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Comparative Statics and Non-expected Utility Preferences*

MARK J. MACHINA

Department of Economics,
University of California, San Diego,
La Jolla, California 92093

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Researchers have shown that many, though not all, of the basic results of expected utility analysis can be more or less directly generalized to non-expected utility preferences. This paper describes the essential difference between results which can be extended in this manner and those which cannot and shows that an important family of comparative statics theorems falls into the former class. *Journal of Economic Literature* Classification Number: 026. © 1989 Academic Press, Inc.

I. EXTENDING EXPECTED UTILITY RESULTS TO NON-EXPECTED UTILITY PREFERENCES

In the last several years, researchers such as Chew [2], Chew, Epstein, and Zilcha [3], Chew, Karni, and Safra [4], Dekel [5], Fishburn [12], Karni [14, 15], Machina [19], Machina and Neilson [22], and Neilson [26, 27] have shown that many of the fundamental concepts, tools, and results of expected utility analysis can be extended to general non-expected utility preferences over probability distributions. The general idea behind this approach is a simple one and is essentially an application of standard calculus: Expected utility preferences over probability distributions exhibit the property of "linearity in the probabilities." If an individual's preferences over probability distributions are *not* linear in the probabilities, but are at least "smooth," then at any distribution there will exist a linear approximation to these preferences which, by virtue of its linearity, can be viewed as a "local expected utility" approximation to the individual's preferences in the neighborhood of that distribution and represented by a "local utility function." If this local expected utility approximation is risk averse—that is, if the local utility function at this distribution is concave—then the

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individual will be averse to all small (in the sense of infinitesimal) mean preserving increases in risk in the neighborhood of this distribution. If one non-expected utility maximizer's local utility function at a given distribution is more risk averse (i.e., more concave) than a second individual's, he or she will exhibit more risk averse behavior in the neighborhood of this distribution than will the second individual, and so forth.

Of course, as with any linear approximation, the local expected utility approximation to an individual's preferences at a given probability distribution will only be exact for infinitesimally small changes from this distribution. However, as in standard calculus, this approach can be extended to obtain exact *global* generalizations of many expected utility results. For example, if an individual's local expected utility approximations at every distribution are risk averse, or in other words, if his or her utility functions are all concave, then he or she will be *globally* risk averse in the sense of being averse to all small or large mean preserving increases in risk. It is not necessary that the approximation to preferences at each distribution consist of the *same* risk averse expected utility ranking (i.e., that the local utility function at every distribution be the same concave function), merely that each approximation be *some* risk averse preference ranking (i.e., that each local utility function be some concave function). Since this condition turns out to be both necessary and sufficient for the property of global risk aversion, it implies that the "expected utility" characterization of risk aversion by the concavity of a utility function is completely general, in the sense that the *only* way for smooth non-expected utility preferences to be risk averse is to possess concave local utility functions.[1]

This approach, known as "generalized expected utility analysis," can be formalized as follows:[2] Consider the set $\mathscr{D}[a, b]$ of cumulative distribution functions $F(\cdot)$ over some outcome interval $[a, b]$. If the individual is an expected utility maximizer, his or her maximand or "preference function" over these distributions will take the form

$$v(F) \equiv \int_a^b u(x)\, dF(x), \tag{1}$$

where $u(\cdot)$ is the individual's von Neumann–Morgenstern utility function.

[1] To take an analogy from univariate calculus, we know that the linear function $g(z) \equiv a + b \cdot z$ is nondecreasing provided $b \geqslant 0$. The local extension of this result to a nonlinear but "smooth" (differentiable) function $h(z)$ is that $h(\cdot)$ will be locally nondecreasing at z_0 provided its local linear approximation $h(z_0) + h'(z_0) \cdot (z - z_0)$ is nondecreasing (i.e., provided $h'(z_0) \geqslant 0$). The global extension of this result is that $h(\cdot)$ will be globally nondecreasing provided its linear approximations, though generally different at each z, are all nondecreasing (i.e., provided that $h'(z)$ is *some* nonnegative number for each z).

