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# Almost-objective uncertainty*

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**Summary.** Every subjective state space with Euclidean structure contains *almost-objective events* which arbitrarily closely approximate the properties of objectively uncertain events for all individuals with event-smooth betting preferences – whether or not they are expected utility, state-independent, or probabilistically sophisticated. These properties include *unanimously agreed-upon* revealed likelihoods, statistical independence from other subjective events, probabilistic sophistication over almost-objective bets, and linearity of state-independent and state-dependent expected utility in almost-objective likelihoods and mixtures. Most physical randomization devices are based on events of this form. Even in the presence of state-dependence, ambiguity, and ambiguity aversion, an individual's betting preferences over almost-objective events are based solely on their attitudes toward objective risk, and can fully predict (or be predicted from) their behavior in an idealized casino.

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## 1 Introduction

The theory of individual choice under uncertainty allows for a wide variation in the specification of uncertain outcomes, which can consist of univariate wealth levels, multivariate commodity bundles, continuous-time consumption streams, or most

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generally, abstract objects $x \in \mathcal{X}$. The theory also admits two different ways of representing *uncertainty itself*, namely:

- *Objective uncertainty*, which is represented by well-defined, additive, numerical probabilities $p \in [0,1]$, and where the objects of choice consist of *lotteries* $\mathbf{P} = (x_1, p_1; \ldots; x_n, p_n)$ which yield outcome $x_i \in \mathcal{X}$ with probability $p_i$, under the standard condition $\sum_{i=1}^{n} p_i = 1$.[1]
- *Subjective uncertainty*, which is represented by mutually exclusive and exhaustive *states of nature* $s \in \mathcal{S}$, and where the objects of choice consist of *bets* or *acts* $f(\cdot) = [x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n]$ which yield outcome $x_i \in \mathcal{X}$ if the realized state lies in *event* $E_i \subseteq \mathcal{S}$, for some partition $\{E_1, \ldots, E_n\}$ of $\mathcal{S}$.

One strength of the objective framework is that it allows us to draw on the tremendous body of results in probability theory, such as the Central Limit Theorem, Law of Large Numbers or Chebyshev's Inequality. Furthermore, since objective probabilities are part of the objects of choice themselves, such results hold *independently* of the individual's particular attitudes toward risk, and in this sense they have the same character as arbitrage-based arguments in finance, which also hold independently of risk preferences and hence yield extremely powerful results.

But in some sense, this strength of the objective framework is also its greatest weakness: it imposes *too much* conformity of beliefs across individuals, and in many cases, too much structure on each individual's own beliefs. In most real-world uncertain prospects, such as real investments, financial assets, or insurance contracts, uncertainty does not appear in the form of unanimously agreed-upon numerical probabilities, but rather, in terms of states or events. In contrast with objective lotteries, preferences over such real-world subjective prospects exhibit the following properties:

- Individuals may have different subjective likelihoods for the relevant events (*diverse beliefs*).
- Individuals' beliefs may not be representable by probabilities at all, with some or all events being considered "ambiguous" (*absence of probabilistic sophistication*).
- Outcome preferences may depend upon the source of uncertainty itself (*state-dependence*).

All of this argues for modeling uncertainty via the subjective approach, which is rightly considered the more "foundational" of the two approaches.

In order to obtain as many of the benefits of probability theory as possible within the subjective framework, several researchers have proposed conditions on preferences over subjective acts which imply that an individual's beliefs can be represented by *subjective probabilities* (sometimes called *personal probabilities*) over events. Examples include:

- Anscombe and Aumann (1963), who combine a subjective state space $\mathcal{S} = \{s_1, \ldots, s_n\}$ (a *horse race*) with an exogenous objective randomization device (a *roulette wheel*), assume expected utility maximization, and then derive von

[1] Throughout this paper, we restrict attention to finite-outcome uncertain prospects.

Neumann-Morgenstern outcome utilities $\{U(x) \,|\, x \in \mathcal{X}\}$ and subjective state probabilities $\{\mu(s_1), \ldots, \mu(s_n)\}$. While elegant, this two-stage[2] approach still ultimately relies upon exogenous objective uncertainty, via the roulette wheel.

- Ramsey (1931), who considers a family of subjective events which includes an *ethically neutral event* $\bar{E}$ for which the individual's betting preferences satisfy $[x^*$ on $\bar{E}; x$ on $\sim\bar{E}] \sim [x$ on $\bar{E}; x^*$ on $\sim\bar{E}]$ for all outcomes $x^*$ and $x$, assumes expected utility maximization, and then derives the utility of each outcome and subjective probability of each event. This approach is thus completely subjective, relying just on the existence of some subjective event $\bar{E}$ whose betting properties essentially correspond to those of an exogenous objective event with probability $\frac{1}{2}$.
- Savage (1954), who takes a general (infinite) state space $\mathcal{S}$, assumes that preferences over finite-outcome acts satisfy the *Sure-Thing Principle* (separability across mutually exclusive events), consistent comparative likelihoods ($[x^*$ on $A; x$ on $\sim A] \succcurlyeq [x^*$ on $B; x$ on $\sim B]$ for $x^* \succ x$ implies $[y^*$ on $A; y$ on $\sim A] \succcurlyeq [y^*$ on $B; y$ on $\sim B]$ for all $y^* \succ y$), and a version of event-continuity, and then derives expected utility risk preferences and a subjective probability measure $\mu(\cdot)$ over $\mathcal{S}$.
- Machina and Schmeidler (1992), who adopt a Savage type setting, and derive the *probabilistically sophisticated* form $V(x_1, \mu(E_1); \ldots; x_n, \mu(E_n))$ for some subjective probability measure $\mu(\cdot)$ and general (i.e., not necessarily expected utility) risk preference function $V(x_1, p_1; \ldots; x_n, p_n)$. Machina and Schmeidler (1995) give a similar derivation in an Anscombe-Aumann setting.

Such formulations succeed in bringing additive numerical probabilities into the subjective framework. However, the *subjective* probabilities they generate do not exhibit all of the properties of exogenous *objective* probabilities. Since subjective probabilities needn't exist for individuals who violate the assumptions of these formulations, they do not exhibit the objective-probability property of *universal existence across agents*. Since they needn't be the same for all individuals even when they do exist, they do not exhibit the objective property of *unanimity across agents*.

But perhaps surprisingly, some subjective events turn out to be more "objective" than others. An early example of this was offered by Poincaré (1912), who considered a Euclidean state space $\mathcal{S} = [\underline{s}, \bar{s}] \subset R^1$ and constructed a sequence of events $\{\bar{E}_m\}_{m=1}^{\infty}$ such that every subjective probability measure $\mu(\cdot)$ over $\mathcal{S}$ with a bounded-derivative density $\nu(\cdot)$ exhibits $\lim_{m\to\infty} \mu(\bar{E}_m) = \frac{1}{2}$. Thus as $m \to \infty$ the events $\bar{E}_m$ converge to the likelihood properties of an objective 50:50 coin, for every individual with a bounded-density subjective probability measure.

The purpose of this paper is to extend the definition of such events, enlarge the set of objective properties that they arbitrarily closely approximate, and expand the universe of agents for whom these properties hold. Specifically, we show that:

- Euclidean structural assumptions on a purely subjective state space $\mathcal{S}$, and the property that subjective act preferences are "smooth in the events," imply

[2] Sarin and Wakker (1997) offer a single-stage version of the Anscombe-Aumann results, which retains their assumptions of joint objective$\times$subjective uncertainty and expected utility risk preferences.

the existence of what we term *almost-objective events* in $\mathcal{S}$. Such events will approximate – and in the limit attain – all the properties of idealized objective events for every individual with event-smooth act preferences, whether or not they possess a subjective probability measure $\mu(\cdot)$.

- These limiting properties include unanimously agreed-upon revealed likelihoods that are independent of the occurrence/non-occurrence of other subjective events, state-independent and probabilistically sophisticated betting preferences over such events (even for individuals who are not state-independent or probabilistically sophisticated in general), and linearity in these likelihoods for all state-independent and state-dependent expected utility maximizers.
- Even in the presence of state-dependence, ambiguity and ambiguity aversion, an individual's preferences over almost-objective acts will be based solely on their attitudes toward objective risk, and can thus fully predict – or be predicted from – their behavior in an idealized casino.

Since they converge to the properties of objective events for all event-smooth individuals, almost-objective events and the acts and mixtures based on them can also serve as completely subjective substitutes for many of the standard analytical uses of objective uncertainty. For example:

- Almost-objective events can replace an exogenously specified objective roulette wheel to allow for a direct Anscombe-Aumann style derivation of exact subjective probability within a completely subjective framework.
- Almost-objective mixtures of subjective acts can also replace objective probability mixtures to allow for an extension of the analysis of Machina (1982) to subjective uncertainty, yielding a joint generalization of expected utility analysis and subjective probability analysis to all event-smooth individuals, whether or not they satisfy the expected utility hypothesis or the hypothesis of probabilistic sophistication.

The following section gives the analytical setting and offers an example ("almost-ethically-neutral events") which illustrates the general idea of the approach. Section 3 presents the notions of an almost-objective event, an almost-objective act, and an almost-objective mixture of subjective acts, and examines the revealed likelihood and betting properties of these objects. Section 4 presents the above-mentioned applications, and a discussion of why some sources of gains from trade under uncertainty lead to bets on almost-objective events, whereas others lead to bets on general subjective events. Section 5 offers comparisons with the literature and analytical extensions, and Section 6 discusses some implications for the modeling of uncertainty and uncertain choice. Mathematical background and proofs of theorems are in an Appendix.

## 2 Setting and illustration

### *2.1 Setting*

Although Section 5.2 shows how it can be developed more generally, the formal analysis of this paper is conducted within the following framework:

| | |
|---|---|
| $\mathcal{X} = \{\ldots, x, \ldots\}$ | arbitrary space of *outcomes* |
| $\mathcal{S} = [\underline{s}, \bar{s}] \subset R^1$ | set of *states of nature*, with uniform Lebesgue measure $\lambda(\cdot)$ |
| $\mathcal{E} = \{\ldots, E, \ldots\}$ | algebra of *events* (each a finite union of intervals) in $\mathcal{S}$ |
| $f(\cdot) = [x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n]$ | finite-outcome *act*, yielding outcome $x_i$ in event $E_i$, for some $\mathcal{E}$-measurable partition $\{E_1, \ldots, E_n\}$ of $\mathcal{S}$ |
| $\mathcal{A} = \{\ldots, f(\cdot), \ldots\}$ | set of all finite-outcome, $\mathcal{E}$-measurable acts on $\mathcal{S}$ |
| $W(\cdot)$ and $\succcurlyeq$ | *preference function* and its corresponding *preference relation* over $\mathcal{A}$ |

An event $E$ is said to be *null* for preference function $W(\cdot)$ if $W(x^* \text{ on } E; f(\cdot) \text{ on } {\sim}E) = W(x \text{ on } E; f(\cdot) \text{ on } {\sim}E)$ for all $x^*$, $x$ and $f(\cdot)$, and our event-continuity assumption (defined in the Appendix) implies that every event $E$ with zero Lebesgue measure will be null. We assume that preferences are *outcome-monotonic* in the sense that if two outcomes satisfy $x^* \succ x$ (i.e., if $W(x^* \text{ on } \mathcal{S}) > W(x \text{ on } \mathcal{S})$), then $W(x^* \text{ on } E; f(\cdot) \text{ on } {\sim}E) > W(x \text{ on } E; f(\cdot) \text{ on } {\sim}E)$ for all $f(\cdot)$ and all nonnull $E$.

It is important to note that the above framework is one of completely subjective uncertainty. In particular, almost-objective events will all be subsets of $\mathcal{S}$ (in fact, elements of $\mathcal{E}$), and almost-objective acts and almost-objective mixtures of subjective acts will be mappings from $\mathcal{S}$ to $\mathcal{X}$ (elements of $\mathcal{A}$). No "objective randomization device" will be appended to this structure.

A preference function $W(\cdot)$ over subjective acts $f(\cdot) = [x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n]$ may or may not be expected utility, it may or may not be state-dependent, and it may or may not be probabilistically sophisticated. That is, it may or may not take one of the following forms:

| | |
|---|---|
| $W_{SEU}(f(\cdot)) = \int_{\mathcal{S}} U(f(s)) \cdot d\mu(s) = \sum_{i=1}^n U(x_i) \cdot \mu(E_i)$ | *state-independent expected utility*, for some utility function $U(\cdot)$ and subjective probability measure $\mu(\cdot)$ |
| $W_{SDEU}(f(\cdot)) = \int_{\mathcal{S}} U(f(s)\|s) \cdot d\mu(s) = \sum_{i=1}^n \int_{E_i} U(x_i\|s) \cdot d\mu(s)$ | *state-dependent expected utility*, for some utility function $U(\cdot\|\cdot)$ and subjective probability measure $\mu(\cdot)$ |
| $W_{PS}(f(\cdot)) = V(x_1, \mu(E_1); \ldots; x_n, \mu(E_n))$ | *probabilistically sophisticated*, for some preference function $V(\cdot)$ and subjective probability measure $\mu(\cdot)$ |

As mentioned, we also assume that each preference function $W(\cdot)$ is "smooth in the events." Event-differentiability is a generalization of the standard mathematical idea of differentiability of a set function, and versions of this idea have been developed and applied by Epstein (1999) in his characterization of uncertainty aversion, and by Machina (2002) in an analysis of the robustness of the classical model of risk preferences and beliefs.[3] Intuitively, event-differentiability is simply the property that $W(f(\cdot)) = W(x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n)$ responds smoothly to differential changes in the *events* $E_1, \ldots, E_n$, as opposed to differential changes in the *outcomes* $x_1, \ldots, x_n$. Thus, for the two-parameter family of subjective acts $f_{\alpha,\beta}(\cdot) = [x_1 \text{ on } [\underline{s}, \alpha); x_2 \text{ on } [\alpha, \beta); x_3 \text{ on } [\beta, \bar{s}]]$ over $\mathcal{S} = [\underline{s}, \bar{s}]$, $W(f_{\alpha,\beta}(\cdot))$ would vary smoothly in the values $\alpha$ and $\beta$. The above three forms will be event-smooth provided they satisfy the following conditions:

| | |
|---|---|
| $W_{SEU}(\cdot)$ : | its subjective probability measure $\mu(\cdot)$ has a continuous density $\nu(\cdot)$ over $\mathcal{S}$ |
| $W_{SDEU}(\cdot)$ : | for each $x$, its *evaluation measure* $\Phi_x(E) \equiv \int_E U(x\|s) \cdot d\mu(s)$ has a continuous density |
| $W_{PS}(\cdot)$ : | $\mu(\cdot)$ has a continuous density $\nu(\cdot)$, and $V(\cdot)$ is differentiable in the probabilities |

Formal definitions of event-differentiability and event-smoothness, based on Machina (2002), are given in the Appendix. Although required for the proofs of the theorems, this mathematical material is not invoked in either the intuitive or the formal discussion of the text.

## 2.2 *Six properties of purely objective events*

*Purely objective events* – as generated by an idealized coin, die or roulette wheel – exhibit the following characteristic properties, either in isolation or in the presence of subjective uncertainty:

1. *Unanimous, outcome-invariant revealed likelihoods:* All individuals exhibit identical, outcome-invariant revealed likelihoods over purely objective events – corresponding, of course, to their objective probabilities. In contrast, betting preferences over any pair of *subjective* events can differ across individuals (when they have different subjective beliefs), can depend upon the prizes assigned to the events (under state-dependence), or can even depend upon the prizes assigned to mutually exclusive events (as in the Ellsberg (1961) Paradox).
2. *Independence from subjective events:* Under joint objective$\times$subjective uncertainty, purely objective event likelihoods are independent of the realization/non-realization of any given subjective event, whether or not that event happens to be assigned a subjective probability. That is, the event likelihoods for an exogenous objective coin, die or roulette wheel are invariant to whether any given subjective event $E$ does or does not occur.
3. *Probabilistic sophistication over objective lotteries:* It is almost a truism that all individuals evaluate objective lotteries $\mathbf{P} = (x_1, p_1; \ldots; x_n, p_n)$ solely according to their outcomes and corresponding objective likelihoods, via some preference function of the form $V(x_1, p_1; \ldots; x_n, p_n)$.

[3] See also Rosenmuller (1972), as well as Epstein and Marinacci's (2001) application to the core of large games.

4. *Reduction of objective×subjective uncertainty:* Standard *reduction of compound uncertainty* assumptions[4] imply that all individuals – whether or not they are expected utility, state-independent or probabilistically sophisticated – evaluate any *objective mixture* $(f(\cdot),\alpha;f^*(\cdot),1-\alpha)$ of subjective acts $f(\cdot)=[x_1$ on $E_1;\ldots;x_n$ on $E_n]$ and $f^*(\cdot)=[x_1^*$ on $E_1^*;\ldots;x_{n^*}^*$ on $E_{n^*}^*]$ solely according to its induced map $[\ldots;(x_i,\alpha;x_j^*,1-\alpha)$ on $E_i\cap E_j^*;\ldots]$ from events to lotteries, so that any other mixture $(f_0(\cdot),\alpha_0;f_0^*(\cdot),1-\alpha_0)$ which induces the same map will be indifferent.[5]

Besides the above properties, which apply to all individuals, exogenous objectively uncertain events exhibit two more specialized properties: one for all probabilistically sophisticated individuals, and the other for all state-independent or state-dependent expected utility maximizers:

5. *Under probabilistic sophistication, two-way independence of objective and subjective likelihoods:* Whenever subjective event likelihoods *are* all well-defined – that is, whenever the individual is probabilistically sophisticated – these subjective event likelihoods are independent of the realization/non-realization of exogenous objective events, and vice versa.
6. *Under expected utility, linearity in objective likelihoods:* Under objective uncertainty, expected utility is linear in objective probabilities (i.e., $V_{EU}(x_1,p_1;\ldots;x_n,p_n)\equiv\sum_{i=1}^n U(x_i)\cdot p_i)$ and in objective mixtures of lotteries $(V_{EU}(\mathbf{P},\alpha;\mathbf{P}^*,1-\alpha)\equiv\alpha\cdot V_{EU}(\mathbf{P})+(1-\alpha)\cdot V_{EU}(\mathbf{P}^*))$. Under objective×subjective uncertainty, both state-independent and state-dependent expected utility are linear in objective mixtures of subjective acts $(W_{SEU}(f(\cdot),\alpha;f^*(\cdot),1-\alpha)=\alpha\cdot W_{SEU}(f(\cdot))+(1-\alpha)\cdot W_{SEU}(f^*(\cdot))$, and $W_{SDEU}(f(\cdot),\alpha;f^*(\cdot),1-\alpha)=\alpha\cdot W_{SDEU}(f(\cdot))+(1-\alpha)\cdot W_{SDEU}(f^*(\cdot)))$.

