beta),\beta}(\cdot))/\partial\beta < \gamma + (1+\varepsilon)\cdot\overline{\Phi}^{*}_{x',x,B}$ at all but a finite number of values of β.

The above inequalities imply that at all but a finite number of values of β, we have

$$\begin{aligned} &\frac{1-\varepsilon}{1+\varepsilon}\cdot\alpha'(\beta)\cdot\partial W^{*}\big(f^{m}_{\alpha(\beta),\beta}(\cdot)\big)/\partial\alpha + \frac{1+\varepsilon}{1-\varepsilon}\cdot\partial W^{*}\big(f^{m}_{\alpha(\beta),\beta}(\cdot)\big)/\partial\beta \\ &\quad = \alpha'(\beta)\cdot\partial W^{*}\big(f^{m}_{\alpha(\beta),\beta}(\cdot)\big)/\partial\alpha + \partial W^{*}\big(f^{m}_{\alpha(\beta),\beta}(\cdot)\big)/\partial\beta \\ &- \frac{2\cdot\varepsilon}{1+\varepsilon}\cdot\alpha'(\beta)\cdot\partial W^{*}\big(f^{m}_{\alpha(\beta),\beta}(\cdot)\big)/\partial\alpha + \frac{2\cdot\varepsilon}{1-\varepsilon}\cdot\partial W^{*}\big(f^{m}_{\alpha(\beta),\beta}(\cdot)\big)/\partial\beta \\ &\quad > \gamma - \frac{2\cdot\varepsilon}{1+\varepsilon}\cdot\big[\gamma + (1+\varepsilon)\cdot\overline{\Phi}^{*}_{x',x,B}\big] - \frac{2\cdot\varepsilon}{1-\varepsilon}\cdot(1+\varepsilon)\cdot\overline{\Phi}^{*}_{x',x,B} \\ &\quad = \gamma - \frac{2\cdot\varepsilon}{1+\varepsilon}\cdot\gamma - 2\cdot\varepsilon\cdot\overline{\Phi}^{*}_{x',x,B} - 2\cdot\varepsilon\cdot\frac{1+\varepsilon}{1-\varepsilon}\cdot\overline{\Phi}^{*}_{x',x,B} \\ &\quad = \gamma - \frac{2\cdot\varepsilon}{1+\varepsilon}\cdot\gamma - \frac{4\cdot\varepsilon}{1-\varepsilon}\cdot\overline{\Phi}^{*}_{x',x,B} > \gamma - \gamma/2 - \gamma/2 = 0 \end{aligned} \tag{A.80}$$

The above inequalities, the likelihood properties of A and B, and (57) then imply that at all but a finite number of values of β, we have

$$\begin{aligned} &\alpha'(\beta)\cdot\partial W\big(f^{m}_{\alpha(\beta),\beta}(\cdot)\big)/\partial\alpha + \partial W\big(f^{m}_{\alpha(\beta),\beta}(\cdot)\big)/\partial\beta \\ &> (1-\varepsilon)\cdot\alpha'(\beta)\cdot[\Phi_{x''}(A; f^{m}_{\alpha(\beta),\beta}) - \Phi_{x'}(A; f^{m}_{\alpha(\beta),\beta})] \\ &\quad -(1+\varepsilon)\cdot[\Phi_{x'}(B; f^{m}_{\alpha(\beta),\beta}) - \Phi_{x}(B; f^{m}_{\alpha(\beta),\beta})] \end{aligned}$$

$$= \frac{(1-\varepsilon)\cdot\alpha'(\beta)\cdot\dfrac{\Phi_{x''}(A;f^m_{\alpha(\beta),\beta})-\Phi_{x'}(A;f^m_{\alpha(\beta),\beta})}{\Phi_{x'}(B;f^m_{\alpha(\beta),\beta})-\Phi_{x}(B;f^m_{\alpha(\beta),\beta})}-(1+\varepsilon)}{1/[\Phi_{x'}(B;f^m_{\alpha(\beta),\beta})-\Phi_{x}(B;f^m_{\alpha(\beta),\beta})]}$$

$$\geq \frac{(1-\varepsilon)\cdot\alpha'(\beta)\cdot\dfrac{p_a}{p_b}\cdot\dfrac{\Phi_{x''}(\mathcal{S};f^m_{\alpha(\beta),\beta})-\Phi_{x'}(\mathcal{S};f^m_{\alpha(\beta),\beta})}{\Phi_{x'}(\mathcal{S};f^m_{\alpha(\beta),\beta})-\Phi_{x}(\mathcal{S};f^m_{\alpha(\beta),\beta})}-(1+\varepsilon)}{1/[\Phi_{x'}(B;f^m_{\alpha(\beta),\beta})-\Phi_{x}(B;f^m_{\alpha(\beta),\beta})]}$$