[2] See Machina [19] for a more formal treatment.

Such an individual will rank the probability distributions $F(\cdot)$ and $F_0(\cdot)$ on the basis of the sign of

$$v(F) - v(F_0) \equiv \int_a^b u(x)[dF(x) - dF_0(x)]. \qquad (2)$$

Now consider a preference function $V(\cdot)$ which is not linear in the probabilities (i.e., in $F(\cdot)$) but is at least smooth (formally, "Fréchet differentiable") in the sense that at each distribution $F_0(\cdot)$ it possesses a first order linear approximation of the form

$$V(F) - V(F_0) \equiv L[F(\cdot) - F_0(\cdot); F_0] + o(\| F - F_0 \|), \qquad (3)$$

where $L[\cdot; F_0]$ is linear in its first argument, $o(\cdot)$ denotes a function which is zero at zero and of higher order than its argument, and $\| F - F_0 \|$ is the standard L^1 distance function $\int_a^b |F(x) - F_0(x)|\, dx$ between cumulative distribution functions. Just as it is possible to represent any linear function of a vector as a weighted sum, we can represent the linear functional $L[F(\cdot) - F_0(\cdot); F_0]$ as a weighted integral, to obtain

$$V(F) - V(F_0) \equiv \int_a^b \psi(x; F_0)[F(x) - F_0(x)]\, dx + o(\| F - F_0 \|) \qquad (4)$$

which, after integration by parts, can be represented in the form

$$V(F) - V(F_0) \equiv \int_a^b U(x; F_0)[dF(x) - dF_0(x)] + o(\| F - F_0 \|). \qquad (5)$$

Comparing (5) with (2), we see that an individual with a smooth non-expected utility preference functional $V(\cdot)$ will rank differential changes from the probability distribution $F_0(\cdot)$, that is, changes for which the first order term in (5) swamps the higher order term, according to their effect on the expectation of the local utility function $U(\cdot; F_0)$. As mentioned above, this approach can also be used to obtain global extensions of many of the basic results of expected utility theory (see Machina [19, Section 3.2] for a formal treatment).

The aforementioned researchers and others have shown that this approach can be used to more or less directly extend the expected utility characterizations of the cardinality of the von Neumann–Morgenstern utility function, first-order stochastic dominance preference, risk aversion, the Arrow–Pratt characterization of comparative risk aversion between individuals, the Ross characterization of comparative risk aversion, the Kihlstrom–Mirman characterization of comparative multivariate risk aversion, and even generic expected utility first-order conditions in choices over sets of probability distributions.

II. First-Order vs Second-Order Properties of Preferences

While this approach has accordingly proven quite useful, it cannot be applied to every result in expected utility theory. A simple example is nonincreasing absolute risk aversion in the sense of Arrow [1] and Pratt [28], as characterized by the condition that the Arrow–Pratt index of absolute risk aversion $-u_{xx}(x)/u_x(x)$ be nonincreasing in x.[3] In Machina [19, pp. 300–301] it was noted that the analogue of this condition, namely,

$$d[-U_{xx}(x; F)/U_x(x; F)]/dx \leqslant 0 \qquad \text{for all } x \text{ and } F(\cdot) \tag{6}$$

is *not* strong enough to imply that a non-expected utility maximizer's risk premium for an additive risk $\tilde{\varepsilon}$ about an initial wealth of x will be nonincreasing in x, even for infinitesimal risks $\tilde{\varepsilon}$.