As shown in Theorems 1–6 below, even though almost-objective events, acts and mixtures are completely subjective, they will approximate – and in the limit attain – each of these properties.

### *2.3 Almost-ethically-neutral events*

Of course, there exists no subjective event $\bar{E}\subseteq\mathcal{S}$ that is viewed as being "objectively ethically neutral" by *all* individuals with event-smooth preferences – that is, no $\bar{E}$ that satisfies

$$W\big(x^* \text{ on } \bar{E}; x \text{ on} \sim\!\bar{E}\big) \underset{\text{all } x,x^*\in\mathcal{X}}{\equiv} W\big(x \text{ on } \bar{E}; x^* \text{ on} \sim\!\bar{E}\big) \tag{1}$$

for *every* event-smooth $W(\cdot)$ over $\mathcal{A}$. This is neither unexpected nor inappropriate – as noted above, one of the primary motivations for the subjective approach is precisely to *allow* for such diversity (or even nonexistence) of likelihood beliefs.

[4] E.g., Anscombe and Aumann (1963), Machina and Schmeidler (1995).

[5] For example, since the mixtures $([x_1$ on $E_1;x_2$ on $E_2],\frac{1}{2};[x_0$ on $\mathcal{S}],\frac{1}{2})$ and $([x_0$ on $E_1;x_2$ on $E_2],\frac{1}{2};[x_1$ on $E_1;x_0$ on $E_2],\frac{1}{2})$ induce the same map $[(x_1,\frac{1}{2};x_0,\frac{1}{2})$ on $E_1;(x_2,\frac{1}{2};x_0,\frac{1}{2})$ on $E_2]$ from events to lotteries, they will be indifferent.

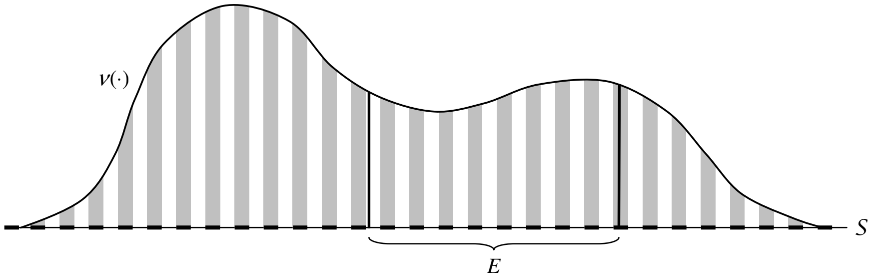


**Figure 1** Almost-ethically-neutral events $\bar{E}_m$ satisfy $\mu(\bar{E}_m) \approx \frac{1}{2}$, as well as $\mu(\bar{E}_m \cap E) \approx \frac{1}{2} \cdot \mu(E)$, for every measure $\mu(\cdot)$ with a continuous density function $\nu(\cdot)$

Nevertheless, some subjective events come "closer" to being objectively ethically neutral than others. For each $m \geq 1$, partition the state space $\mathcal{S} = [\underline{s}, \bar{s}]$ into the $m$ equal-length intervals

$$\left[\underline{s},\ \underline{s} + \tfrac{\lambda_{\mathcal{S}}}{m}\right), \ldots, \left[\underline{s} + i \cdot \tfrac{\lambda_{\mathcal{S}}}{m},\ \underline{s} + (i+1) \cdot \tfrac{\lambda_{\mathcal{S}}}{m}\right), \ldots, \left[\underline{s} + (m-1) \cdot \tfrac{\lambda_{\mathcal{S}}}{m},\ \bar{s}\right] \qquad (2)$$

where $\lambda_{\mathcal{S}} = \lambda(\mathcal{S}) = \bar{s} - \underline{s}$, and define the event $\bar{E}_m \subset \mathcal{S}$ by

$$\bar{E}_m \;=\; \bigcup_{i=0}^{m-1} \left[\underline{s} + i \cdot \tfrac{\lambda_{\mathcal{S}}}{m},\ \underline{s} + (i + \tfrac{1}{2}) \cdot \tfrac{\lambda_{\mathcal{S}}}{m}\right] \qquad (3)$$

that is, as the union of the *left halves* of the intervals in (2). For uniform Lebesgue measure $\lambda(\cdot)$ on $\mathcal{S}$, it is clear that $\bar{E}_m$ exactly satisfies $\lambda(\bar{E}_m) = \frac{1}{2} \cdot \lambda(\mathcal{S})$ for each $m$.

Consider the *subjective probabilities* of such events.[6] Given an arbitrary continuous-density subjective probability measure $\mu(\cdot)$, say from a preference function $W_{SEU}(\cdot)$, $W_{SDEU}(\cdot)$ or $W_{PS}(\cdot)$, the events $\bar{E}_m$ generally *do not* satisfy $\mu(\bar{E}_m) = \frac{1}{2}$. However, as $m \to \infty$ the events $\bar{E}_m$ and their complements $\sim\bar{E}_m$ will *arbitrarily closely approximate* this property – that is, they will satisfy

$$\lim_{m \to \infty} \mu\big(\bar{E}_m\big) \;=\; \lim_{m \to \infty} \mu\big(\sim\bar{E}_m\big) \;=\; \tfrac{1}{2} \qquad (4)$$

This result can be illustrated in Figure 1, where the event $\bar{E}_m$ appears as the union of the solid intervals along the entire horizontal line (the state space $\mathcal{S}$). For any subjective probability measure $\mu(\cdot)$ with a continuous density function $\nu(\cdot)$, the probability $\mu(\bar{E}_m)$ consists of the total shaded area lying below the density function, which for large $m$ is seen to approach *half* of the total area below the density function – or in other words, to approach the value $\frac{1}{2}$.

As mentioned, examples like (3) date back at least to Poincaré (1912, Sects. 92–93), who thought of the state space $\mathcal{S} = [\underline{s}, \bar{s}]$ and events $\bar{E}_m, \sim\bar{E}_m$ as a spinning wheel divided into a large number of equal-sized alternating red and black sectors of angular width $\varepsilon$. Denoting the total number of angular rotations of the wheel by $\omega$, Poincaré gave the following result linking $\omega$'s subjective density $\nu(\cdot)$ with the probability that the spin results in red (or black):

[6] All properties of the events $\bar{E}_m$ and $\sim\bar{E}_m$ reported below follow from the theorems in Section 3.

**Theorem (Poincaré, 1912).** *For any differentiable density function* $\nu(\cdot)$ *with support lying in an interval* $[\underline{s}, \bar{s}]$, *and such that* $|\nu'(\cdot)| < \nu_{max}$ *over* $[\underline{s}, \bar{s}]$ *:*

$$\big|2 \cdot \text{prob(red)} - 1\big| \;<\; \nu_{max} \cdot (\bar{s} - \underline{s}) \cdot \varepsilon \tag{5}$$

Other authors have provided similar results, such as:

- Feller (1971, pp. 62–63), who considers the case when $\nu(\cdot)$ is unimodal and small at its mode
- Kemperman (1975), who considers the case of $k$'th order differentiable densities
- Good (1986, pp. 162–164), who considers the case when $\nu(\cdot)$ is a mixture of distributions
- Diaconis and Engel (1986), who consider a two-dimensional state space

Similar results and/or discussions appear in Fréchet (1952, pp. 3–8,16), Diaconis and Keller (1989) and the posthumously published lecture of Savage (1973), which contains a version of Figure 1.

Results such as equation (4), Poincaré's Theorem, and the above extensions show that as $m \to \infty$, the events $\bar{E}_m$ and $\sim\!\bar{E}_m$ approximate the *equal subjective probability* property with respect to any suitably regular (e.g., continuous-density) subjective probability measure $\mu(\cdot)$ on $\mathcal{S}$. However, these results fall short of establishing that $\bar{E}_m$ and $\sim\!\bar{E}_m$ approximate the *objective ethical neutrality* property (1) for all event-smooth $W(\cdot)$, for two reasons:

- Since they *assume the existence* of a prior subjective probability measure $\mu(\cdot)$ (or subjective density $\nu(\cdot)$) on $\mathcal{S}$, these results can only be applied to preference functions that are *based on* such measures, such as the forms $W_{SEU}(\cdot)$, $W_{SDEU}(\cdot)$ or $W_{PS}(\cdot)$, as opposed to a general event-smooth $W(\cdot)$.
- Even for a measure-based form such as $W_{SDEU}(\cdot)$, the equal subjective probability condition $\mu(\bar{E}) = \mu(\sim\!\bar{E})$ need not imply the ethical-neutrality property $W(x^* \text{ on } \bar{E};\, x \text{ on } \sim\bar{E}) \underset{x,x^*}{\equiv} W(x \text{ on } \bar{E}; x^* \text{ on } \sim\bar{E})$, for reasons of state-dependence.

However, by strengthening the above results, it is possible to show that the events $\bar{E}_m$ and $\sim\!\bar{E}_m$ in fact *do* approximate the property of objective ethical neutrality, as well as all other properties of objective 50:50 events, for every event-smooth preference function $W(\cdot)$ over $\mathcal{A}$. Consider first the *revealed likelihood properties* of the events $\bar{E}_m$ and $\sim\bar{E}_m$. Since each event-smooth $W_{SDEU}(\cdot)$ will possess continuous-density evaluation measures $\varPhi_x(E) \equiv \int_E U(x|s) \cdot d\mu(s)$, all event-smooth $W_{SEU}(\cdot)$, $W_{SDEU}(\cdot)$ and $W_{PS}(\cdot)$ will exhibit the following properties for all $x, x^* \in \mathcal{X}$:

$$\lim_{m\to\infty} W_{SEU}\begin{pmatrix} x^* \text{ on } & \bar{E}_m \\ x \;\text{ on } & \sim\!\bar{E}_m \end{pmatrix} = \frac{U(x^*)+U(x)}{2} = \lim_{m\to\infty} W_{SEU}\begin{pmatrix} x \;\text{ on } & \bar{E}_m \\ x^* \text{ on } & \sim\!\bar{E}_m \end{pmatrix} \tag{6}$$

$$\lim_{m\to\infty} W_{SDEU}\begin{pmatrix} x^* \text{ on } & \bar{E}_m \\ x \;\text{ on } & \sim\!\bar{E}_m \end{pmatrix} = \frac{\varPhi_{x^*}(\mathcal{S})+\varPhi_x(\mathcal{S})}{2} = \lim_{m\to\infty} W_{SDEU}\begin{pmatrix} x \;\text{ on } & \bar{E}_m \\ x^* \text{ on } & \sim\!\bar{E}_m \end{pmatrix}$$

$$\lim_{m\to\infty} W_{PS}\begin{pmatrix} x^* \text{ on } & \bar{E}_m \\ x \text{ on } & \sim\bar{E}_m \end{pmatrix} = V(x^*, \tfrac{1}{2}; x, \tfrac{1}{2}) = \lim_{m\to\infty} W_{PS}\begin{pmatrix} x \text{ on } & \bar{E}_m \\ x^* \text{ on } & \sim\bar{E}_m \end{pmatrix}$$

so $\bar{E}_m$ and $\sim\bar{E}_m$ can be said to be *almost-ethically-neutral* events for these preference functions. More generally, $\bar{E}_m$ and $\sim\bar{E}_m$ will be shown to satisfy the almost-ethical-neutrality condition

$$\lim_{m\to\infty} W\begin{pmatrix} x^* \text{ on } & \bar{E}_m \\ x \text{ on } & \sim\bar{E}_m \end{pmatrix} \underset{\text{all } x,x^*\in\mathcal{X}}{\equiv} \lim_{m\to\infty} W\begin{pmatrix} x \text{ on } & \bar{E}_m \\ x^* \text{ on } & \sim\bar{E}_m \end{pmatrix} \tag{7}$$

for *all* event-smooth preference functions $W(\cdot)$ on $\mathcal{A}$.

In addition, the almost-ethically-neutral events $\bar{E}_m$ and $\sim\bar{E}_m$ also approximate the objective property of *independence with respect to the realization of any fixed subjective event* $E \subseteq \mathcal{S}$, in the sense that, conditional on $E$, the joint events $\bar{E}_m \cap E$ and $\sim\bar{E}_m \cap E$ will also have unanimous, outcome-independent, equal limiting revealed likelihoods. That is, these events satisfy

$$\begin{aligned} \lim_{m\to\infty} \mu\big(\bar{E}_m \cap E\big) &= \tfrac{1}{2}\cdot\mu(E) = \lim_{m\to\infty} \mu\big(\sim\bar{E}_m \cap E\big) \\ \lim_{m\to\infty} \Phi_x\big(\bar{E}_m \cap E\big) &= \tfrac{1}{2}\cdot\Phi_x(E) = \lim_{m\to\infty} \Phi_x\big(\sim\bar{E}_m \cap E\big) \end{aligned} \tag{8}$$

for all continuous-density measures $\mu(\cdot)$ or $\Phi_x(\cdot)$, as well as the limiting equal-revealed-likelihood identities

$$\lim_{m\to\infty} W\begin{pmatrix} x^* \text{ on } & \bar{E}_m\cap E \\ x \text{ on } & \sim\bar{E}_m\cap E \\ f(\cdot) \text{ elsewhere} & \end{pmatrix} \underset{\substack{\text{all } x,x^*\in\mathcal{X} \\ \text{all } f(\cdot)\in\mathcal{A}}}{\equiv} \lim_{m\to\infty} W\begin{pmatrix} x \text{ on } & \bar{E}_m\cap E \\ x^* \text{ on } & \sim\bar{E}_m\cap E \\ f(\cdot) \text{ elsewhere} & \end{pmatrix} \tag{9}$$

for all event-smooth $W(\cdot)$ on $\mathcal{A}$.

The limiting measure property $\lim_{m\to\infty} \mu(\bar{E}_m\cap E) = \frac{1}{2}\cdot\mu(E)$ from (8) is also illustrated in Figure 1 for an arbitrary interval event $E$. For any probability measure $\mu(\cdot)$ with continuous density $\nu(\cdot)$, the probability $\mu(\bar{E}_m\cap E)$ consists of the shaded area lying within the boundaries of the interval $E$ and below the density function, which for large $m$ is seen to approach half the total area within the boundaries of $E$ and below the density function, or in other words, to approach $\frac{1}{2}\cdot\mu(E)$.

It is worth noting the following properties of almost-ethically-neutral events, which will turn out to hold for all almost-objective events: First, since $\bar{E}_m$ and $\sim\bar{E}_m$ are subsets of $\mathcal{S}$, they are completely subjective events. Second, the two key assumptions needed to generate them and their properties are that the state space $\mathcal{S}$ have a Euclidean structure and that preferences be event-smooth. Third, although the *limiting event* "$\bar{E}_\infty$" from (3) does not actually exist, event-smoothness will imply that the *limiting preference function values* in (6), (7) and (9) always exist. Fourth, although results such as (6) and (7) imply equality of $W(\cdot)$'s *limiting valuations* of the bets $[x^* \text{ on } \bar{E}_m;\, x \text{ on } \sim\bar{E}_m]$ and $[x \text{ on } \bar{E}_m;\, x^* \text{ on } \sim\bar{E}_m]$ – as measured, for example, by their limiting certainty equivalents – they yield no information on $W(\cdot)$'s *limiting ranking* of these two bets, even for arbitrarily large $m$. (We will, however, be able to obtain limiting rankings for such bets, once we consider events with *unequal* limiting likelihoods.) Finally, just as with the convergence of linear approximations or of Taylor series, the *rate* of convergence in results like (4) – (9) will generally depend upon the specifics of the measures $\mu(\cdot)$ or $\Phi_x(\cdot)$, or the function $W(\cdot)$.

The special structure of the almost-ethically-neutral events $\bar{E}_m$ and $\sim\bar{E}_m$ might suggest that they are not very relevant "in practice". In fact, they constitute a fully operational and frequently used method of generating "objective" uncertainty. For example, if the subjective state $s \in [0°, 100°]$ is the temperature at Times Square at noon tomorrow, then the events $\bar{E}_{100}$ and $\sim\bar{E}_{100}$ correspond to the temperature being even versus odd, and $\bar{E}_{1,000}$ and $\sim\bar{E}_{1,000}$ correspond to its *first decimal digit* being even versus odd, etc. At this point, virtually all bettors with event-smooth act preferences – regardless of their differing beliefs, state-dependence or ambiguity aversion – will be indifferent between betting on $\bar{E}_{1,000}$ versus $\sim\bar{E}_{1,000}$. Similar computer-generated events, based on the $k$'th decimal of the clock time at which the return key is pressed, will also generate revealed likelihoods which are unanimously viewed as equal, as well as being independent of any fixed subjective event (such as tomorrow's temperature falling in the range $[40°, 50°]$). And as seen in Poincaré's example, even the classic "objective" events of $\{$red, black$\}$ on a roulette wheel, or $\{$heads, tails$\}$ in a coin flip, are *precisely* almost-ethically-neutral events, where the subjective state $s$ is simply the physical force with which the wheel is spun, or the coin is flipped.