$$\geq \frac{(1-\varepsilon)\cdot\alpha'(\beta)\cdot\dfrac{p_a}{p_b}\cdot\dfrac{\Phi^*_{x''}(\mathcal{S};f^m_{\alpha(\beta),\beta})-\Phi^*_{x'}(\mathcal{S};f^m_{\alpha(\beta),\beta})}{\Phi^*_{x'}(\mathcal{S};f^m_{\alpha(\beta),\beta})-\Phi^*_{x}(\mathcal{S};f^m_{\alpha(\beta),\beta})}-(1+\varepsilon)}{1/[\Phi_{x'}(B;f^m_{\alpha(\beta),\beta})-\Phi_{x}(B;f^m_{\alpha(\beta),\beta})]}$$

$$\geq \frac{(1-\varepsilon)\cdot\alpha'(\beta)\cdot\dfrac{\Phi^*_{x''}(A;f^m_{\alpha(\beta),\beta})-\Phi^*_{x'}(A;f^m_{\alpha(\beta),\beta})}{\Phi^*_{x'}(B;f^m_{\alpha(\beta),\beta})-\Phi^*_{x}(B;f^m_{\alpha(\beta),\beta})}-(1+\varepsilon)}{1/[\Phi_{x'}(B;f^m_{\alpha(\beta),\beta})-\Phi_{x}(B;f^m_{\alpha(\beta),\beta})]} \qquad \text{(A.81)}$$

$$= \frac{(1-\varepsilon)\cdot\alpha'(\beta)\cdot[\Phi^*_{x''}(A;f^m_{\alpha(\beta),\beta})-\Phi^*_{x'}(A;f^m_{\alpha(\beta),\beta})]}{[\Phi^*_{x'}(B;f^m_{\alpha(\beta),\beta})-\Phi^*_{x}(B;f^m_{\alpha(\beta),\beta})]/[\Phi_{x'}(B;f^m_{\alpha(\beta),\beta})-\Phi_{x}(B;f^m_{\alpha(\beta),\beta})]}$$

$$- \frac{(1+\varepsilon)\cdot[\Phi^*_{x'}(B;f^m_{\alpha(\beta),\beta})-\Phi^*_{x}(B;f^m_{\alpha(\beta),\beta})]}{[\Phi^*_{x'}(B;f^m_{\alpha(\beta),\beta})-\Phi^*_{x}(B;f^m_{\alpha(\beta),\beta})]/[\Phi_{x'}(B;f^m_{\alpha(\beta),\beta})-\Phi_{x}(B;f^m_{\alpha(\beta),\beta})]}$$

$$> \frac{\dfrac{1-\varepsilon}{1+\varepsilon}\cdot\alpha'(\beta)\cdot\dfrac{\partial W^*(f^m_{\alpha(\beta),\beta})}{\partial\alpha}+\dfrac{1+\varepsilon}{1-\varepsilon}\cdot\dfrac{\partial W^*(f^m_{\alpha(\beta),\beta})}{\partial\beta}}{[\Phi^*_{x'}(B;f^m_{\alpha(\beta),\beta})-\Phi^*_{x}(B;f^m_{\alpha(\beta),\beta})]/[\Phi_{x'}(B;f^m_{\alpha(\beta),\beta})-\Phi_{x}(B;f^m_{\alpha(\beta),\beta})]} > 0$$

Since $\alpha'(\beta)\cdot\partial W(f^m_{\alpha(\beta),\beta}(\cdot))/\partial\alpha + \partial W(f^m_{\alpha(\beta),\beta}(\cdot))/\partial\beta > 0$ at all but a finite number of values of β, we have

$$W\big(f_L(\cdot)\big)-W\big(f_R(\cdot)\big)=W\big(f^m_{1,1}(\cdot)\big)-W\big(f^m_{0,0}(\cdot)\big)= \qquad \text{(A.82)}$$
$$\int_0^1\frac{dW(f^m_{\alpha(\beta),\beta})}{d\beta}\cdot d\beta=\int_0^1\Big[\alpha'(\beta)\cdot\frac{\partial W(f^m_{\alpha(\beta),\beta})}{\partial\alpha}+\frac{\partial W(f^m_{\alpha(\beta),\beta})}{\partial\beta}\Big]\cdot d\beta>0$$

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