An analysis of why this result does not extend reveals an inherent limitation on the extent to which expected utility results can be generalized (or at least *directly* generalized) to non-expected utility preferences: From (5) we have that if $V(\cdot)$ is a smooth non-expected utility preference functional, then the premium $\pi(\tilde{\varepsilon}\,|\,x)$ that the individual would pay to avoid an infinitesimal zero-mean risk $\tilde{\varepsilon}$ about an initial wealth of x is given by

$$\pi(\tilde{\varepsilon}\,|\,x) \cong \tfrac{1}{2} \cdot \operatorname{var}(\tilde{\varepsilon}) \cdot [-U_{xx}(x; \delta_x)/U_x(x; \delta_x)], \tag{7}$$

where $\delta_x(\cdot)$ denotes the degenerate distribution which yields x with certainty (i.e., the individual's initial wealth distribution). Differentiating, we have that the condition for an increase in x to preserve or lower $\pi(\tilde{\varepsilon}\,|\,x)$ for all infinitesimal risks $\tilde{\varepsilon}$ is accordingly

$$d[-U_{xx}(\omega; \delta_x)/U_x(\omega; \delta_x)]/d\omega\,|_{\omega=x} + d[-U_{xx}(x; \delta_\omega)/U_x(x; \delta_\omega)]/d\omega\,|_{\omega=x} \leqslant 0. \tag{8}$$

From the two terms on the left-hand side of this condition, we see that a change in initial wealth x has two effects on the infinitesimal risk premium (7). The first comes from the change in the *outcome level* x in the generalized Arrow–Pratt index $-U_{xx}(x; \delta_x)/U_x(x; \delta_x)$ for fixed $\delta_x(\cdot)$, and pertains to the shape of the local utility function $U(\cdot; \delta_x(\cdot))$, just as in expected utility theory. The second effect, however, comes from the change in the *distribution* $\delta_x(\cdot)$ at which the local utility function, and hence its Arrow–Pratt index, are evaluated, and this effect does *not* have any analogue in expected utility theory.

What is it about nonincreasing absolute risk aversion that makes it different from the other, directly generalizable results mentioned above? The key distinction is between what we may term "first-order" versus "second-

[3] Throughout this paper, first and higher order derivatives with respect to x and/or α will be denoted by the subscripts $_x$ and $_\alpha$.

order" properties of preferences. First-order stochastic dominance preference and risk aversion, for example, refer to an individual's attitudes toward a given type of change (a first-order stochastically dominating shift or a mean preserving increase in risk) from a *given* probability distribution. Comparative risk aversion, in the sense of either Arrow–Pratt or Ross [29], applies to individuals' relative attitudes toward changes from some given probability distribution. Since they describe attitudes toward such changes *per se* (i.e., their sign or comparative strengths), such aspects of preferences may be termed "first-order" (or more precisely, "first-order in the probabilities"), and it is no surprise that the charactersics of their first-order (i.e., expected utility) approximations would determine the global properties of (smooth) non-expected utility preferences.

The property of increasing or decreasing absolute (or relative) risk aversion, on the other hand, does not pertain to the sign or strength of an individual's attitude toward risk at a given distribution, but rather how this attitude *changes* when evaluated at different distributions (in the above example, at different initial wealth distributions $\delta_x(\cdot)$). These properties of preferences may accordingly be termed "second-order" (or more precisely, "second-order in the probabilities"). Since such properties are determined by the manner in which the local expected utility approximation to preferences *changes* when we evaluate it at different probability distributions, their non-expected utility characterizations will generally involve additional components which have no analogue in expected utility theory, as exemplified by the second term in condition (8) above. This is not to say that theoretical results cannot be derived for non-expected utility preferences, only that they may well be more complicated than their expected utility counterparts. Viewed in another manner, we have less assurance that expected utility results involving second-order properties of preferences (such as increasing or decreasing absolute or relative risk aversion) will be robust to departures from the hypothesis of linearity in the probabilities.