## 3 Almost-objective events, acts and mixtures under completely subjective uncertainty

The above ideas can be extended to a general specification of *almost-objective uncertainty* which is based solely on the subjective state space $\mathcal{S} = [\underline{s}, \bar{s}]$, the family of subjective acts $f(\cdot) \in \mathcal{A}$, and the family of event-smooth preference functions $W(\cdot)$ over $\mathcal{A}$. In the limit, it will exhibit each of the six properties of objective uncertainty listed in Section 2.2 above.[7]

### *3.1 Almost-objective events: Construction and measure properties*

Although the almost-ethically-neutral events $\bar{E}_m$ from (3) were constructed from the *left halves* of the equal-length intervals (2), they could also have been constructed from the *central halves* $[\underline{s} + (i + \frac{1}{4}) \cdot \frac{\lambda_{\mathcal{S}}}{m}, \underline{s} + (i + \frac{3}{4}) \cdot \frac{\lambda_{\mathcal{S}}}{m}]$ of these intervals, or from *any* finite interval union $\wp \subset [0, 1]$ with Lebesgue measure $\lambda(\wp) = \frac{1}{2}$. Similarly, events constructed from any $\wp \subset [0, 1]$ with $\lambda(\wp) = \frac{1}{3}$ will approximate the properties of an objective event with probability $\frac{1}{3}$, etc. For any finite

[7] Theorems 1–6 below will be numbered to correspond to these six properties.

interval union $\wp \subseteq [0,1]$ and any positive integer $m$, we define the *almost-objective event* $\wp \underset{m}{\times} \mathcal{S} \subseteq \mathcal{S}$ by

$$\wp \underset{m}{\times} \mathcal{S} \;=\; \bigcup_{i=0}^{m-1} \left\{ \underline{s} + (i+\omega)\cdot\tfrac{\lambda_{\mathcal{S}}}{m} \;\middle|\; \omega \in \wp \right\} \tag{10}$$

that is, as the union of the natural images of $\wp$ into each of $\mathcal{S}$'s equal-length intervals from (2).

For each $m$, the mapping $\wp \to \wp \underset{m}{\times} \mathcal{S}$ from subsets of $[0,1]$ to almost-objective events in $\mathcal{S}$ is seen to satisfy $\lambda(\wp \underset{m}{\times} \mathcal{S}) = \lambda(\wp)\cdot\lambda(\mathcal{S})$, as well as the following preservation properties

| | |
|---|---|
| *preservation of relative Lebesgue measure:* | $\lambda(\wp \underset{m}{\times} \mathcal{S})/\lambda(\wp' \underset{m}{\times} \mathcal{S}) \;\equiv\; \lambda(\wp)/\lambda(\wp')$ |
| *preservation of disjointness/ non-disjointness:* | $\wp \cap \wp' = \emptyset \;\Leftrightarrow\; (\wp \underset{m}{\times} \mathcal{S}) \cap (\wp' \underset{m}{\times} \mathcal{S}) = \emptyset$ (11) |
| *preservation of finite unions and intersections:* | $(\wp \cup \wp') \underset{m}{\times} \mathcal{S} \equiv (\wp \underset{m}{\times} \mathcal{S}) \cup (\wp' \underset{m}{\times} \mathcal{S})$ <br> $(\wp \cap \wp') \underset{m}{\times} \mathcal{S} \equiv (\wp \underset{m}{\times} \mathcal{S}) \cap (\wp' \underset{m}{\times} \mathcal{S})$ |

Thus, each partition $\{\wp_1,\ldots,\wp_n\}$ of $[0,1]$ induces the *almost-objective partition* of $\mathcal{S}$ defined by

$$\{\wp_1,\ldots,\wp_n\} \underset{m}{\times} \mathcal{S} \;=\; \left\{ \wp_1 \underset{m}{\times} \mathcal{S},\ldots,\wp_n \underset{m}{\times} \mathcal{S} \right\} \tag{12}$$

Given any *fixed subjective event* $E \in \mathcal{E}$, we can similarly define its *almost-objective subevents* and *almost-objective partitions* by

$$\wp \underset{m}{\times} E = \left(\wp \underset{m}{\times} \mathcal{S}\right) \cap E \quad \text{and} \quad \{\wp_1,\ldots,\wp_n\} \underset{m}{\times} E = \left\{ \wp_1 \underset{m}{\times} E,\ldots,\wp_n \underset{m}{\times} E \right\} \tag{13}$$

Although the mapping $\wp \to \wp \underset{m}{\times} E$ preserves both disjointness/non-disjointness as well as finite unions and intersections in the same manner as the mapping $\wp \to \wp \underset{m}{\times} \mathcal{S}$, standard "lumpiness" considerations imply that it will *not* exactly satisfy $\lambda(\wp \underset{m}{\times} E) = \lambda(\wp)\cdot\lambda(E)$, even when both $\wp$ and $E$ are intervals, so it does not exactly preserve relative Lebesgue measure. However, almost-objective subevents will turn out to satisfy the *limiting* form of this property, namely that $\lambda(\wp \underset{m}{\times} E)$ converges to $\lambda(\wp)\cdot\lambda(E)$ as $m \to \infty$. More generally, the limiting measure properties of *almost-ethically-neutral* events and subevents generalize to *almost-objective* events and subevents, as formalized in the following generalization of Poincaré's Theorem:

**Theorem 0 (Limiting measure properties of almost-objective events and subevents).** *For every finite interval union* $\wp \subseteq [0,1]$ *and every event* $E \in \mathcal{E}$, *the almost-objective events* $\wp \underset{m}{\times} \mathcal{S}$ *and subevents* $\wp \underset{m}{\times} E = (\wp \underset{m}{\times} \mathcal{S}) \cap E$ *satisfy*

$$\lim_{m\to\infty} K\left(\wp \underset{m}{\times} \mathcal{S}\right) = \lambda(\wp)\cdot K(\mathcal{S}) \quad \text{and} \quad \lim_{m\to\infty} K\left(\wp \underset{m}{\times} E\right) = \lambda(\wp)\cdot K(E) \tag{14}$$

*for every continuous-density signed measure* $K(\cdot)$ *over* $\mathcal{S}$. *For any family of signed measures* $\{K(\cdot;\tau)\,|\,\tau \in T\}$ *over* $\mathcal{S}$ *with uniformly bounded and uniformly continuous densities, such convergence will be uniform.*

Since almost-objective events $\wp \underset{m}{\times} \mathcal{S}$ and subevents $\wp \underset{m}{\times} E$ are subsets of the original subjective state space $\mathcal{S}$, they can be used to define "almost-objective" acts within the original subjective act space $\mathcal{A}$, as well as "almost-objective" *mixtures* of arbitrary subjective acts, which will also be elements of $\mathcal{A}$. For any partition $\{\wp_1, \ldots, \wp_n\}$ of $[0, 1]$ (where each $\wp_i$ is a finite interval union) and collection of outcomes $x_1, \ldots, x_n \in \mathcal{X}$, we define the *almost-objective act*

$$\left[x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\right] \in \mathcal{A} \tag{15}$$

For any family of subjective acts $f_1(\cdot), \ldots, f_n(\cdot) \in \mathcal{A}$, we define the *almost-objective mixture*

$$\left[f_1(\cdot) \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; f_n(\cdot) \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\right] \in \mathcal{A} \tag{16}$$

which is seen to yield outcome $x$ on the event $\wp_1 \underset{m}{\times} f_1^{-1}(x) \cup \cdots \cup \wp_n \underset{m}{\times} f_n^{-1}(x)$.

### 3.2 Revealed beliefs over almost-objective events

*Property 1: Unanimous, outcome-invariant limiting revealed likelihoods.* In the temperature example, it is reasonable to suppose that most individuals, regardless of their beliefs, and even if their preferences are temperature-dependent, would be indifferent between staking a prize on the almost-objective events $[.5, .6)_{10,000}^{\times}\mathcal{S}$ versus $[.6, .7)_{10,000}^{\times}\mathcal{S}$ (i.e., on the second decimal of the temperature being a 5 versus a 6), and would prefer staking it on the union of these events than on the event $[.8, .9)_{10,000}^{\times}\mathcal{S}$. That is, as $m \to \infty$ individuals exhibit unanimous revealed likelihoods for the almost-objective events $\wp \underset{m}{\times} \mathcal{S}$, corresponding to the value $\lambda(\wp)$. Formally, we have

**Theorem 1 (Unanimous, outcome-invariant limiting revealed likelihoods).** *For all disjoint finite interval unions $\wp, \wp' \subseteq [0, 1]$ with $\lambda(\wp) > (=) \lambda(\wp')$, the almost-objective events $\wp \underset{m}{\times} \mathcal{S}$ and $\wp' \underset{m}{\times} \mathcal{S}$ satisfy*

$$\lim_{m\to\infty} W\begin{pmatrix} x^* & \text{on } \wp \underset{m}{\times} \mathcal{S} \\ x & \text{on } \wp' \underset{m}{\times} \mathcal{S} \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \underset{\substack{\text{all } x^* \succ x \\ \text{all } f(\cdot) \in \mathcal{A}}}{> (=)} \lim_{m\to\infty} W\begin{pmatrix} x & \text{on } \wp \underset{m}{\times} \mathcal{S} \\ x^* & \text{on } \wp' \underset{m}{\times} \mathcal{S} \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \tag{17}$$

*for each event-smooth, outcome-monotonic preference function $W(\cdot)$ over $\mathcal{A}$.*

Thus whenever $\lambda(\wp) > \lambda(\wp')$, for any outcomes $x^* \succ x$ and any subjective act $f(\cdot)$, the left act in (17) will be strictly preferred to the right act for all sufficiently large $m$. Whenever $\lambda(\wp) = \lambda(\wp')$, all event-continuous *equivalency measures*[8] of these two acts will approach equality as $m \to \infty$.

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[8] When the outcome space $\mathcal{X}$ is a continuum, a standard equivalency measure of $f(\cdot)$ is its *certainty equivalent*. If $\mathcal{X}$ is not a continuum but has a most and least preferred outcome $\bar{x}$ and $\underline{x}$, another equivalency measure would be any *event-equivalent*, such as the state $s_{f(\cdot)} \in \mathcal{S}$ that solves $W(\bar{x}$ on $[\underline{s}, s_{f(\cdot)}]$; $\underline{x}$ on $(s_{f(\cdot)}, \bar{s}]) = W(f(\cdot))$.

*Property 2: Limiting independence from other subjective events.* The limiting revealed likelihoods of almost-objective events also exhibit the objective property of being *independent of the realization of any fixed subjective event $E$*, in the same sense in which the probability of an Anscombe-Aumann objective roulette wheel event is independent of the outcome of their subjective horse race. Similarly, an individual's preference for betting on the second decimal of the temperature being a 5 or 6, versus it being an 8, will continue to hold even if the bet is made conditional on the temperature *also* being at least 60°. Even though individuals may differ in their subjective probabilities of the conditioning event $E = [60°, 100°]$, some may consider it ambiguous, and others may have temperature-dependent utility, virtually everyone with event-smooth preferences who views $E$ as nonnull will view the pair of *joint events* $\{$second decimal of $s$ is 5 or 6$\} \cap E$ versus $\{$second decimal of $s$ is 8$\} \cap E$ as having the same likelihood ratio as the pair of *unconditional events* $\{$second decimal of $s = 5$ or 6$\}$ versus $\{$second decimal of $s = 8\}$, namely, a likelihood ratio of 2:1. Formally, we have

**Theorem 2 (Limiting independence of almost-objective likelihoods from subjective realizations).** *For all disjoint finite interval unions $\wp, \wp' \subseteq [0,1]$ with $\lambda(\wp) > (=)\, \lambda(\wp')$ and nonnull events $E \in \mathcal{E}$, the events $(\wp \underset{m}{\times} \mathcal{S}) \cap E$ and $(\wp' \underset{m}{\times} \mathcal{S}) \cap E$ satisfy*

$$\lim_{m\to\infty} W\begin{pmatrix} x^* & \text{on } (\wp \underset{m}{\times} \mathcal{S}) \cap E \\ x & \text{on } (\wp' \underset{m}{\times} \mathcal{S}) \cap E \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \underset{\substack{\text{all } x^* \succ x \\ \text{all } f(\cdot) \in \mathcal{A}}}{>(=)} \lim_{m\to\infty} W\begin{pmatrix} x & \text{on } (\wp \underset{m}{\times} \mathcal{S}) \cap E \\ x^* & \text{on } (\wp' \underset{m}{\times} \mathcal{S}) \cap E \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \quad (18)$$

*for each event-smooth, outcome-monotonic preference function $W(\cdot)$ over $\mathcal{A}$.*

### *3.3 Betting preferences over almost-objective acts and mixtures*

*Property 3: Limiting probabilistic sophistication over almost-objective acts.* Given that as $m \to \infty$ all event-smooth $W(\cdot)$ exhibit unanimous, outcome-invariant revealed likelihoods for *almost-objective events*, it is no surprise that as $m \to \infty$ they will also exhibit probabilistic sophistication over *almost-objective acts*. That is, each event-smooth $W(\cdot)$ on $\mathcal{A}$ has an associated preference function $V_W(\cdot)$ over objective lotteries, such that $W(\cdot)$'s limiting evaluation of any almost-objective act $f(\cdot) = [x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}]$ is determined solely by $V_W(\cdot)$'s evaluation of its outcomes $x_1, \ldots, x_n$ and their limiting likelihoods $\lambda(\wp_1), \ldots, \lambda(\wp_n)$. Formally, we have:

**Theorem 3 (Limiting probabilistic sophistication over almost-objective acts).** *For each event-smooth, outcome-monotonic preference function $W(\cdot)$ over $\mathcal{A}$ there exists a preference function $V_W(\cdot)$ over objective lotteries, satisfying strict first order stochastic dominance preference, such that*

$$\begin{aligned} &\lim_{m\to\infty} W\big(x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S};\ \ldots\ ; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\big) \\ &\qquad \equiv\ V_W\big(x_1, \lambda(\wp_1); \ldots; x_n, \lambda(\wp_n)\big) \end{aligned} \quad (19)$$

*for all almost-objective acts in $\mathcal{A}$.*

Thus, if two partitions $\{\wp_1,\ldots,\wp_n\}$ and $\{\wp'_1,\ldots,\wp'_n\}$ of $[0,1]$ satisfy $\lambda(\wp_i) = \lambda(\wp'_i)$ for each $i$, then as $m, m' \to \infty$ every event-smooth $W(\cdot)$ will approach indifference between the almost-objective acts $[x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S};\ldots;x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}]$ and $[x_1 \text{ on } \wp'_1 \underset{m'}{\times} \mathcal{S};\ldots;x_n \text{ on } \wp'_n \underset{m'}{\times} \mathcal{S}]$, and we refer to such pairs as *probabilistically equivalent* almost-objective acts. The role of $V_W(\cdot)$ as a complete summary and predictor of $W(\cdot)$'s "objective risk preferences" is explored further in Section 4.2 below.

*Property 4: Reduction of almost-objective×subjective uncertainty.* Even though they are completely subjective acts, the mixtures $[f_1(\cdot) \text{ on } \wp_1 \underset{m}{\times} \mathcal{S};\ldots;f_n(\cdot) \text{ on } \wp_n \underset{m}{\times} \mathcal{S}]$ from (16) are seen to be the almost-objective analogues of classic Anscombe-Aumann *roulette/horse lotteries* $(f_1(\cdot),p_1;\ldots;f_n(\cdot),p_n)$, which can be viewed as objective mixtures of the subjective acts $f_1(\cdot),\ldots,f_n(\cdot)$. Anscombe and Aumann also considered *horse/roulette lotteries* $[\mathbf{P}_1 \text{ on } E_1;\ldots;\mathbf{P}_n \text{ on } E_n]$, which can be viewed as subjective mixtures of the objective lotteries $\mathbf{P}_1,\ldots,\mathbf{P}_n$, or equivalently, as mappings from subjective events to objective lotteries. As noted in Section 2.2, researchers have postulated that preferences across these distinct types of prospects are linked by the *reduction of compound uncertainty principle*, which states that all prospects that imply the same mapping from events to lotteries will be viewed as indifferent.

By definition, each horse/roulette lottery $[\mathbf{P}_1 \text{ on } E_1;\ldots;\mathbf{P}_n \text{ on } E_n]$ already is a mapping from events to lotteries. For each roulette/horse lottery $(f_1(\cdot),p_1;\ldots;f_n(\cdot),p_n)$, its implied mapping from events to lotteries can be derived by expressing its component acts $f_1(\cdot),\ldots,f_n(\cdot)$ in terms of the *common refinement* $\{E^*_1,\ldots,E^*_K\}$ of their respective underlying partitions – that is, by writing

$$\begin{aligned} f_1(\cdot) &= \big[x_{1,1} \text{ on } E^*_1;\ldots;x_{1,K} \text{ on } E^*_K\big] \\ &\ \vdots \\ f_n(\cdot) &= \big[x_{n,1} \text{ on } E^*_1;\ldots;x_{n,K} \text{ on } E^*_K\big] \end{aligned} \tag{20}$$

so we can write the roulette/horse lottery $(f_1(\cdot),p_1;\ldots;f_n(\cdot),p_n)$ as

$$\begin{aligned} \big([x_{1,1} \text{ on } E^*_1;\ldots;x_{1,K} \text{ on } E^*_K],p_1\,;\ldots \\ \ldots\,;\,[x_{n,1} \text{ on } E^*_1;\ldots;x_{n,K} \text{ on } E^*_K],p_n\big) \end{aligned} \tag{21}$$

and its implied mapping from events to lotteries is accordingly

$$\big[(x_{1,1},p_1;\ldots;x_{n,1},p_n)\text{ on }E_1^*;\ldots;(x_{1,K},p_1;\ldots;x_{n,K},p_n)\text{ on }E_K^*\big] \quad (22)$$

The reduction of compound uncertainty principle thus asserts that the original roulette/horse lottery $(f_1(\cdot),p_1;\ldots;f_n(\cdot),p_n)$ would be viewed as indifferent to its implied horse/roulette lottery (22), and furthermore, that *all other* roulette/horse lotteries that yield the same (or probabilistically equivalent) implied mappings from events to lotteries would also be viewed as indifferent.

In an Anscombe-Aumann setting of objective×subjective uncertainty, it is necessary to *exogenously impose* the reduction of compound uncertainty principle as an additional restriction on preferences. However in our present completely subjective framework, as $m\to\infty$ every event-smooth preference function $W(\cdot)$ will *inherently exhibit* the reduction principle for almost-objective×subjective uncertainty:

Given an almost-objective mixture $[f_1(\cdot)\text{ on }\wp_1\underset{m}{\times}\mathcal{S};\ldots;f_n(\cdot)\text{ on }\wp_n\underset{m}{\times}\mathcal{S}]$ of subjective acts $f_1(\cdot),\ldots,f_n(\cdot)$, we can derive its implied mapping from events to almost-objective acts by again expressing $f_1(\cdot),\ldots,f_n(\cdot)$ as in (20), so we can write the mixture $[f_1(\cdot)\text{ on }\wp_1\underset{m}{\times}\mathcal{S};\ldots;f_n(\cdot)\text{ on }\wp_n\underset{m}{\times}\mathcal{S}]$ as

$$\begin{aligned}
&\big[x_{1,1}\text{ on }\wp_1\underset{m}{\times}E_1^*;\ldots;x_{1,K}\text{ on }\wp_1\underset{m}{\times}E_K^*;\ldots\ldots;\\
&\qquad\qquad x_{n,1}\text{ on }\wp_n\underset{m}{\times}E_1^*;\ldots;x_{n,K}\text{ on }\wp_n\underset{m}{\times}E_K^*\big]\\
=\ &\big[x_{1,1}\text{ on }\wp_1\underset{m}{\times}E_1^*;\ldots;x_{n,1}\text{ on }\wp_n\underset{m}{\times}E_1^*;\ldots\ldots;\\
&\qquad\qquad x_{1,K}\text{ on }\wp_1\underset{m}{\times}E_K^*;\ldots;x_{n,K}\text{ on }\wp_n\underset{m}{\times}E_K^*\big]\\
=\ &\big[[x_{1,1}\text{ on }\wp_1\underset{m}{\times}\mathcal{S};\ldots;x_{n,1}\text{ on }\wp_n\underset{m}{\times}\mathcal{S}]\text{ on }E_1^*;\ldots\ldots;\\
&\qquad\qquad [x_{1,K}\text{ on }\wp_1\underset{m}{\times}\mathcal{S};\ldots;x_{n,K}\text{ on }\wp_n\underset{m}{\times}\mathcal{S}]\text{ on }E_K^*\big]
\end{aligned} \quad (23)$$

The final expression in (23) is seen to be the almost-objective analogue of the implied event-to-lottery mapping (22), with the same subjective partition $\{E_1^*,\ldots,E_K^*\}$, and with (22)'s event-contingent *objective lotteries* $(x_{1,1},p_1;\ldots;x_{n,1},p_n)$ … $(x_{1,K},p_1;\ldots;x_{n,K},p_n)$ replaced by (23)'s event-contingent *almost-objective acts* $[x_{1,1}\text{ on }\wp_1\underset{m}{\times}\mathcal{S};\ldots;x_{n,1}\text{ on }\wp_n\underset{m}{\times}\mathcal{S}]$ … $[x_{1,K}\text{ on }\wp_1\underset{m}{\times}\mathcal{S};\ldots;x_{n,K}\text{ on }\wp_n\underset{m}{\times}\mathcal{S}]$. Under objective×subjective uncertainty, the original roulette/horse lottery $(f_1(\cdot),p_1;\ldots;f_n(\cdot),p_n)$ and its implied mapping (22) (which is a horse/roulette lottery) are distinct types of prospects, so any reduction principle ensuring their indifference must be exogenously imposed. But, in the present purely subjective framework, the final item in (23) is simply a *reexpression* of the original almost-objective mixture $[f_1(\cdot)\text{ on }\wp_1\underset{m}{\times}\mathcal{S};\ldots;f_n(\cdot)\text{ on }\wp_n\underset{m}{\times}\mathcal{S}]$, and is thus automatically indifferent to it. The following result shows that as $m\to\infty$, *all* almost-objective mixtures that imply probabilistically equivalent mappings from events to almost-objective acts will be viewed as indifferent by each event-smooth $W(\cdot)$ – again without the need to impose any additional exogenous reduction condition:

**Theorem 4 (Reduction of almost-objective×subjective uncertainty).** *If the almost-objective mixtures* $[f_1(\cdot)$ on $\wp_1 \underset{m}{\times} \mathcal{S}; \ldots; f_n(\cdot)$ on $\wp_n \underset{m}{\times} \mathcal{S}]$ *and* $[\hat{f}_1(\cdot)$ on $\hat{\wp}_1 \underset{m}{\times} \mathcal{S}; \ldots; \hat{f}_{\hat{n}}(\cdot)$ on $\hat{\wp}_{\hat{n}} \underset{m}{\times} \mathcal{S}]$ *imply probabilistically equivalent almost-objective acts over each event in the common refinement of* $\{f_1(\cdot), \ldots, f_n(\cdot), \hat{f}_1(\cdot), \ldots, \hat{f}_{\hat{n}}(\cdot)\}$, *then*

$$\begin{aligned} &\lim_{m\to\infty} W\big(f_1(\cdot) \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; f_n(\cdot) \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\big) \\ &= \lim_{m\to\infty} W\big(\hat{f}_1(\cdot) \text{ on } \hat{\wp}_1 \underset{m}{\times} \mathcal{S}; \ldots; \hat{f}_{\hat{n}}(\cdot) \text{ on } \hat{\wp}_{\hat{n}} \underset{m}{\times} \mathcal{S}\big) \end{aligned} \tag{24}$$

*for each event-smooth, outcome-monotonic preference function* $W(\cdot)$ *over* $\mathcal{A}$.

*Property 5: Under probabilistic sophistication, limiting independence of almost-objective and subjective events.* Although Theorem 2 demonstrates that the limiting revealed likelihood ranking of a pair of almost-objective events $\wp \underset{m}{\times} \mathcal{S}$, $\wp' \underset{m}{\times} \mathcal{S}$ is invariant to whether or not they are conditioned on a fixed subjective event $E$, the opposite is generally *not* true. That is, an event-smooth preference function $W(\cdot)$ could prefer to bet on a subjective event $E$ versus $E'$, yet reverse these preferences when they are conditioned on some almost-objective event $\wp \underset{m}{\times} \mathcal{S}$, even for large $m$. The reason is that without any additional regularity on beliefs, properties such as ambiguity or state-dependence allow practically any configuration of betting preferences over arbitrary subjective events.

However, for individuals whose subjective beliefs *are* regular enough to be represented by well-defined subjective probabilities – that is, whose preference functions take the probabilistically sophisticated form $W_{PS}(f(\cdot)) \equiv V(x_1,\mu(E_1); \ldots; x_n,\mu(E_n))$ – almost-objective and subjective events will exhibit the limiting property of being *mutually independent*. Thus, in addition to the first of the following two properties, which by Theorem 2 is already displayed by every event-smooth $W(\cdot)$, every event-smooth $W_{PS}(\cdot)$ will also exhibit the second property:

- for any pair of almost-objective events $\wp \underset{m}{\times} \mathcal{S}$, $\wp' \underset{m}{\times} \mathcal{S}$ and fixed nonnull subjective event $E \in \mathcal{E}$, the limiting revealed likelihood ratio for the joint events $(\wp \underset{m}{\times} \mathcal{S}) \cap E$ versus $(\wp' \underset{m}{\times} \mathcal{S}) \cap E$ is the same as for $\wp \underset{m}{\times} \mathcal{S}$ versus $\wp' \underset{m}{\times} \mathcal{S}$ – namely $\lambda(\wp) : \lambda(\wp')$
- for any pair of fixed subjective events $E$, $E' \in \mathcal{E}$ and almost-objective event $\wp \underset{m}{\times} \mathcal{S}$ with $\lambda(\wp) > 0$, the limiting revealed likelihood ratio for the joint events $(\wp \underset{m}{\times} \mathcal{S}) \cap E$ versus $(\wp \underset{m}{\times} \mathcal{S}) \cap E'$ is the same as for $E$ versus $E'$ – namely $\mu(E) : \mu(E')$

The following result subsumes these two properties into the general "limiting independence" property that a probabilistically sophisticated individual's limiting likelihood of any joint event $(\wp \underset{m}{\times} \mathcal{S}) \cap E$ is given by the unanimously-held limiting likelihood $\lambda(\wp)$ of the almost-objective event $\wp \underset{m}{\times} \mathcal{S}$, multiplied by their own subjective probability $\mu(E)$ of the subjective event $E$:

**Theorem 5 (Under probabilistic sophistication, limiting independence of almost-objective and subjective likelihoods).** *For each event-smooth, outcome-monotonic, probabilistically sophisticated preference function* $W_{PS}(\cdot)$ *over* $\mathcal{A}$*, if the almost-objective events* $\wp \underset{m}{\times} \mathcal{S}$, $\wp' \underset{m}{\times} \mathcal{S}$ *and disjoint subjective events* $E, E' \in \mathcal{E}$ *satisfy* $\lambda(\wp) \cdot \mu(E) > (=)\ \lambda(\wp') \cdot \mu(E')$*, then the joint events* $(\wp \underset{m}{\times} \mathcal{S}) \cap E$ *and* $(\wp' \underset{m}{\times} \mathcal{S}) \cap E'$ *satisfy*

$$\lim_{m\to\infty} W_{PS}\begin{pmatrix} x^* & \text{on } (\wp \underset{m}{\times} \mathcal{S}) \cap E \\ x & \text{on } (\wp' \underset{m}{\times} \mathcal{S}) \cap E' \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \underset{\substack{\text{all } x^* \succ x \\ \text{all } f(\cdot) \in \mathcal{A}}}{> (=)} \lim_{m\to\infty} W_{PS}\begin{pmatrix} x & \text{on } (\wp \underset{m}{\times} \mathcal{S}) \cap E \\ x^* & \text{on } (\wp' \underset{m}{\times} \mathcal{S}) \cap E' \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \tag{25}$$

*Property 6: Under expected utility, limiting linearity in almost-objective likelihoods and mixtures.* Expected utility is linear in objective probabilities, and both state-independent and state-dependent expected utility are linear in objective mixtures of subjective acts. But whereas state-independent expected utility $W_{SEU}(f(\cdot)) \equiv \sum_{i=1}^{n} U(x_i) \cdot \mu(E_i)$ is also linear in the *subjective* probabilities $\mu(E_1), \ldots, \mu(E_n)$, this is generally *not* true of state-dependent expected utility $W_{SDEU}(f(\cdot)) \equiv \sum_{i=1}^{n} \int_{E_i} U(x_i|s) \cdot d\mu(s)$. That is, even if $W_{SDEU}(\cdot)$'s subjective probabilities for the acts $f'(\cdot) = [x_1 \text{ on } E'_1; \ldots; x_n \text{ on } E'_n]$, $f''(\cdot) = [x_1 \text{ on } E''_1; \ldots; x_n \text{ on } E''_n]$ and $f'''_m(\cdot) = [x_1 \text{ on } E'''_1; \ldots; x_n \text{ on } E'''_n]$ should satisfy $\mu(E''_i) = \alpha \cdot \mu(E'_i) + (1-\alpha) \cdot \mu(E'''_i)$ for each $i$, its state-dependence will typically imply

$$W_{SDEU}\big(f''(\cdot)\big) \neq \alpha \cdot W_{SDEU}\big(f'(\cdot)\big) + (1-\alpha) \cdot W_{SDEU}\big(f'''(\cdot)\big) \tag{26}$$

But even though almost-objective events are subsets of the subjective state space $\mathcal{S}$, their properties in this regard are more similar to those of objective events than to regular subjective events. Thus, if the limiting event likelihoods in the almost-objective acts $f'_m(\cdot) = [x_1 \text{ on } \wp'_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp'_n \underset{m}{\times} \mathcal{S}]$, $f''_m(\cdot) = [x_1 \text{ on } \wp''_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp''_n \underset{m}{\times} \mathcal{S}]$ and $f'''_m(\cdot) = [x_1 \text{ on } \wp'''_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp'''_n \underset{m}{\times} \mathcal{S}]$ satisfy $\lambda(\wp''_i) = \alpha \cdot \lambda(\wp'_i) + (1-\alpha) \cdot \lambda(\wp'''_i)$ for each $i$, then every event-smooth state-independent *and* state-dependent expected utility preference function will exhibit the limiting equalities

$$\begin{aligned} \lim_{m\to\infty} W_{SEU}\big(f''_m(\cdot)\big) &= \\ &\alpha \cdot \lim_{m\to\infty} W_{SEU}\big(f'_m(\cdot)\big) + (1-\alpha) \cdot \lim_{m\to\infty} W_{SEU}\big(f'''_m(\cdot)\big) \\ \lim_{m\to\infty} W_{SDEU}\big(f''_m(\cdot)\big) &= \\ &\alpha \cdot \lim_{m\to\infty} W_{SDEU}\big(f'_m(\cdot)\big) + (1-\alpha) \cdot \lim_{m\to\infty} W_{SDEU}\big(f'''_m(\cdot)\big) \end{aligned} \tag{27}$$

Formally, we have:

**Theorem 6 (Under expected utility, limiting linearity in almost-objective likelihoods and mixtures).** *For each event-smooth, outcome-monotonic expected utility preference function* $W_{SEU}(f(\cdot)) \equiv \int_S U(f(s))\cdot d\mu(s)$ *or* $W_{SDEU}(f(\cdot)) \equiv \int_S U(f(s)|s)\cdot d\mu(s)$ *over* $\mathcal{A}$*, preferences over almost-objective acts satisfy*

$$\lim_{m\to\infty} W_{SEU}\big(x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots ; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\big) \equiv \sum_{i=1}^{n} \lambda(\wp_i)\cdot U(x_i)$$

$$\lim_{m\to\infty} W_{SDEU}\big(x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots ; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\big) \equiv \sum_{i=1}^{n} \lambda(\wp_i)\cdot \int_{\mathcal{S}} U(x_i|s)\cdot d\mu(s) \qquad (28)$$

*and preferences over almost-objective mixtures of purely subjective acts satisfy*

$$\lim_{m\to\infty} W_{SEU}\left(f_1(\cdot) \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots ; f_n(\cdot) \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\right) \equiv \sum\nolimits_{i=1}^{n} \lambda(\wp_i)\cdot \int_{\mathcal{S}} U(f_i(s))\cdot d\mu(s)$$

$$\lim_{m\to\infty} W_{SDEU}\left(f_1(\cdot) \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots ; f_n(\cdot) \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\right) \equiv \sum\nolimits_{i=1}^{n} \lambda(\wp_i)\cdot \int_{\mathcal{S}} U(f_i(s)|s)\cdot d\mu(s) \qquad (29)$$

### *3.4 Ordinal implications of limiting preference function values*

Since almost-objective events $\wp \underset{m}{\times} \mathcal{S}$ and almost-objective acts $f_m(\cdot) = [x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots ; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}]$ do not exist at $m = \infty$, we have used their *limiting preference function values* $\lim_{m\to\infty} W(f_m(\cdot))$ – which under event-smoothness *do* exist – to represent the individual's limiting beliefs and preferences toward such objects. But since the limiting objects themselves do not exist, it is worth clarifying what these limiting preference function values do and do not imply about the individual's *limiting ordinal rankings* of almost-objective acts, both with respect to each other and with respect to other subjective acts in $\mathcal{A}$.

*Taken individually*, each limiting preference function value $w = \lim_{m\to\infty} W(f_m(\cdot))$ corresponds to a unique, nonempty indifference class $I_w$ of subjective acts in $\mathcal{A}$. To see this, let $\underline{x}$ and $\bar{x}$ be the least and most preferred outcomes in $\{x_1, \ldots, x_n\}$, so outcome monotonicity implies $W(\bar{x} \text{ on } \mathcal{S}) \geq w \geq W(\underline{x} \text{ on } \mathcal{S})$, and event-continuity implies some state $s_w \in \mathcal{S}$ such that the act $f_w(\cdot) = [\bar{x} \text{ on } [\underline{s}, s_w]; \underline{x} \text{ on } (s_w, \bar{s}]]$ satisfies $W(f_w(\cdot)) = w$. But since $W(f_m(\cdot))$ could converge to the value $w$ from above, from below, or by damped oscillation, the almost-objective acts $f_m(\cdot)$ needn't converge to a stable preference ranking with respect to $f_w(\cdot)$ or any other act in the class $I_w$. However, the acts $f_m(\cdot)$ *do* converge to a stable ranking with respect to *every other act* in $\mathcal{A}$. That is, for every act $f(\cdot) \in \mathcal{A} - I_w$, $\lim_{m\to\infty} W(f_m(\cdot)) > (<)\ W(f(\cdot))$ will imply $f_m(\cdot) \succ (\prec)\ f(\cdot)$ for all sufficiently large $m$.

*Taken in pairwise comparisons*, if the limiting preference function values for the acts $f_m(\cdot) = [x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots ; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}]$ and $f^*_m(\cdot) = [x^*_1 \text{ on } \wp^*_1 \underset{m}{\times} \mathcal{S}; \ldots ; x_{n*}$

on $\wp^*_{n^*} \underset{m}{\times} \mathcal{S}]$ satisfy the *inequality* $\lim_{m\to\infty} W(f^*_m(\cdot)) > \lim_{m\to\infty} W(f_m(\cdot))$, then $f^*_m(\cdot) \succ f_m(\cdot)$ for all sufficiently large $m$. Although the *equality* $\lim_{m\to\infty} W(f^*_m(\cdot)) = \lim_{m\to\infty} W(f_m(\cdot)) = w$ needn't imply a stable limiting ranking of $f^*_m(\cdot)$ versus $f_m(\cdot)$ (since $W(f^*_m(\cdot)) - W(f_m(\cdot))$ could approach zero from above, below, or by oscillation), it does imply that $f^*_m(\cdot)$ and $f_m(\cdot)$ both converge to the *same* limiting ranking with respect to every act $f(\cdot) \in \mathcal{A} - I_w$.