III. Extending Comparative Statics Results to Non-expected Utility Preferences

The implications of the previous section for comparative statics analysis should be clear. Since the canonical comparative statics result in economics involves *second-order* properties of an agent's maximand,[4] there is reason

[4] Using scalars for simplicity, the general maximization problem in exonomics can be described as maximizing an objective function $\xi(z, \alpha)$ by choosing a control variable α out of some opportunity set A, subject to a parameter z. If ξ is concave in α to ensure a maximum, then the derivative of the optimal value $\alpha(z)$ with respect to z is given by the expression $-\xi_{\alpha z}(z, \alpha(z))/\xi_{\alpha\alpha}(z, \alpha(z))$, which involves second-order derivatives of $\xi(\cdot, \cdot)$.

to be concerned that these important types of results cannot be directly extended from expected utility to non-expected utility preferences. We have already seen that the comparative statics of wealth changes on risk attitudes does not directly generalize.

The purpose of this paper is to show that while not all of expected utility's comparative statics results will directly generalize, an important class of them will. This family of results was first considered by Rothshild and Stiglitz [31, p. 67], who examined the effect of an increase in the riskiness of some underlying economic variable upon the optimal value of a (scalar) control variable. It was subsequently extended by Diamond and Stiglitz [6] to the effect of *compensated* increases in risk upon control variables and can be further extended to include first- and third-order stochastically dominating shifts. Thus, whenever a comparative statics result can be expressed in this general framework, it can be directly generalized from the expected utility to the non-expected utility case.

This family of expected utility results all involve a maximization problem of the form

$$\max_{\alpha} \int_a^b u(x, \alpha)\, dF(x), \tag{9}$$

where α is a control variable and $\tilde{x}$ is an economically relevant random variable with cumulative distribution function $F(\cdot)$.[5] Given the second-order condition

$$u_{\alpha\alpha}(x, \alpha) < 0 \qquad \text{for all } x \text{ and } \alpha, \tag{10}$$

the unique optimal value of the control variable α given $F(\cdot)$, i.e., $\alpha(F)$, is determined by the solution to the first-order condition

$$\int_a^b u_\alpha(x, \alpha(F))\, dF(x) = 0. \tag{11}$$

Since $u_{\alpha\alpha}(x, \alpha)$ is everywhere negative, any change in the distribution $F(\cdot)$ which raises (lowers) the left-hand integral will require a rise (drop) in $\alpha(F)$ to reestablish the equality in (11). Say, for example, that $u_\alpha(x, \alpha)$ is increasing in x (i.e., $u_{\alpha x}(x, \alpha) > 0$). In this case any first-order stochastically dominating shift in the distribution $F(\cdot)$ will raise the integral in (11), and hence raise $\alpha(F)$. If $u_\alpha(x, \alpha)$ is decreasing in x (if $u_{\alpha x}(x, \alpha) < 0$), any first-order stochastically dominating shift in $F(\cdot)$ will lower $\alpha(F)$. If $u_\alpha(x, \alpha)$ is a strictly concave (convex) function of x (if $u_{\alpha xx}(x, \alpha) <\ (>)\ 0$), then any

[5] Since they consider maximands of the form $\int u(z(x, \alpha))\, dF(x)$, the comparative statics analyses of Ekern [9], Feder [11], Katz [16], Kraus [17], and Meyer and Ormiston [23] can also be fitted into this framework.

mean preserving increase in risk will lower (raise) the integral and hence lower (raise) $\alpha(F)$.[6]

The result of Diamond and Stiglitz [6, Theorem 2] concerns a parametrized family of distributions $\{F(\cdot, r)\}$, where increases in r induce what they term "mean *utility* preserving increases in risk." By this they mean that an infinitesimal increase in r from r_0 preserves the expectation $\int u(x, \alpha(F(\cdot; r_0)))dF(x; r)$ and leads to an infinitesimal mean preserving increase in risk in the distribution of the random variable $u(\tilde{x}, \alpha(F(\cdot; r_0)))$ when $\tilde{x}$ has the distribution $F(\cdot; r)$. By the envolope theorem it follows that $d[\int u(x, \alpha(F(\cdot; r)))\, dF(\cdot; r)]/dr = 0$ for all r, so that *global* increases in r will also preserve expected utility when the individual is allowed to remaximize with respect to α. Diamond and Stiglitz [6] demonstrated that if increases in α cause $u(x, \alpha)$ to become a more (less) risk averse function of x, i.e., if

$$d[-u_{xx}(x, \alpha)/u_x(x, \alpha)]/d\alpha > (<)\, 0 \tag{12}$$

then any mean utility preserving increase in risk will lead to a drop (rise) in $\alpha(F(\cdot; r))$. As these authors (p. 344) have phrased it, "the optimal response to a mean utility preserving increase in risk is to adjust the control variable so as to make u show less risk aversion."

The non-expected utility version of the maximization problem (9) is

$$\max_{\alpha} V(F, \alpha), \tag{13}$$

where $V(\cdot, \cdot)$ is assumed to be twice continuously Fréchet differentiable (i.e., both its local utility function $U(\cdot; F, \alpha)$ and $V_\alpha(F, \alpha) = \partial V(F, \alpha)/\partial\alpha$ are continuously differentiable in (F, α)).[7] The second-order condition for this maximization problem is that

$$V_{\alpha\alpha}(F, \alpha) < 0 \qquad \text{for all } F(\cdot) \text{ and } \alpha \tag{14}$$

and the optimal value of the control variable is the value $\alpha(F)$ which solves

$$V_\alpha(F, \alpha(F)) = 0. \tag{15}$$

Our analogues of the expected utility conditions on the signs of the derivatives $u_{\alpha x}(x, \alpha)$ and $u_{\alpha xx}(x, \alpha)$ will be the corresponding conditions on the signs of the derivatives $U_{\alpha x}(x; F, \alpha)$ and $U_{\alpha xx}(x; F, \alpha)$. The natural

[6] Technically, a function need not possess an everywhere positive derivative to be strictly increasing (consider x^3 at 0) nor an everywhere negative second derivative to be strictly concave (consider $-x^4$ at 0). In our theorem these strict properties and inequalities are replaced by their weak counterparts, for which there *is* equivalence.

[7] Since it considers maximands of the form $V(G(\cdot;\, \alpha;\, F))$, the non-expected utility comparative statics analysis of Neilson [27] can, with appropriate modification for the formal notion of smoothness, also be fitted into this framework.

analogue of the Diamond–Stiglitz notion of a mean utility preserving increase in risk, which we shall term a "*compensated* increase in risk," involves a family of distributions $\{F(\cdot, r)\}$ such that for each r_0, an infinitesimal increase in r from r_0 preserves the value of $V(F(\cdot, r), \alpha(F(\cdot; r_0)))$ (or equivalently, preserves the expectation of the local utility function $U(\cdot; F(\cdot; r_0), \alpha(F(\cdot; r_0))))$ and causes an infinitesimal mean preserving increase in risk in the distribution of $U(\tilde{x}; F(\cdot; r_0), \alpha(F(\cdot; r_0)))$ when $\tilde{x}$ has the distribution $F(\cdot; r)$.[8] By the envelope theorem, it again follows that global changes in r will preserve the value of the remaximized preference functional $V(F(\cdot; r), \alpha(F(\cdot; r)))$. Given this, our comparative statics result for non-expected utility preferences is:

THEOREM. *Let $V(\cdot, \cdot)$ be a twice continuously Fréchet differentiable preference functional over $\mathscr{D}[a, b] \times A$ with local utility function $U(\cdot; F, \alpha)$ and satisfying $V_{\alpha\alpha}(F, \alpha) < 0$ for all $F(\cdot)$ and α, and let $\alpha(F) =$* $\operatorname{argmax}\{V(F, \alpha) \mid \alpha \in A\}$. *Then*:

(i) *If $U_{\alpha x}(x; F, \alpha) \geqslant (\leqslant)\, 0$ for all x, $F(\cdot)$ and α, then $\alpha(F^*) \geqslant (\leqslant)$ $\alpha(F)$ whenever $F^*(\cdot)$ first-order stochastically dominates $F(\cdot)$*;