*Taken collectively*, the family of limiting values $\lim_{m\to\infty} W(x_1$ on $\wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n$ on $\wp_n \underset{m}{\times} \mathcal{S})$ for all almost-objective acts in $\mathcal{A}$, or the family of limiting preference function values for all almost-objective×subjective acts in $\mathcal{A}$, can be used to obtain exact, global results for *all* subjective acts in $\mathcal{A}$, as seen in Theorem 7, where a condition on these limiting values is shown to imply exact probabilistic sophistication over all subjective acts in $\mathcal{A}$. As outlined in Section 4.3, additional results linking these limiting preference function values to exact global properties of preferences over purely subjective acts, including characterizations of comparative subjective likelihood, relative subjective likelihood, and comparative risk aversion, are given in Machina (2002).

## 4 Applications

Since almost-objective events, acts and mixtures have essentially the same properties as their objective counterparts, their advantage does not lie in generating *new* applications, but in extending existing applications of objective uncertainty to completely subjective settings. This is similar to the analyses of Savage (1954) and Anscombe and Aumann (1963), which also did not generate any new applications of probability theory, but rather, extended it to subjective settings.[9]

### *4.1 Anscombe-Aumann without objective uncertainty*

In their classic paper, Anscombe and Aumann (1963) provided a characterization of probabilistic sophistication that was much simpler and more direct than that of Savage (1954). Their three formal assumptions were: existence of exogenous objective uncertainty (an objective×subjective framework), "independence of order of resolution" of the objective and subjective uncertainty, and expected utility risk preferences. Machina and Schmeidler (1995) retained the objective×subjective framework and independence of order of resolution, but showed that the expected utility hypothesis could be replaced by a weaker property on act preferences, as

[9] In addition to the following applications, Joel Watson has observed that almost-objective events can be used to approximate *purified strategies* (e.g., Harsanyi, 1973) in games when players do not have common beliefs, or even knowledge of each other's beliefs: In a game of matching pennies, if Player I announces the strategy [Heads on $[0,\frac{1}{2}] \underset{m}{\times} \mathcal{S}$; Tails on $(\frac{1}{2},1] \underset{m}{\times} \mathcal{S}$] and Player II announces [Heads on $([0,\frac{1}{4}] \cup (\frac{3}{4},1]) \underset{m}{\times} \mathcal{S}$; Tails on $(\frac{1}{4},\frac{3}{4}] \underset{m}{\times} \mathcal{S}$], then every continuous-density subjective probability measure on $\mathcal{S}$ will assign approximately $\frac{1}{4}$ probability to each combination of plays. Eddie Dekel has observed that a similar construction can approximate *correlated equilibria*, and Quiggin (2002) provides an application of almost-objective events to the value of information under ambiguity.

described below. By replacing objective×subjective uncertainty (that is, the combination of a roulette wheel and a horse race) with almost-objective×subjective uncertainty over a single subjective state space $\mathcal{S} = [\underline{s}, \bar{s}]$, we can obtain a direct Anscombe-Aumann type characterization of exact probabilistic sophistication without *any* of their original three assumptions – that is, without exogenous objective uncertainty, independence of order of resolution,[10] or expected utility risk preferences.

Machina and Schmeidler (1995) obtained probabilistic sophistication under objective×subjective uncertainty by replacing the expected utility hypothesis with the following property:

**Horse/Roulette Replacement Axiom.** *For any partition* $\{E_1, \ldots, E_n\}$ *of* $\mathcal{S}$, *if*

$$\begin{bmatrix} x^* \text{ on } E_i \\ x \text{ on } E_j \\ x \text{ on } E_k \;\; k \neq i,j \end{bmatrix} \sim \begin{bmatrix} (x^*, \alpha; x, 1-\alpha) \text{ on } E_i \\ (x^*, \alpha; x, 1-\alpha) \text{ on } E_j \\ x \text{ on } E_k \;\; k \neq i,j \end{bmatrix} \tag{30}$$

*for some outcomes* $x^* \succ x$, *probability* $\alpha \in [0,1]$, *and pair of events* $E_i$ *and* $E_j$, *then*

$$\begin{bmatrix} \mathbf{P}_i \text{ on } E_i \\ \mathbf{P}_j \text{ on } E_j \\ \mathbf{P}_k \text{ on } E_k \;\; k \neq i,j \end{bmatrix} \sim \begin{bmatrix} (\mathbf{P}_i, \alpha; \mathbf{P}_j, 1-\alpha) \text{ on } E_i \\ (\mathbf{P}_i, \alpha; \mathbf{P}_j, 1-\alpha) \text{ on } E_j \\ \mathbf{P}_k \text{ on } E_k \;\; k \neq i,j \end{bmatrix} \tag{31}$$

*for all objective lotteries* $\mathbf{P}_1, \ldots, \mathbf{P}_n$.

This axiom states that the ratio $\alpha : (1-\alpha)$ at which the individual is willing to replace a pair of distinct prizes (either pure outcomes or objective lotteries) over the events $E_i$ and $E_j$ with an objective probability mixture of these same prizes over $E_i \cup E_j$ will not depend upon the specific prizes involved, or upon the prize assigned to any other event $E_k$. Of course, the ratio $\alpha : (1-\alpha)$ will turn out to equal the individual's subjective probability ratio $\mu(E_i) : \mu(E_j)$ for the two events.

To convert this axiom from objective×subjective uncertainty to almost-objective×subjective uncertainty over a single state space $\mathcal{S} = [\underline{s}, \bar{s}]$, we replace its objective lotteries $(x^*, \alpha; x, 1-\alpha)$ by the corresponding almost-objective acts $[x^* \text{ on } [0,\alpha] \underset{m}{\times} \mathcal{S}; x \text{ on } (\alpha, 1] \underset{m}{\times} \mathcal{S}]$, its objective lotteries $\mathbf{P}_i = (x_{i,1}, p_{i,1}; \ldots; x_{i,K_i}, p_{i,K_i})$ by $f^i_m(\cdot) = [x_{i,1} \text{ on } \wp_{i,1} \underset{m}{\times} \mathcal{S}; \ldots; x_{i,K_i} \text{ on } \wp_{i,K_i} \underset{m}{\times} \mathcal{S}]$, and its mixtures $(\mathbf{P}_i, \alpha; \mathbf{P}_j, 1-\alpha)$ $= (x_{i,1}, \alpha \cdot p_{i,1}; \ldots; x_{i,K_i}, \alpha \cdot p_{i,K_i}; x_{j,1}, (1-\alpha) \cdot p_{j,1}; \ldots; x_{j,K_j}, (1-\alpha) \cdot p_{j,K_j})$ by the mixtures $\alpha \cdot f^i_m(\cdot) \oplus (1-\alpha) \cdot f^j_m(\cdot) \equiv [x_{i,1} \text{ on } (\wp_{i,1} \underset{1}{\times} [0,\alpha]) \underset{m}{\times} \mathcal{S}; \ldots; x_{i,K_i}$

[10] Independence of order of resolution is not needed since almost-objective×subjective acts are completely subjective and hence just involve a single stage of uncertainty, namely the realization of the subjective state $s \in \mathcal{S}$.

on $(\wp_{i,K_i} \underset{1}{\times} [0,\alpha]) \underset{m}{\times} \mathcal{S}; x_{j,1}$ on $(\wp_{j,1} \underset{1}{\times} (\alpha,1]) \underset{m}{\times} \mathcal{S}; \ldots ; x_{j,K_j}$ on $(\wp_{j,K_j} \underset{1}{\times} (\alpha,1]) \underset{m}{\times} \mathcal{S}]$, to obtain:[11]

**Almost-Objective/Subjective Replacement Axiom.** *For any partition $\{E_1,\ldots,E_n\}$ of $\mathcal{S}$, if*

$$W\begin{pmatrix} x^* \text{ on } E_i \\ x \text{ on } E_j \\ x \text{ on } E_k \;\; k \neq i,j \end{pmatrix} = \lim_{m\to\infty} W\begin{pmatrix} \left[x^* \text{ on } [0,\alpha] \underset{m}{\times} \mathcal{S}; x \text{ on } (\alpha,1] \underset{m}{\times} \mathcal{S}\right] & \text{on } E_i \\ \left[x^* \text{ on } [0,\alpha] \underset{m}{\times} \mathcal{S}; x \text{ on } (\alpha,1] \underset{m}{\times} \mathcal{S}\right] & \text{on } E_j \\ x & \text{on } E_k \;\; k \neq i,j \end{pmatrix} \tag{32}$$

*for some outcomes $x^* \succ x$, value $\alpha \in [0,1]$, and pair of events $E_i$ and $E_j$, then*

$$\lim_{m\to\infty} W\begin{pmatrix} f_m^i(\cdot) \text{ on } E_i \\ f_m^j(\cdot) \text{ on } E_j \\ f_m^k(\cdot) \text{ on } E_k \;\; k \neq i,j \end{pmatrix} = \lim_{m\to\infty} W\begin{pmatrix} \alpha\cdot f_m^i(\cdot) \oplus (1-\alpha)\cdot f_m^j(\cdot) & \text{on } E_i \\ \alpha\cdot f_m^i(\cdot) \oplus (1-\alpha)\cdot f_m^j(\cdot) & \text{on } E_j \\ f_m^k(\cdot) & \text{on } E_k \;\; k \neq i,j \end{pmatrix} \tag{33}$$

*for all almost-objective acts $f_m^1(\cdot),\ldots,f_m^n(\cdot)$.*

An argument parallel to that of Machina and Schmeidler (1995) then yields the following characterization of probabilistically sophisticated preferences over *all* subjective acts $f(\cdot) \in \mathcal{A}$, where $\alpha : (1-\alpha)$ again turns out to be the individual's subjective probability ratio $\mu(E_i) : \mu(E_j)$:[12]

**Theorem 7 (Anscombe-Aumann without objective uncertainty).** *If an event-smooth, outcome-monotonic preference function $W(\cdot)$ over $\mathcal{A}$ satisfies the Almost-Objective/Subjective Replacement Axiom, then it takes the globally probabilistically sophisticated form $W_{PS}(f(\cdot)) \equiv V(\ldots; x_i, \mu(E_i); \ldots)$ for some finitely-additive probability measure $\mu(\cdot)$.*

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[11] The mixture $\alpha \cdot f_m^i(\cdot) \oplus (1-\alpha) \cdot f_m^j(\cdot)$ is neither a "two-stage" prospect nor an almost-objective mixture of the almost-objective acts $f_m^i(\cdot)$ and $f_m^j(\cdot)$, but simply the standard (single-stage) almost-objective act on $\mathcal{S}$ generated by the $K_i+K_j$ outcomes $\{x_{i,1},\ldots,x_{i,K_i},x_{j,1},\ldots,x_{j,K_j}\}$ and the $(K_i+K_j)$-element partition $\{\wp_{i,1} \underset{1}{\times} [0,\alpha],\ldots,\wp_{i,K_i} \underset{1}{\times} [0,\alpha], \wp_{j,1} \underset{1}{\times} (\alpha,1],\ldots,\wp_{j,K_j} \underset{1}{\times} (\alpha,1]\}$ of $[0,1]$. The following axiom can be stated directly in terms of the preference relation $\succsim$ by replacing each equality by the condition that its left and right acts have the same limiting indifference class in $\mathcal{A}$.

[12] Adding the Sure-Thing Principle to the assumptions of Theorem 7 would yield an Anscombe-Aumann type *joint characterization* of expected utility and subjective probability under completely subjective uncertainty.

### *4.2 Separating objective risk preferences from state-dependence, ambiguous beliefs, and attitudes toward ambiguity*

In this section we consider the following question:

> "When can we recover an individual's *objective risk preferences* – that is, the preferences they would exhibit in an idealized casino – from their preferences over purely subjective acts?"

For a state-independent expected utility maximizer with subjective preference function $W_{SEU}(f(\cdot)) \equiv \int_S U(f(s)) \cdot d\mu(s)$ the answer is straightforward – the von Neumann-Morgenstern utility function $U(\cdot)$ derived from their preferences over purely subjective acts would presumably also represent their preferences over purely objective lotteries. For a state-dependent expected utility maximizer with subjective preference function $W_{SDEU}(f(\cdot)) \equiv \int_S U(f(s)|s) \cdot d\mu(s)$ the answer is also straightforward: Since the expected utility of winning prize $x$ in a casino (and payable in every state of nature) is given by $\int_S U(x|s) \cdot d\mu(s)$, the expected utility of any objective lottery $(x_1, p_1; \ldots; x_n, p_n)$ is presumably just $\sum_{i=1}^{n} \int_S U(x_i|s) \cdot d\mu(s) \cdot p_i$. Finally, a probabilistically sophisticated individual with subjective preference function $W_{PS}(f(\cdot)) \equiv V(x_1, \mu(E_1); \ldots; x_n, \mu(E_n))$ would presumably rank objective lotteries according to their risk preference function $V(x_1, p_1; \ldots; x_n, p_n)$.

The above derivations of objective risk preferences from subjective act preferences are possible because each of the forms $W_{SEU}(\cdot), W_{SDEU}(\cdot)$ and $W_{PS}(\cdot)$ incorporates a separation of "beliefs" (as represented by $\mu(\cdot)$) from "risk preferences" (as represented by $U(\cdot), U(\cdot|\cdot)$ or $V(\cdot)$). However, a *general* subjective preference function $W(\cdot)$ might not have such a well-defined separation, it might not involve probabilistic beliefs at all but instead reflect ambiguity, and it might exhibit an aversion (or preference) toward such ambiguity on top of any underlying objective risk preferences. Complicating matters further is the fact that in a world of *two* subjectively uncertain variables $s \in \mathcal{S}$ and $t \in \mathcal{T}$ (say temperature and air pressure), an individual might exhibit *vastly different* state-dependence and/or ambiguity properties with respect to these two variables.

But by use of almost-objective uncertainty, it is possible to recover the underlying objective risk preferences of any event-smooth $W(\cdot)$ – even in the presence of state-dependence, ambiguity or ambiguity aversion – while remaining in a completely subjective setting. Furthermore, whereas other features of an individual's subjective act preferences – their subjective beliefs, state-dependence, ambiguity and ambiguity attitudes – may well depend upon the *source* of subjective uncertainty (that is, upon the particular subjective variable and state space), their objective risk preferences will turn out to be *invariant* to the source of subjective uncertainty, and will carry over unchanged from one state space to another. In particular, an event-smooth individual's preferences over *almost-objective bets* in any subjective setting – even one that involves state-dependence, ambiguity or ambiguity aversion – will serve to completely predict their objective risk preferences, and hence their preferences over *all bets* in any idealized casino.

Theorem 3 showed that every event-smooth $W(\cdot)$ over subjective acts – even if it exhibits state-dependence, ambiguity, or ambiguity aversion – has an associated function $V_W(\cdot)$ that represents its limiting preferences over almost-objective acts, in the sense that $\lim_{m\to\infty} W(x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}) \equiv V_W(x_1, \lambda(\wp_1); \ldots; x_n, \lambda(\wp_n))$. To see why $V_W(\cdot)$ might be viewed as representing $W(\cdot)$'s objective risk preferences, recall that as $m \to \infty$ each event $\wp_i \underset{m}{\times} \mathcal{S}$ in an almost-objective act will exhibit all the belief and betting properties of an objective event with probability $\lambda(\wp_i)$. Thus as $m \to \infty$ we would expect the act $[x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}]$ to be viewed as indifferent to the objective lottery $(x_1, \lambda(\wp_1); \ldots; x_n, \lambda(\wp_n))$, so that the preference level of this objective lottery would also be given by the expression $V_W(x_1, \lambda(\wp_1); \ldots; x_n, \lambda(\wp_n))$.

In order to establish that $V_W(\cdot)$ actually represents $W(\cdot)$'s objective risk preferences, we must formally demonstrate that it satisfies two properties:

1. $V_W(\cdot)$ does not reflect or embody any of the state-dependence or ambiguity properties associated with the particular source of subjective uncertainty (that is, with the current state space).
2. $V_W(\cdot)$ completely predicts – and can be completely predicted from – the individual's behavior in any alternative setting they view as having completely probabilistic uncertainty, free of any state-dependence or ambiguity, such as an idealized casino.

Property 1 states that even if the individual should face two subjective variables $s \in \mathcal{S}$ and $t \in \mathcal{T}$ with *different* state-dependence and ambiguity properties, the limiting preference function values $\lim_{m\to\infty} W(x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S})$ and $\lim_{m\to\infty} W(x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{T}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{T})$ will both be given by $V_W(x_1, \lambda(\wp_1); \ldots; x_n, \lambda(\wp_n))$. Property 2 states that if their betting preferences over one of these variables (say $t$) exhibited *no* state-dependence or ambiguity, and were in fact probabilistically sophisticated with respect to a *uniform* subjective probability measure $\lambda_T(\cdot)$ on $\mathcal{T}$, then $t$ could be viewed as an idealized "casino variable," and the individual's preferences over *all bets* on $t$ would be represented by $V_W(\cdot)$, via the formula $W(x_1 \text{ if } t \in E_1; \ldots; x_n \text{ if } t \in E_n) \equiv V_W(x_1, \lambda_T(E_1); \ldots; x_n, \lambda_T(E_n))$.