(ii) *if $U_{\alpha xx}(x; F, \alpha) \geqslant (\leqslant)\, 0$ for all x, $F(\cdot)$ and α, then $\alpha(F^*) \geqslant (\leqslant)\, \alpha(F)$ whenever $F^*(\cdot)$ differs from $F(\cdot)$ by a mean preserving increase in risk*;

(iii) *if $U_{\alpha xxx}(x; F, \alpha) \geqslant 0$ for all x, $F(\cdot)$ and α, then $\alpha(F^*) \geqslant \alpha(F)$ whenever $F^*(\cdot)$ differs from $F(\cdot)$ by a third-order stochastically dominating shift*; and

(iv) *if $d[-U_{xx}(x; F, \alpha)/U_x(x; F, \alpha)]/d\alpha \geqslant (\leqslant)\, 0$ for all x, $F(\cdot)$ and α, then $\alpha(F(\cdot; r^*)) \leqslant (\geqslant)\, \alpha(F(\cdot; r))$ whenever $r^* \geqslant r$ and increases in r represent compensated increases in risk with respect to the preference function $V(\cdot, \cdot)$.*[9]

Proof in Appendix.[10]

Examples of problems which fit into the maximization framework (13) and to which this theorem may accordingly be applied, include[11]:

[8] From Machina [19, p. 315], it follows that when the preference functional $V(F)$ does not depend upon any control variable, this definition includes the concept of a *simple compensated spread* [19, p. 281] as a special case.

[9] Since it treats the effect of a change in a probability distribution upon the individual's endogenous level of risk aversion, result (iv) is distinct from the non-expected utility comparative statics results of Karni [15, Theorem 2] and Machina [19, Theorem 4], which treat the effect of a change in risk aversion upon the choice of a probability distribution.

[10] By judicious choice of counterexample, the reader may verify that if the phrase "for all x, $F(\cdot)$ and α" is replaced by "for all x, $F(\cdot)$ and $\alpha = \alpha(F)$," then these conditions on the derivatives of $U(x; F, \alpha)$ are also necessary for their respective implications on $\alpha(F)$.

[11] For additional applications, see Hadar and Russell [13, pp. 302–309].

labor supply with an uncertain wage or uncertain non-wage income, where α denotes leisure and $F(\cdot)$ is the distribution of the wage rate or of non-wage income;

two-period consumption/savings decisions with an uncertain rate of return or uncertain second-period income, where α denotes (nonstochastic) first-period consumption and $F(\cdot)$ is the distribution of the rate of return or of second period income; or more generally;

any situation of "temporal risk" (e.g., Drèze and Modigliani [8], Epstein [10], Kreps and Porteus [18], Machina [21], Mossin [24], Rossman and Selden [30], Spence and Zeckhauser [32]), where $F(\cdot)$ is the distribution of a delayed-resolution random variable and $V(F, \alpha)$ represents the individual's "induced" preferences over $(F(\cdot), \alpha)$ pairs after maximizing out control variables other than α.[12]

Since comparative statics problems intrinsically involve second-order properties of preferences (see Footnote 4), what is it about the general maximization framework (13) that allows for this direct generalization of expected utility results, when other comparative statics results such as increasing or decreasing absolute or relative risk aversion do *not* directly generalize? The key difference is what may be termed the "functional separation of the probability distribution $F(\cdot)$ from the control variable α" in the objective function $V(F, \alpha)$ of (13). In the decreasing absolute risk aversion case, a change in base wealth x (that is, a change in the distribution $\delta_x(\cdot)$) does not cause a change in the individual's preference ordering over the space of probability distributions over final wealth levels, but rather a relocation *within* this space, so that the individual's attitude toward purchasing insurance against the risk $\tilde{\varepsilon}$ is now governed by his or her preference ranking over some different region in the space of distributions. This effect therefore involves how the local expected utility approximation to a *given* preference ranking changes when evaluated at a *different* location in the space of distributions, as represented by the term in condition (8) that possesses no expected utility analogue. On the other hand, since α and $F(\cdot)$ enter as separate arguments in $V(F, \alpha)$, changes in α (whether exogenous or endogenous) cause a change in the individual's ranking over the entire space of probability distributions rather than a relocation within this space. This corresponds to the case of *comparative* risk aversion, which, as noted above, does admit of direct generalizations of expected utility results. Put another way, the comparative statics effect of the probabilities $F(\cdot)$ upon the control variable α are governed by their cross-effect on $V(F, \alpha)$ or, in otherwords, by the (functional) cross partial

[12] These researchers have shown that such derived preferences will typically not be linear in the probabilities, even when the agent's underlying preferences are expected utility.

derivative $\partial^2 V(F, \alpha)/\partial F(\cdot)\partial\alpha$ of $V(F, \alpha)$ with respect to $F(\cdot)$ and α.[13] While this cross-effect is second-order in the probabilities *and* the control variable, it is only *first-order* in the probabilities themselves. It is this functional separation of the probabilities and the control variable, also present in the non-expected utility formulation of Karni [15, Theorem 2], which accordingly allows for the direct extension of expected utility comparative statics results to non-expected utility preferences.

APPENDIX—PROOFS

We shall make use of the following result:

LEMMA. *If $V(F, \alpha)$ is twice continuously Fréchet differentiable over $\mathscr{D}[a, b] \times A$ with local utility function $U(\cdot; F, \alpha)$, then the local utility function of $V_\alpha(F, \alpha)$ is given by $U_\alpha(\cdot; F, \alpha)$.*

Proof. Defining $\phi(\cdot; F, \alpha)$ as the local utility function of $V_\alpha(F, \alpha)$, we have

$$\begin{aligned}
&\phi(x; F_0, \alpha_0) - \int_a^b \phi(\omega; F_0, \alpha_0)\, dF_0(\omega) \\
&\underset{x}{\equiv} \frac{d}{d\beta}\left[V_\alpha(\beta\delta_x + (1-\beta)F_0, \alpha_0)\right]\Big|_{\beta=0} \\
&\underset{x}{\equiv} \frac{d}{d\beta}\left[\frac{d}{d\alpha}\left[V(\beta\delta_x + (1-\beta)F_0, \alpha)\right]\Big|_{\alpha=\alpha_0}\right]\Bigg|_{\beta=0} \\
&\underset{x}{\equiv} \frac{d}{d\alpha}\left[\frac{d}{d\beta}\left[V(\beta\delta_x + (1-\beta)F_0, \alpha)\right]\Big|_{\beta=0}\right]\Bigg|_{\alpha=\alpha_0} \\
&\underset{x}{\equiv} \frac{d}{d\alpha}\left[U(x; F_0, \alpha) - \int_a^b U(\omega; F_0, \alpha)\, dF_0(\omega)\right]\Bigg|_{\alpha=\alpha_0} \\
&\underset{x}{\equiv} U_\alpha(x; F_0, \alpha_0) - \int_a^b U_\alpha(\omega; F_0, \alpha_0)\, dF_0(\omega).
\end{aligned} \tag{16}$$

Proof of Theorem. (i) If $U_{\alpha x}(x; F, \alpha) \geqslant 0$ for all x, $F(\cdot)$ and α so that $U_\alpha(x; F, \alpha)$ is nondecreasing in x, it follows from the above lemma and Machina [19, Theorem 1] that if $F^*(\cdot)$ first-order stochastically dominates $F(\cdot)$ then $V_\alpha(F^*, \alpha) \geqslant V_\alpha(F, \alpha)$ for any α, so that $V_\alpha(F^*, \alpha(F)) \geqslant V_\alpha(F, \alpha(F)) = 0$. Since $V_{\alpha\alpha}(F, \alpha) < 0$ for all $F(\cdot)$ and α, this implies that the

[13] In the following lemma we demonstrate that this cross partial derivative, i.e., the local utility function of $V_\alpha(F, \alpha)$, is given by $U_\alpha(\cdot; F, \alpha)$.

value $\alpha(F^*)$ which solves the first-order condition $V_\alpha(F^*, \alpha(F^*))=0$ must be greater than or equal to $\alpha(F)$. A similar argument applies to the case when $U_{\alpha x}(x; F, \alpha) \leqslant 0$.