The following result establishes Property 1 by showing that if $W(\cdot)$ is event-smooth over the set $\mathcal{A}_{\mathcal{S}\times\mathcal{T}}$ of subjective acts $f(\cdot,\cdot)$ on a state space $\mathcal{S} \times \mathcal{T} = [\underline{s}, \bar{s}] \times [\underline{t}, \bar{t}]$,[13] then even though the individual's beliefs, state-dependence and ambiguity properties may be different for $s$ and $t$, their limiting preferences over almost-objective bets on $s$ and almost-objective bets on $t$ will be represented by the same $V_W(\cdot)$ function. In this sense, an individual's objective risk preferences are "more inherent" than either their state-independence/state-dependence or their probabilistic vs. ambiguous belief properties, each of which can vary across different subjective state variables.[14]

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[13] Formally, $\mathcal{A}_{\mathcal{S}\times\mathcal{T}}$ is the set of finite-outcome acts whose events consist of finite unions of rectangles in $\mathcal{S} \times \mathcal{T}$. Event-smoothness over this bivariate act space is formally defined in the Appendix.

[14] The objective risk preferences for $W_{SEU}(\cdot)$, $W_{SDEU}(\cdot)$ and $W_{PS}(\cdot)$ listed in the first paragraph of this section all follow as special cases of Theorem 8.

**Theorem 8 (Invariance of objective risk preferences to the source of subjective uncertainty).** *For each event-smooth, outcome-monotonic preference function $W(\cdot)$ over $\mathcal{A}_{\mathcal{S}\times\mathcal{T}}$ there exists a preference function $V_W(\cdot)$ over objective lotteries, satisfying strict first order stochastic dominance preference, such that*

$$\begin{aligned}\lim_{m\to\infty} & W\big(x_1 \text{ on } (\wp_1 \underset{m}{\times} \mathcal{S}) \times \mathcal{T}; \ldots; x_n \text{ on } (\wp_n \underset{m}{\times} \mathcal{S}) \times \mathcal{T}\big) \\ & \equiv \quad V_W\big(x_1, \lambda(\wp_1); \ldots; x_n, \lambda(\wp_n)\big) \quad \equiv \\ \lim_{m\to\infty} & W\big(x_1 \text{ on } \mathcal{S} \times (\wp_1 \underset{m}{\times} \mathcal{T}); \ldots; x_n \text{ on } \mathcal{S} \times (\wp_n \underset{m}{\times} \mathcal{T})\big)\end{aligned} \tag{34}$$

*for all almost-objective acts in $\mathcal{A}_{\mathcal{S}\times\mathcal{T}}$ that depend solely on a single subjective variable ($s$ or $t$).*

To see that this result also implies Property 2, consider an event-smooth $W(\cdot)$ defined over acts on a given state space $\mathcal{S} = [\underline{s}, \bar{s}]$, which may exhibit state-dependence, ambiguity and ambiguity aversion, and its associated $V_W(\cdot)$ function. By the theorem, every event-smooth *extension* of $W(\cdot)$ to acts on any state space $\mathcal{S}\times\mathcal{T} = [\underline{s}, \bar{s}]\times[\underline{t}, \bar{t}]$ will inherit the same $V_W(\cdot)$ function for acts that depend solely on $t$. Say the extension is such that the individual is probabilistically sophisticated for bets on $t$, with a uniform subjective probability measure $\lambda_{\mathcal{T}}(\cdot)$ over $\mathcal{T}$. In such a case, $t$ can be viewed as an idealized casino variable, involving no state-dependence or ambiguity, so that the individual's betting preferences on $t$ will only reflect their underlying objective risk preferences.[15] Probabilistic sophistication over $\mathcal{T}$ and Theorem 8 then imply

$$W\big(x_1 \text{ if } t \in E_1; \ldots; x_n \text{ if } t \in E_n\big) = V_W\big(x_1, \lambda_{\mathcal{T}}(E_1); \ldots; x_n, \lambda_{\mathcal{T}}(E_n)\big) \tag{35}$$

that is, the same function $V_W(\cdot)$ that represents $W(\cdot)$'s preferences over *almost-objective bets* on the state-dependent and ambiguous state variable $s$ will also represent preferences over *all bets* on the state-independent and completely probabilistic "casino variable" $t$.

It is fair to say that the real world involves many subjectively uncertain variables $s \in \mathcal{S}$, $t \in \mathcal{T}$, $r \in \mathcal{R}$, ... (or equivalently, a single high-dimensional state space $\mathcal{S}\times\mathcal{T}\times\mathcal{R}\times\ldots$) and that a typical subjective prospect only depends upon one or a few of these variables. The results of this section show that under event-smoothness, the individual's *objective risk preferences* will be the same toward each state variable (or subset of state variables), even though their *subjective beliefs, state-dependence, ambiguity* and *ambiguity attitudes* may vary from variable to variable.

[15] For exogeneity, we could also assume that beliefs on $\mathcal{T}$ are uniform even when conditioned on any event in $\mathcal{S}$.

### *4.3 Generalized expected utility / subjective probability analysis under subjective uncertainty*

The approach of "generalized expected utility analysis" developed in Machina (1982)[16] provides a means by which the basic concepts, tools and result of expected utility analysis under objective uncertainty can be extended to general probability-smooth preference functions $V(\cdot)$ over objective lotteries. The key ideas in this extension are that the observations that

- the classical expected utility form $V_{EU}(\mathbf{P}) \equiv \sum_{i=1}^{n} U(x_i) \cdot p_i$ is *linear in the probabilities*
- standard expected utility concepts and results can thus be phrased in terms of $V_{EU}(\cdot)$'s *probability coefficients* $\{U(x)\,|\,x \in \mathcal{X}\}$, where $U(x)$ = coefficient of prob($x$)
- standard calculus arguments allow us to extend most of these probability coefficient results to the *probability derivatives* $U(x;\mathbf{P}) \equiv \partial V(\mathbf{P})/\partial \text{prob}(x)$ of a general probability-smooth $V(\cdot)$

The key step in extending coefficient-based results to derivative-based results lies in integrating along the "straight line path" between two objective lotteries $\mathbf{P}$ and $\mathbf{P}^*$ – that is, the probability mixture path $\{\mathbf{P}_\alpha\,|\,\alpha \in [0,1]\} = \{(\mathbf{P}^*, \alpha; \mathbf{P}, 1-\alpha)\,|\,\alpha \in [0,1]\}$. For $V_{EU}(\cdot)$, this line integral takes the form

$$\begin{aligned} V_{EU}(\mathbf{P}^*) - V_{EU}(\mathbf{P}) &= \int_0^1 \frac{dV_{EU}(\mathbf{P}^*, \alpha; \mathbf{P}, 1-\alpha)}{d\alpha} \cdot d\alpha \\ &= \int_0^1 \sum_{x \in \mathcal{X}} U(x) \cdot \big[\text{prob}^*(x) - \text{prob}(x)\big] \cdot d\alpha \end{aligned} \tag{36}$$

where the integrand denotes how for each outcome $x$, its probability change $\text{prob}^*(x) - \text{prob}(x)$ in going from $\mathbf{P}$ to $\mathbf{P}^*$ is evaluated by $V_{EU}(\cdot)$'s corresponding probability coefficient $U(x)$. For a general probability-smooth $V(\cdot)$, this line integral takes the analogous form

$$\begin{aligned} V(\mathbf{P}^*) - V(\mathbf{P}) &= \int_0^1 \frac{dV(\mathbf{P}^*, \alpha; \mathbf{P}, 1-\alpha)}{d\alpha} \cdot d\alpha \\ &= \int_0^1 \sum_{x \in \mathcal{X}} U(x;\mathbf{P}_\alpha) \cdot \big[\text{prob}^*(x) - \text{prob}(x)\big] \cdot d\alpha \end{aligned} \tag{37}$$

where the integrand now denotes how each of these probability changes is evaluated by $V(\cdot)$'s corresponding probability derivative $U(x;\mathbf{P}_\alpha)$ at each distribution along the path $\{\mathbf{P}_\alpha\,|\,\alpha \in [0,1]\}$. Thus the condition that $U(x;\mathbf{P})$ be concave in $x$ at each $\mathbf{P}$ is both necessary and sufficient for the integrand in (37), hence its integrated value $V(\mathbf{P}^*) - V(\mathbf{P})$, to be nonpositive whenever $\mathbf{P}^*$ differs from $\mathbf{P}$ by a mean-preserving increase in risk, so that the expected utility characterization of risk

[16] See also the extensions and applications of Allen (1987), Bardsley (1993), Chew, Epstein and Zilcha (1988), Chew, Karni and Safra (1987), Chew and Nishimura (1992), Karni (1987,1989) and Wang (1993).

aversion by the concavity of a cardinal function extends to general probability-smooth $V(\cdot)$. Similar extensions hold for a large body of expected utility results under objective uncertainty.

Machina (2002) extends this approach to subjective uncertainty, via the parallel arguments that

- the classical expected utility/subjective probability forms $W_{SEU}(f(\cdot)) \equiv \sum_{i=1}^{n} U(x_i)\cdot\mu(E_i)$ and $W_{SDEU}(f(\cdot)) \equiv \sum_{i=1}^{n} \int_{E_i} U(x_i|s)\cdot d\mu(s)$ are both *additive in the events*
- standard expected utility/subjective probability results can be phrased in terms of these functions' *evaluation measures* $\Phi_x(E) \equiv U(x)\cdot\mu(E)$ for $W_{SEU}(\cdot)$ and $\Phi_x(E) \equiv \int_E U(x|s)\cdot d\mu(s)$ for $W_{SDEU}(\cdot)$
- standard calculus arguments allow us to extend most of these evaluation measure results to the *event-derivatives* ("local evaluation measures") $\{\Phi_x(\cdot;f)\,|\,x\in\mathcal{X}\}$ of a general event-smooth $W(\cdot)$

The analogue of (36)/(37) would be a line integral along a path $\{f_\alpha(\cdot)\,|\,\alpha\in[0,1]\}$ from $f(\cdot)$ to $f^*(\cdot)$ whose integrand would denote how the global event changes $\{f^{*-1}(x) - f^{-1}(x)\,|\,x\in\mathcal{X}\}$ are evaluated by $W(\cdot)$'s local evaluation measures $\{\Phi_x(\cdot;f_\alpha)\,|\,x\in\mathcal{X}\}$ at each act along the path. But as shown in Machina (2002), paths with this exact property cannot exist in the space $\mathcal{A}$, essentially because one cannot take exact "convex combinations" of subjective events. However as $m\to\infty$, *almost-objective mixture paths* $\{f_\alpha^m(\cdot)\,|\,\alpha\in[0,1]\}_{m=1}^{\infty} = \{[f^*(\cdot) \text{ on } [0,\alpha] \underset{m}{\times} \mathcal{S}; f(\cdot) \text{ on } (\alpha,1] \underset{m}{\times} \mathcal{S}]\,|\,\alpha\in[0,1]\}_{m=1}^{\infty}$ will converge to this property, to yield exact global extensions of the most basic classical results, including characterizations of comparative subjective likelihood, relative subjective likelihood, and comparative risk aversion.

### *4.4 Markets for almost-objective bets versus purely subjective bets*

Individuals facing subjective uncertainty have at least four sources of gains from trade: attitudes toward risk (either differences in risk aversion or outright risk preference); differences in beliefs; hedging against uncertain endowments; and diversification of multiple risks. Of these four sources, the first most naturally leads to trade in almost-objective bets, and the latter three most naturally lead to trade in purely subjective bets.

For example, if individuals $W(\cdot)$ and $W^*(\cdot)$ have nonrandom endowments $x_0$ and $x_0^*$ and associated risk preference functions $V_W(\cdot)$ and $V_{W^*}(\cdot)$, at least one of which is risk loving, then there exists an objective lottery $\hat{\mathbf{P}} = (\ldots;\hat{x}_i,\hat{p}_i;\ldots)$ with $V_W(\ldots;x_0+\hat{x}_i,\hat{p}_i;\ldots) > V_W(x_0,1)$ and $V_{W^*}(\ldots;x_0^*-\hat{x}_i,\hat{p}_i;\ldots) > V_{W^*}(x_0^*,1)$, and hence an almost-objective bet $\hat{f}_m(\cdot) = [\ldots;\hat{x}_i \text{ on } \hat{\wp}_i \underset{m}{\times} \mathcal{S};\ldots]$ such that $W(\ldots;x_0+\hat{x}_i \text{ on } \hat{\wp}_i \underset{m}{\times} \mathcal{S};\ldots) > W(x_0 \text{ on } \mathcal{S})$ and $W^*(\ldots;x_0^*-\hat{x}_i \text{ on } \hat{\wp}_i \underset{m}{\times} \mathcal{S};\ldots) > W^*(x_0^* \text{ on } \mathcal{S})$. This phenomenon underlies the extensive amount of real-world betting on coins, roulette wheels and similar mechanisms, which as noted above, are examples of almost-objective bets generated from subjective driving variables. Since the limiting event likelihoods for such bets are unaffected by differences in subjective beliefs, trade in them is not subject to the usual difficulties that arise from asymmetric information or moral hazard. If one individual has an almost-objective endowment and both are risk averse but not equally so, a similar opportunity for trade in almost-objective bets will arise.

Given the prevalence of betting on such almost-objective events, one might ask why *financial markets* don't offer what might be termed *almost-objective securities or options*. Such instruments would not be difficult to create – they could be defined to pay off based on the $k$'th decimal of any security price, market index or publicly observed variable. But it is precisely *because* of their objective-like properties that such instruments could not serve any of the standard purposes of financial instruments, which are the second, third and fourth of the above reasons for trade:

- Since they would approximate the objective property of *unanimous likelihoods*, such almost-objective instruments *would not permit gains from trade based on differences in beliefs*, which (as the saying goes) are a primary reason for horse races, as well as for trade in financial instruments such as individual securities or standard (i.e., purely subjective) options.
- Since they would approximate the objective property of *independence of the realization of subjective events*, almost-objective instruments based on a particular index or security price *could not serve to hedge* against downside risk in that variable, which is another primary purpose of options and related instruments.
- Since they would approximate *objective probability mixtures* rather than *payoff mixtures*, almost-objective mixtures of securities *could not serve to diversify* independent risks. Thus, if a probabilistically sophisticated investor believed two securities to be independent and identically distributed, he or she would view any almost-objective mixture of them as having *approximately the same distribution* as each individual security, rather than one with a lower variance.

In other words, financial markets do not involve trade in almost-objective securities for the same reasons that they do not involve trade in bets based on physical randomizing devices. Although we do not do so here, a formalization of the above arguments would involve showing that for individuals with degenerate or almost-objectively uncertain endowments, different risk preferences, and identical beliefs, trade in almost-objective bets would (in the limit) weakly Pareto dominate trade in purely subjective bets, and for individuals with degenerate or purely subjectively uncertain endowments, identical risk preferences, and different beliefs, trade in purely subjective bets would weakly Pareto dominate trade in almost-objective bets.

## 5 Comparison and extensions

### *5.1 Comparison with related results under subjective uncertainty*

*Comparison with Ghirardato-Maccheroni-Marinacci-Siniscalchi mixtures:* Ghirardato, Maccheroni, Marinacci and Siniscalchi (2003) give an alternative sense in which one purely subjective act can be thought of as a cardinal mixture of two other subjective acts. In their notation (where $xEy$ denotes the act [$x$ on $E$; $y$ on $\sim E$]), if the outcomes $x$, $y$, $z$, $c_{xEz}$, $c_{zEy}$ satisfy the preference conditions

$$c_{xEz} \sim xEz \qquad c_{zEy} \sim zEy \qquad xEy \sim c_{xEz}Ec_{zEy} \tag{38}$$

then the outcome $z$ can be described as a 50:50 "preference average" of outcomes $x$ and $y$.[17] A 75:25 preference average of $x$ and $y$ can then be obtained by finding a 50:50 preference average of $x$ and $z$, etc. These authors then define a 50:50 mixture of two subjective acts $f(\cdot)$ and $g(\cdot)$ as any subjective act whose outcome in each state $s$ is a 50:50 preference average of the outcomes $f(s)$ and $g(s)$ in that state, and 75:25 (and other) mixtures of $f(\cdot)$ and $g(\cdot)$ are defined similarly.

Unlike an objective mixture $(f(\cdot), \alpha; g(\cdot), 1-\alpha)$ or an almost-objective mixture $[f(\cdot)$ on $\wp \underset{m}{\times} \mathcal{S}; g(\cdot)$ on $\sim\wp \underset{m}{\times} \mathcal{S}]$ – which are defined prior to and independently of any individual's preferences – a GMMS mixture of two subjective acts $f(\cdot)$ and $g(\cdot)$ involves generating individual-specific *preference-based mixtures* of each statewise outcome pair $f(s)$, $g(s)$. GMMS mixtures can accordingly be described as preference-based "subjective cardinal mixtures," and these authors have shown how they can provide an alternative basis for axiomatizing the *SEU*, Choquet and maxmin models of preferences over subjective acts.

*Comparison with the Liapunov Convexity Theorem:* Liapunov (1940) provides an important result which, though not explicitly choice-theoretic, can still be interpreted as implying the existence of a family of events with unanimously agreed-upon subjective probabilities for a given population of individuals. Most generally, it states that the range of every finite, vector-valued, non-atomic, countably additive measure $\Psi(\cdot) = (\psi_1(\cdot), \ldots, \psi_K(\cdot))$ is closed and convex.[18] In our present setting, this implies the result:

> "Given a family of nonatomic probability measures $\{\mu_1(\cdot), \ldots, \mu_K(\cdot)\}$ over $\mathcal{S}$, for each probability $p \in [0,1]$ there exists an event $\bar{E}_p \subseteq \mathcal{S}$ such that $\mu_k(\bar{E}_p) = p$ for $k = 1, \ldots, K$."