(ii) If $U_{\alpha xx}(x; F, \alpha) \geqslant 0$ for all x, $F(\cdot)$ and α so that $U_\alpha(x; F, \alpha)$ is convex in x, it follows from the lemma and Machina [19, Theorem 2] that if $F^*(\cdot)$ differs from $F(\cdot)$ by a mean preserving increase in risk, then $V_\alpha(F^*, \alpha) \geqslant V_\alpha(F, \alpha)$ for any α, so that $V_\alpha(F^*, \alpha(F)) \geqslant V_\alpha(F, \alpha(F))=0$. Since $V_{\alpha\alpha}(F, \alpha)<0$ for all $F(\cdot)$ and α, this implies that the value $\alpha(F^*)$ which solves the first-order condition $V_\alpha(F^*, \alpha(F^*))=0$ must be greater than or equal to $\alpha(F)$. A similar argument applies to the case when $U_{\alpha xx}(x; F, \alpha) \leqslant 0$.

(iii) If $U_{\alpha xxx}(x; F, \alpha) \geqslant 0$ for all x, $F(\cdot)$ and α, it follows from the lemma, Whitmore [33], and an argument analogous to those in Machina [19, Theorems 1 and 2] that if $F^*(\cdot)$ differs from $F(\cdot)$ by a third-order stochastically dominating shift, then then $V_\alpha(F^*, \alpha) \geqslant V_\alpha(F, \alpha)$ for any α, so that $V_\alpha(F^*, \alpha(F)) \geqslant V_\alpha(F, \alpha(F))=0$. Since $V_{\alpha\alpha}(F, \alpha)<0$ for all $F(\cdot)$ and α, this implies that the value $\alpha(F^*)$ which solves the first-order condition $V(F^*, \alpha(F^*))=0$ must be greater than or equal to $\alpha(F)$.

(iv) From the lemma and Eq. (5) it follows that

$$\frac{d}{dr}\left[V_\alpha(F(\cdot; r), \alpha(F(\cdot; r_0)))\right]\Big|_{r=r_0}$$

$$= \frac{d}{dr}\left[\int_a^b U_\alpha(x; F(\cdot; r_0), \alpha(F(\cdot; r_0)))\, dF(x; r)\right]\Bigg|_{r=r_0} \tag{17}$$

for all r_0. Since an increase in r represents a mean utility preserving increase in risk with respect to the function $U(\cdot; F(\cdot; r_0), \alpha(F(\cdot; r_0)))$ and since

$$\frac{d}{d\alpha}\left[\frac{-U_{xx}(x; F(\cdot; r_0), \alpha)}{U_x(x; F(\cdot; r_0), \alpha)}\right] \geqslant (\leqslant)\, 0, \tag{18}$$

we have from the integration by parts result of Diamond and Stiglitz [6, p. 344] that the right side of (17) will be nonpositive (nonnegative). Since the first-order condition $V_\alpha(F(\cdot; r), \alpha(F(\cdot; r))) \equiv 0$ implies

$$\frac{d\alpha(F(\cdot; r))}{dr}\Bigg|_{r=r_0} = \frac{-d/dr\left[V_\alpha(F(\cdot; r), \alpha(F(\cdot; r_0)))\right]|_{r=r_0}}{V_{\alpha\alpha}(F(\cdot; r_0), \alpha(F(\cdot; r_0)))} \tag{19}$$

and since the denominator of the right side of (19) is negative, we have that $d\alpha(F(\cdot; r))/dr$ will be nonpositive (nonnegative) for all r, which implies that $\alpha(F(\cdot; r^*)) \leqslant (\geqslant)\, \alpha(F(\cdot; r))$ whenever $r^* \geqslant r$.

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