This result is analogous to Theorem 1 in that it identifies events that are unanimously assigned a common likelihood by every member of some prespecified family of individuals. One aspect of this result that is *stronger* than in Theorem 1 is that the values $\mu_k(\bar{E}_p) = p$ are *exact* for each individual $k$ and event $\bar{E}_p$, whereas the revealed likelihoods in Theorem 1 are approximate and only hold exactly in the limit. On the other hand, an aspect of this result that is *weaker* than in Theorem 1 is its restriction to a *finite* family of individual measures $\{\mu_1(\cdot), \ldots, \mu_K(\cdot)\}$.[19]

Another aspect of this result that is weaker than in Theorem 1 is its restriction to *measures* $\{\mu_1(\cdot), \ldots, \mu_K(\cdot)\}$ as the "inputs". Thus, for it to apply to a family $\{W_1(\cdot), \ldots, W_K(\cdot)\}$ of preference functions over subjective acts, it would have to be a family of *probabilistically sophisticated* preference functions $W_k(f(\cdot)) \equiv V_k(\ldots; x_i, \mu_k(E_i); \ldots)$ or some other measure-based form. This contrasts with Theorem 1, which applies to all event-smooth preference functions $W(\cdot)$.

[17] For *SEU* or rank-dependent expected utility preferences, (38) implies $U(z) = \frac{1}{2}\cdot U(x) + \frac{1}{2}\cdot U(y)$.

[18] See also the simplified proofs of Halmos (1948) and Lindenstrauss (1966).

[19] Halmos (1948, Note 1) credits Liapunov (1946) with showing that neither convexity nor closure necessarily follows in the case of an *infinite* number of measures.

### 5.2 Analytical extensions

*Extension to more general events:* Although the almost-ethically-neutral and almost-objective events defined above have an exactly periodic structure over the state space $\mathcal{S}$, the analysis of this paper will extend to more general types of events. For example, instead of events $\wp \underset{m}{\times} \mathcal{S}$ based on a fixed subset $\wp \subseteq [0,1]$ and $m$ *equal-length* intervals, we could define events $E'_m$ based on $m$ *unequal-length* intervals, such as

$$\big[\underline{s},\, \underline{s}+\xi(\tfrac{1}{m})\big), \ldots, \big[\underline{s}+\xi(\tfrac{i}{m}),\, \underline{s}+\xi(\tfrac{i+1}{m})\big), \ldots, \big[\underline{s}+\xi(\tfrac{m-1}{m}),\, \bar{s}\big] \tag{39}$$

for some increasing, onto and suitably regular[20] function $\xi(\cdot):[0,1] \to [0, \lambda_{\mathcal{S}}]$, where the union $\wp \underset{m}{\times} \mathcal{S} = \bigcup_{i=0}^{m-1} \{\underline{s} + (i+\omega)\cdot\frac{\lambda_{\mathcal{S}}}{m} \mid \omega \in \wp\}$ from (10) is replaced by

$$E'_m = \bigcup\nolimits_{i=0}^{m-1} \big\{\underline{s} + \xi(\tfrac{i}{m}) + \omega\cdot\big(\xi(\tfrac{i+1}{m}) - \xi(\tfrac{i}{m})\big) \,\big|\, \omega \in \wp\big\} \tag{40}$$

For each $m$, the Lebesgue measure of $E'_m$ will continue to be exactly $\lambda(\wp)\cdot\lambda(\mathcal{S})$. As $m \to \infty$, the proofs of the above theorems can be appropriately adapted, and the events (40) can thus also be described as almost-objective, with a unanimous limiting likelihood of $\lambda(\wp)$.

Conversely, one could also define a sequence of events $E''_m$ based on $m$ equal-length intervals as in (2), but a (regularly) *variable* finite interval subset $\{\wp_m\}_{m=1}^{\infty}$, with the requirement that $\lambda(\wp_m) = \lambda(\wp_1)$ for $m = 2, 3, \ldots$ Defining the events

$$E''_m = \bigcup\nolimits_{i=0}^{m-1} \big\{\underline{s} + (i+\omega)\cdot\tfrac{\lambda_{\mathcal{S}}}{m} \,\big|\, \omega \in \wp_m\big\} \tag{41}$$

again yields a unanimous limiting revealed likelihood, equal to $\lambda(\wp_1)$. Of course, any combination of the generalizations (40) and (41) will yield similar results. Given proper attention to issues of uniform convergence, the most general definition of "almost-objective" events $E_m$ with a limiting likelihood of $\rho \in [0,1]$ would presumably be one satisfying a condition such as

$$\lim_{m\to\infty} \lambda\big(E_m \cap [a,b]\big) \;=\; \rho\cdot(b-a) \qquad \text{for all } [a,b] \subseteq [\underline{s}, \bar{s}] \tag{42}$$

*Extension to more general state spaces:* As seen in Theorem 8, the approach of this paper can also be applied to more general state spaces. Given any multivariate state space $\mathcal{S} = [\underline{s}_1, \bar{s}_1] \times \ldots \times [\underline{s}_K, \bar{s}_K] \subset R^K$, we could replace the $m$ equal-length intervals in (2) by the $m^K$ equal-volume boxes obtained by partitioning $\mathcal{S}$ in each dimension. $\mathcal{S}$ could also be a smooth manifold (such as the surface of a sphere), or any space that can be "tiled" by arbitrarily small but suitably comparable measure spaces.

*Weakening event-smoothness:* Although we have assumed event-smoothness, it is clear that both Poincaré's Theorem and the approach of this paper will also hold for more general preferences, such as $W_{SEU}(f(\cdot)) \equiv \int_{\mathcal{S}} U(f(s))\cdot\nu(s)\cdot ds$ with

[20] $\xi(\cdot)$ would be suitably regular provided $\xi'(\cdot)$ is bounded away from both 0 and $+\infty$ over $[0,1]$.

piecewise-continuous $\nu(\cdot)$, or piecewise event-smooth $W(\cdot)$. However, to see that event-continuity alone is not sufficient, and that some type of absolute continuity condition is needed for both Poincaré's Theorem and the analysis of this paper, observe that the standard Cantor measure[21] $C(\cdot)$ over $\mathcal{S} = [\underline{s}, \bar{s}]$ will *not* satisfy the property $\lim_{m\to\infty} C(E_m) = \frac{1}{3}$ for the events $E_m = (\frac{1}{3}, \frac{2}{3}) \underset{m}{\times} \mathcal{S}$, since $C(E_m)$ will equal 0 whenever $m$ is a power of 3.

*Dropping outcome-monotonicity:* [22] Although the analysis of this paper did not require that subjective act preferences be expected utility, state-independent or probabilistically sophisticated, we did follow most of the literature in assuming that preferences were *outcome-monotonic*, as defined in Section 2.1. Under objective uncertainty, outcome monotonicity is also known as *first order stochastic dominance preference*.

Although outcome-monotonicity is natural for monetary outcomes, it may not be so in other cases (Grant 1995). But without outcome-monotonicity there is no longer a direct relationship between betting preferences and likelihood rankings, even for purely objective events. Thus, if a preference function $V(\cdot)$ over objective lotteries is not outcome-monotonic, it may prefer $x^*$ to $x$ yet prefer $(x, \frac{2}{3}; x^*, \frac{1}{3})$ to $(x^*, \frac{2}{3}; x, \frac{1}{3})$. But while dropping outcome-monotonicity breaks the link between betting preferences and *strict* likelihood rankings for objective events, a link between betting preferences and *equal* likelihood remains: If two mutually exclusive objective events have the same objective likelihood, even individuals who violate outcome-monotonicity will be indifferent to swapping the payoffs assigned to these events. This is generally not true for equal *subjective* likelihoods: the preference function $W_{SDEU}(f(\cdot)) \equiv \int_{\mathcal{S}} U(f(s)|s) \cdot d\mu(s)$, even if it is outcome monotonic, can satisfy both $\mu(E) = \mu(E')$ and $W_{SDEU}(x^* \text{ on } E; x \text{ on } E'; f(\cdot) \text{ elsewhere}) \neq W_{SDEU}(x \text{ on } E; x^* \text{ on } E'; f(\cdot) \text{ elsewhere})$.

But even in the absence of outcome-monotonicity, the limiting properties of almost-objective events will continue to match the properties of objective rather than fixed subjective events. That is, the arguments used in the Appendix can be adapted to show that if two disjoint finite-interval unions $\wp, \wp' \subseteq [0,1]$ satisfy $\lambda(\wp) = \lambda(\wp')$, then $\wp \underset{m}{\times} \mathcal{S}$ and $\wp' \underset{m}{\times} \mathcal{S}$ satisfy

$$\lim_{m\to\infty} W\begin{pmatrix} x^* \ \text{on} \ \wp \underset{m}{\times} \mathcal{S} \\ x \ \text{on} \ \wp' \underset{m}{\times} \mathcal{S} \\ f(\cdot) \ \text{elsewhere} \end{pmatrix} \underset{\substack{\text{all } x^*, x \in \mathcal{X} \\ \text{all } f(\cdot) \in \mathcal{A}}}{\equiv} \lim_{m\to\infty} W\begin{pmatrix} x \ \text{on} \ \wp \underset{m}{\times} \mathcal{S} \\ x^* \ \text{on} \ \wp' \underset{m}{\times} \mathcal{S} \\ f(\cdot) \ \text{elsewhere} \end{pmatrix} \tag{43}$$

for every event-smooth $W(\cdot)$, whether or not it is outcome-monotonic, and they will retain this equal-likelihood betting property even when conditioned on an arbitrary

[21] E.g., Billingsley (1986, pp. 427–429), Feller (1971, pp. 35–36) or Romano and Siegel (1986, pp. 27–28).

[22] I am grateful to Simon Grant for suggesting the key ideas of this section.

fixed subjective event $E$:

$$\lim_{m\to\infty} W\begin{pmatrix} x^* & \text{on } (\wp \underset{m}{\times} \mathcal{S})\cap E \\ x & \text{on } (\wp' \underset{m}{\times} \mathcal{S})\cap E \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \underset{\substack{\text{all } x^*, x\in\mathcal{X} \\ \text{all } f(\cdot)\in\mathcal{A}}}{\equiv} \lim_{m\to\infty} W\begin{pmatrix} x & \text{on } (\wp \underset{m}{\times} \mathcal{S})\cap E \\ x^* & \text{on } (\wp' \underset{m}{\times} \mathcal{S})\cap E \\ f(\cdot) & \text{elsewhere} \end{pmatrix} \quad (44)$$

In addition, if two almost-objective acts $[x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}]$ and $[x_1 \text{ on } \wp_1^* \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n^* \underset{m}{\times} \mathcal{S}]$ satisfy $\lambda(\wp_i) = \lambda(\wp_i^*)$ for each $i$, they will continue to satisfy $\lim_{m\to\infty} W(x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}) = \lim_{m\to\infty} W(x_1 \text{ on } \wp_1^* \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n^* \underset{m}{\times} \mathcal{S})$. Similarly, if two almost-objective mixtures $[f_1(\cdot) \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; f_n(\cdot) \text{ on } \wp_n \underset{m}{\times} \mathcal{S}]$ and $[f_1^*(\cdot) \text{ on } \wp_1^* \underset{m}{\times} \mathcal{S}; \ldots; f_{n^*}^*(\cdot) \text{ on } \wp_{n^*}^* \underset{m}{\times} \mathcal{S}]$ imply probabilistically equivalent almost-objective acts over each event in the common refinement of $\{f_1(\cdot), \ldots, f_n(\cdot), f_1^*(\cdot), \ldots, f_{n^*}^*(\cdot)\}$, they will continue to satisfy a similar limiting equality. Each event-smooth $W(\cdot)$ will continue to have an associated $V_W(\cdot)$ satisfying $\lim_{m\to\infty} W(x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}) \equiv V_W(x_1, \lambda(\wp_1); \ldots; x_n, \lambda(\wp_n))$, although $V_W(\cdot)$ need no longer be outcome-monotonic.

Finally, we can extend this approach to obtain natural definitions of *comparative* objective and almost-objective likelihood, even in the absence of outcome-monotonicity, by defining one event to be at least as likely as another if and only if the first event possesses some *subset* that satisfies the equal-likelihood betting property with respect to the second.

## 6 Modeling uncertainty and modeling uncertain choice: some revised perspectives

### *6.1 Two types of events rather than two types of uncertainty*

Poincaré's theorem and the results of this paper argue for a new perspective on the classic distinction between "objective" and "subjective" uncertainty. Although the traditional view has been to distinguish *objective processes* (such as coin flips or roulette wheels) from *subjective processes* (such as horse races or the temperature), a more unified perspective would be to recognize that virtually all uncertain processes involve *both* completely subjective and (almost-) objective *events*.

We have seen that even in a purely subjective setting where the state is the temperature, some events (such as whether its second decimal is even) will exhibit the virtually unanimous revealed-likelihood properties of idealized objective events. Even in Anscombe-Aumann's canonical example of a "subjective" horse race, where individuals may have diverse or even nonexistent beliefs over how fast the horses may run, virtually everyone will exhibit a 50:50 revealed likelihood for whether a given horse will complete the race in an even versus odd number of milliseconds.

Conversely, the "objective" processes of spinning a roulette wheel or flipping a balanced coin also include both types of events. The uncertain driving variable in such processes is the force of the spin or flip, and we have seen that events such as "red" or "heads" attain their unanimous revealed-likelihood properties because of their structure as almost-objective (i.e., periodic) events in the uncertain state space of force levels. But other events defined on these processes, such as whether the wheel will spin more than 23 times before stopping, or the coin will rotate more than 19 times before landing, correspond to intervals or half-lines in the space of force levels, and if we ask individuals for their subjective likelihoods of *these* events, we can expect the same divergence (or even non-existence) of beliefs characteristic of most subjective events.[23]

Thus, instead of the perspective that God throws *two different types of dice* in the universe, we should adopt the perspective that, for virtually all uncertain physical variables, individuals exhibit very different belief and betting properties toward *interval-based* (i.e. subjective) events defined on these variables than toward *periodic* (i.e. almost-objective) events defined on them. In other words, rather than divide the universe into physical phenomena that *do* satisfy what Anscombe and Aumann (1963) term the "physical theory of chances" and physical phenomena that *don't*, we should recognize that is their *different types of events* (periodic versus interval), rather than *different laws of physics*, that make bets on spinning roulette wheels different from bets on running horses.

### *6.2 Structural assumptions on the choice objects rather than structural assumptions on choice*

The results of this paper also suggest that both the abstract subjective setting of Savage and the objective×subjective setting of Anscombe-Aumann be replaced (or at least supplemented) by the *Euclidean subjective setting*, with its real-valued state space $\mathcal{S} = [\underline{s}, \bar{s}]$ or vector-valued state space $\mathcal{S} \times \mathcal{T} \times \mathcal{R} \times \ldots = [\underline{s}, \bar{s}] \times [\underline{t}, \bar{t}] \times [\underline{r}, \bar{r}] \times \ldots$, as the most analytically fruitful setting for modeling economic choice under uncertainty.

The Euclidean subjective setting is neither as general as that of Savage (who posits an arbitrary infinitely divisible state space $\mathcal{S}$), nor as simple as that of Anscombe-Aumann (who besides their objective roulette wheel, posit only a finite set of states). However, the process of adding structure to an agent's choice space in order to obtain additional results is a standard one in economic analysis: For example, although it is perfectly possible to axiomatize ordinal utility under certainty for very general choice spaces, it is not until the choice space has a Euclidean structure (i.e., becomes a space of commodity bundles) that powerful results like the Slutsky equation will emerge.

The results of this paper have shown how adding structure to the subjective state space – and hence to the objects of choice – allows us to obtain the benefits of probability theory under subjective uncertainty without having to impose the types of

[23] This also holds if the process depends on more than one driving variable: Say that besides the force of the flip, the behavior of the coin also depends on the (unknown) air pressure in the room. In this case, the event "heads" will be a periodic (nonlinear checkerboard-like) event defined over a two-dimensional subjective state space, whereas "rotates more than 19 times" will be a nonlinear half-region of this state space.

strong structural assumptions on act preferences or interpersonal beliefs required by the other two approaches: Adding Euclidean structure to Savage's state space yields a family of subjective events with virtually objective revealed-likelihood and betting properties – not just for individuals who satisfy the Savage (1954) Sure-Thing Principle P2 or the Machina-Schmeidler (1992) Strong Comparative Probability Axiom P4* (both strong global assumptions on act preferences), but for *all* individuals with event-smooth act preferences. In addition, these properties *emerge directly from preferences* (whether or not they are probabilistically sophisticated), in contrast with the objective×subjective approach of Anscombe-Aumann, which requires a prespecified family of events (roulette events) over which all agents must have exogenous, identical *beliefs*.

From a modeling perspective, most situations of physical or economic uncertainty can be represented as being driven by real- or vector-valued underlying state variables: lightning strikes are driven by the electrical potential in the atmosphere; earthquakes by geological pressure and resilience levels; and currency runs by supply, demand and expectation levels. When state spaces are represented in such Euclidean terms, it is reasonable to expect that most individual's betting preferences will be smooth with respect to small changes in the events, so the results of this paper will apply. In other words, the assumption of event-smooth preferences in a Euclidean subjective setting can be viewed as very general in its applicability, as well as providing an underlying behavioral foundation for the theory of "objectively uncertain" events.

While none of the three settings – Savage, Anscombe-Aumann, or Euclidean subjective – scientifically dominates the others, a final scientific advantage of the latter setting over the former two is that is it much easier to *empirically verify* strong structural assumptions on the *state space*, than to verify strong global assumptions on agents' *preferences*, or strong unanimity assumptions on their *beliefs*.

## Appendix

We define a preference function $W(\cdot)$ over $\mathcal{A}$ to be *event-continuous* if

$$\lim_{\delta(f(\cdot),f_0(\cdot))\to 0} W\big(f(\cdot)\big) \underset{\text{all } f_0(\cdot)\in\mathcal{A}}{\equiv} W\big(f_0(\cdot)\big) \tag{A.1}$$

where the distance function $\delta(\cdot,\cdot)$ between two acts is defined by

$$\delta\big(f(\cdot),f_0(\cdot)\big) \;=\; \lambda\big\{s\in\mathcal{S}\,\big|\,f(s)\neq f_0(s)\big\} \tag{A.2}$$

that is, as the Lebesgue measure of the set on which the acts differ. Just as differentiability with respect to real variables can be thought of as "local linearity" in those variables, differentiability with respect to an act $f(\cdot)$'s events $\{f^{-1}(x)\,|\,x\in\mathcal{X}\}$ can be thought of as "local additivity" in these events. Formally, $W(\cdot)$ is *event-additive* if there exists a family of (signed) *evaluation measures* $\{\Phi_x(\cdot)\,|\,x\in\mathcal{X}\}$ such that it takes the form $W(x_1 \text{ on } E_1;\ldots;x_n \text{ on } E_n)\equiv\Phi_{x_1}(E_1)+\ldots+\Phi_{x_n}(E_n)$, that is, where each outcome's event $E_i$ is additively evaluated by that outcome's evaluation measure $\Phi_{x_i}(\cdot)$, and the terms are then summed. We can equivalently write this property as[24]

$$W\big(f(\cdot)\big) \underset{\text{all } f(\cdot)\in\mathcal{A}}{\equiv} \sum_{x\in\mathcal{X}} \varPhi_x\left(f^{-1}(x)\right) \tag{A.3}$$

By event-continuity, each evaluation measure $\varPhi_x(\cdot)$ will be absolutely continuous with respect to Lebesgue measure, and thus have a signed *evaluation density* $\phi_x(\cdot)$. The expected utility forms $W_{SEU}(f(\cdot)) \equiv \sum_{i=1}^{n} U(x_i)\cdot\mu(E_i)$ and $W_{SDEU}(f(\cdot)) \equiv \sum_{i=1}^{n} \int_{E_i} U(x_i|s)\cdot d\mu(s)$ are both event-additive, with

$$\begin{aligned} &\text{for } W_{SEU}(\cdot): \begin{cases} \varPhi_x(E) \equiv U(x)\cdot\mu(E) \\ \phi_x(s) \;\equiv U(x)\cdot\nu(s) \end{cases} \\ &\text{for } W_{SDEU}(\cdot): \begin{cases} \varPhi_x(E) \equiv \int_E U(x|s)\cdot d\mu(s) \\ \phi_x(s) \;\equiv U(x|s)\cdot\nu(s) \end{cases} \end{aligned} \tag{A.4}$$

In fact, as shown in Machina (2002), an event-continuous $W(\cdot)$ will be event-additive *if and only if* it takes the state-independent or state-dependent expected utility form $W_{SEU}(\cdot)$ or $W_{SDEU}(\cdot)$.

Under event-additivity, the *change* in $W(\cdot)$ in going from an act $f_0(\cdot)$ to an act $f(\cdot)$ can be expressed in terms of the *changes* (expansions and contractions) $\varDelta E_x^+ = f^{-1}(x) - f_0^{-1}(x)$ and $\varDelta E_x^- = f_0^{-1}(x) - f^{-1}(x)$ in each outcome's event $E_x$, via the formula

$$\begin{aligned} W\big(f(\cdot)\big) - W\big(f_0(\cdot)\big) &= \sum_{x\in\mathcal{X}} \varPhi_x\big(f^{-1}(x)\big) - \sum_{x\in\mathcal{X}} \varPhi_x\big(f_0^{-1}(x)\big) \\ &= \sum_{x\in\mathcal{X}} \varPhi_x\big(\varDelta E_x^+\big) \;-\; \sum_{x\in\mathcal{X}} \varPhi_x\big(\varDelta E_x^-\big) \end{aligned} \tag{A.5}$$

An event-continuous $W(\cdot)$ is thus said to be *event-differentiable* at act $f_0(\cdot)$ if it is "locally" event-additive there – that is, if there exists a family of absolutely continuous *local evaluation measures* $\{\varPhi_x(\cdot; f_0)\,|\,x\in\mathcal{X}\}$, with corresponding *local evaluation densities* $\{\phi_x(\cdot; f_0)\,|\,x\in\mathcal{X}\}$, such that

$$\begin{aligned} &W\big(f(\cdot)\big) - W\big(f_0(\cdot)\big) \\ &= \sum_{x\in\mathcal{X}} \varPhi_x\big(f^{-1}(x); f_0\big) - \sum_{x\in\mathcal{X}} \varPhi_x\big(f_0^{-1}(x); f_0\big) + o\big(\delta\big(f(\cdot), f_0(\cdot)\big)\big) \\ &= \sum_{x\in\mathcal{X}} \varPhi_x\big(\varDelta E_x^+; f_0\big) - \sum_{x\in\mathcal{X}} \varPhi_x\big(\varDelta E_x^-; f_0\big) + o\big(\delta\big(f(\cdot), f_0(\cdot)\big)\big) \end{aligned} \tag{A.6}$$

To establish our results we must impose some regularity on how much each outcome's local evaluation density $\phi_x(\cdot\,;\cdot)$ can vary in its arguments $s$ and $f(\cdot)$. Although the space of acts $\mathcal{A}$ is not compact with respect to the distance function

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[24] Our restriction to finite-outcome acts ensures that all but a finite number of terms in the following sum will be zero.

$\delta(\cdot,\cdot)$, our regularity conditions consist of the properties that continuous $\phi_x(\cdot\,;\cdot)$ functions would exhibit if their domain $\mathcal{S}\times\mathcal{A}$ *was* compact, namely:[25]

$$\begin{array}{ll} \text{for each outcome } x: & \phi_x(s;f) \text{ is } \textit{uniformly continuous} \text{ over } \mathcal{S}\times\mathcal{A} \\ \text{for each outcome } x: & \phi_x(s;f) \text{ is } \textit{bounded above and below} \text{ on } \mathcal{S}\times\mathcal{A} \\ \text{for each pair } x^*\succ x & \Phi_{x^*}(E;f)-\Phi_x(E;f) \text{ is } \textit{bounded above} \text{ and} \\ \text{and nonnull event } E: & \textit{bounded above } 0\text{, uniformly in } f(\cdot) \end{array} \quad \text{(A.7)}$$

We thus define a general event-differentiable preference function $W(\cdot)$ on $\mathcal{A}$ to be *event-smooth* if it satisfies these properties, which can be stated more formally as:

$$\begin{array}{l} \text{for each } x\in\mathcal{X} \text{ and } \varepsilon>0 \text{ there exists } \delta_{x,\varepsilon}>0 \text{ such that } |s'-s|<\delta_{x,\varepsilon} \\ \text{and } \delta(f'(\cdot),f(\cdot))<\delta_{x,\varepsilon} \text{ implies } |\phi_x(s';f')-\phi_x(s;f)|<\varepsilon \\ \text{for each } x\in\mathcal{X} \text{ there exist } \bar{\phi}_x \text{ and } \underline{\phi}_x \text{ such that } \bar{\phi}_x>\phi_x(s;f)>\underline{\phi}_x \text{ for} \\ \text{all } s\in\mathcal{S} \text{ and all } f(\cdot)\in\mathcal{A} \\ \text{for each pair } x^*\succ x \text{ and nonnull } E \text{ there exist } \bar{\Phi}_{x^*,x,E}>\underline{\Phi}_{x^*,x,E}>0 \\ \text{such that } \bar{\Phi}_{x^*,x,E}>\Phi_{x^*}(E;f)-\Phi_x(E;f)>\underline{\Phi}_{x^*,x,E} \text{ for all } f(\cdot)\in\mathcal{A} \end{array} \quad \text{(A.7}')$$

To see how event-differentiability can be applied to analyze $W(\cdot)$'s ranking of acts, define the *single-sweep path* $\{f_\omega(\cdot)\,|\,\omega\in[\underline{s},\bar{s}]\}$ from the constant act $[x_0 \text{ on } \mathcal{S}]$ to $[x^* \text{ on } \mathcal{S}]$ by

$$f_\omega(\cdot)=\big[x^* \text{ on } [\underline{s},\omega];\,x_0 \text{ on } (\omega,\bar{s}]\big] \quad \text{(A.8)}$$

As $\omega$ runs from $\underline{s}$ to $\bar{s}$, the outcome $x_0$ is seen to be replaced by $x^*$ over an expanding event $[\underline{s},\omega]$ that uniformly "sweeps" across the state space $\mathcal{S}=[\underline{s},\bar{s}]$. Since the change sets in going from act $f_{\hat{\omega}}(\cdot)$ to $f_{\hat{\omega}+\Delta\omega}(\cdot)$ are given by $\Delta E^+_{x^*}=\Delta E^-_{x_0}=(\hat{\omega},\hat{\omega}+\Delta\omega]$, (A.6) and (A.2) imply

$$\begin{aligned} &W\big(f_{\hat{\omega}+\Delta\omega}(\cdot)\big)-W\big(f_{\hat{\omega}}(\cdot)\big) \\ &\quad=\ \Phi_{x^*}\big((\hat{\omega},\,\hat{\omega}+\Delta\omega];f_{\hat{\omega}}\big)-\Phi_{x_0}\big((\hat{\omega},\,\hat{\omega}+\Delta\omega];f_{\hat{\omega}}\big)+o\big(|\Delta\omega|\big) \end{aligned} \quad \text{(A.9)}$$

Dividing both sides by $\Delta\omega$ and letting $\Delta\omega\to 0$ yields

$$\left.\frac{dW\big(f_\omega(\cdot)\big)}{d\omega}\right|_{\omega=\hat{\omega}}\ =\ \phi_{x^*}(\hat{\omega};f_{\hat{\omega}})-\phi_{x_0}(\hat{\omega};f_{\hat{\omega}}) \quad \text{(A.10)}$$

In other words, starting at the act $f_{\hat{\omega}}(\cdot)$, the differential effect of replacing $x_0$ by $x^*$ at state $s=\hat{\omega}$ is given by the term $\phi_{x^*}(\hat{\omega};f_{\hat{\omega}})-\phi_{x_0}(\hat{\omega};f_{\hat{\omega}})$. From (A.4) it follows that for the form $W_{SEU}(\cdot)$ this term reduces to $[U(x^*)-U(x_0)]\cdot\nu(\hat{\omega})$, and for

[25] In line with our approach of placing no restrictions on nature or extent of the outcome space, we impose no uniformity or boundedness conditions on $\phi_x(s;f)$ or $\Phi_{x^*}(E;f)-\Phi_x(E;f)$ as $x$ or $x^*$ range over $\mathcal{X}$. Thus, we still allow *outcomes* to become arbitrarily "far apart" in preference, as with risk neutral preferences over monetary lotteries.

$W_{SDEU}(\cdot)$ it reduces to $[U(x^*|\hat{\omega}) - U(x_0|\hat{\omega})]\cdot\nu(\hat{\omega})$. Applying the Fundamental Theorem of Calculus to (A.10) yields the path integral formula

$$\begin{aligned} W\big(x^* \text{ on } \mathcal{S}\big) - W\big(x_0 \text{ on } \mathcal{S}\big) &= \int_{\underline{s}}^{\bar{s}} \frac{dW\big(f_\omega(\cdot)\big)}{d\omega}\cdot d\omega \\ &= \int_{\underline{s}}^{\bar{s}} \big[\phi_{x^*}(\omega; f_\omega) - \phi_{x_0}(\omega; f_\omega)\big]\cdot d\omega \end{aligned} \tag{A.11}$$

which exactly characterizes $W(\cdot)$'s comparison of the acts $[x_0 \text{ on } \mathcal{S}]$ versus $[x^* \text{ on } \mathcal{S}]$ in terms of its local evaluation densities at the acts $f_\omega(\cdot)$ along the path $\{f_\omega(\cdot)\,|\,\omega\in[\underline{s},\bar{s}]\}$ between them.

We define the single-sweep path between two *nonconstant* acts $f(\cdot)$ and $f^*(\cdot)$ by

$$f_\omega(\cdot) = \big[f^*(\cdot) \text{ on } [\underline{s},\omega]; f(\cdot) \text{ on } (\omega,\bar{s}]\big] \tag{A.12}$$

in which case $W(\cdot)$'s path derivative and path integral formulas are similarly given by

$$\left.\frac{dW\big(f_\omega(\cdot)\big)}{d\omega}\right|_{\omega=\hat{\omega}} = \phi_{f^*(\hat{\omega})}(\hat{\omega}; f_{\hat{\omega}}) - \phi_{f(\hat{\omega})}(\hat{\omega}; f_{\hat{\omega}}) \tag{A.13}$$

$$\begin{aligned} W\big(f^*(\cdot)\big) - W\big(f(\cdot)\big) &= \int_{\underline{s}}^{\bar{s}} \frac{dW\big(f_\omega(\cdot)\big)}{d\omega}\cdot d\omega \\ &= \int_{\underline{s}}^{\bar{s}} \big[\phi_{f^*(\omega)}(\omega; f_\omega) - \phi_{f(\omega)}(\omega; f_\omega)\big]\cdot d\omega \end{aligned} \tag{A.14}$$

which again characterize $W(\cdot)$'s evaluation of $f(\cdot)$ versus $f^*(\cdot)$ in terms of its local evaluation densities at the acts $f_\omega(\cdot)$ along the path $\{f_\omega(\cdot)\,|\,\omega\in[\underline{s},\bar{s}]\}$ between them. $W(\cdot)$'s path derivative along any *almost-objective mixture path* $\{f^m_\alpha(\cdot)|\,\alpha\in[0,1]\} = \{[f^*(\cdot) \text{ on } [0,\alpha] \underset{m}{\times} \mathcal{S}; f(\cdot) \text{ on } (\alpha,1] \underset{m}{\times} \mathcal{S}]\,|\,\alpha\in[0,1]\}$ from $f(\cdot)$ to $f^*(\cdot)$ will be given by the "$m$-sweep" version of (A.13), namely (A.28) below.

**Lemma.** The following result gives conditions under which the integrals of a family of functions on $[\underline{s},\bar{s}]$ can be uniformly approximated by their Riemann sums:

**Lemma.** *Let the family of functions $\{g(\cdot;\tau)\,|\,\tau\in\mathcal{T}\}$ over $[\underline{s},\bar{s}]$ be both uniformly bounded and uniformly continuous over each interval $E_j$ of some finite interval partition $\{E_1,\ldots,E_J\}$ of $[\underline{s},\bar{s}]$. Then for each $\varepsilon>0$ there exists some $m_\varepsilon$ such that*

$$\left|\int_{\underline{s}}^{\bar{s}} g(s;\tau)\cdot ds - \frac{\lambda_{\mathcal{S}}}{m}\cdot\sum_{i=0}^{m-1} g\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m};\tau\big)\right| < \varepsilon \quad \begin{array}{l}\text{all } m\geq m_\varepsilon\\ \text{all } \alpha\in[0,1]\\ \text{all } \tau\in T\end{array} \tag{A.15}$$

*Proof of Lemma.* Since the functions $\{g(\cdot;\tau)|\tau\in T\}$ are uniformly bounded over each interval $E_1,\ldots,E_J$, they are uniformly bounded over $[\underline{s},\bar{s}]$ by some values $\underline{g}<\bar{g}$. Given $\varepsilon>0$, uniform continuity on each interval $E_1,\ldots,E_J$ implies some $\delta_\varepsilon>0$ such that if $s,s'\in E_j$ and $|s'-s|<\delta_\varepsilon$ then $|g(s';\tau)-g(s;\tau)|<\varepsilon/(2\cdot\lambda_{\mathcal{S}})$ for all $\tau\in T$. Select $m_\varepsilon$ such that $\lambda_{\mathcal{S}}/m_\varepsilon<\min\{\delta_\varepsilon,\varepsilon/(2\cdot J\cdot(\bar{g}-\underline{g}))\}$.

Given arbitrary $m\geq m_\varepsilon$ and arbitrary $\alpha\in[0,1)$ (we treat the case of $\alpha=1$ below), partition $[\underline{s},\bar{s}]$ into the following intervals of length $\lambda_{\mathcal{S}}/m$:

$$\begin{aligned} I_0=\big[\underline{s},\underline{s}+\tfrac{\lambda_{\mathcal{S}}}{m}\big)\ ,\ \ \ldots\ ,\ I_i=\big[\underline{s}+\tfrac{i\cdot\lambda_{\mathcal{S}}}{m},\underline{s}+\tfrac{(i+1)\cdot\lambda_{\mathcal{S}}}{m}\big)\ ,\ \ldots \\ \ldots\ ,\ I_{m-1}=\big[\underline{s}+\tfrac{(m-1)\cdot\lambda_{\mathcal{S}}}{m},\bar{s}\big] \end{aligned} \tag{A.16}$$

(For $m=1$, define $I_0=[\underline{s},\bar{s}]$). Observe that, for all $i=0,\ldots,m-1$, the value $\underline{s}+\frac{(i+\alpha)\cdot\lambda_s}{m}$ lies in $I_i$. For each interval $I_i$ that lies wholly within some interval $E_j$ (and there can be up to $m$ such intervals), the above uniform continuity property implies

$$\left|\int_{I_i}\Big[g(s;\tau)-g\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m};\tau\big)\Big]\cdot ds\right| \underset{\text{all }\tau\in T}{<} \frac{\varepsilon}{2\cdot\lambda_{\mathcal{S}}}\cdot\frac{\lambda_{\mathcal{S}}}{m} = \frac{\varepsilon}{2\cdot m} \tag{A.17}$$

For each interval $I_i$ that *does not* lie wholly within one of the intervals $E_1,\ldots,E_J$ (and there can be up to $J-1$ such intervals), uniform boundedness implies

$$\left|\int_{I_i}\Big[g(s;\tau)-g\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m};\tau\big)\Big]\cdot ds\right| \underset{\text{all }\tau\in T}{\leq} (\bar{g}-\underline{g})\cdot\frac{\lambda_{\mathcal{S}}}{m} < \frac{\varepsilon}{2\cdot J} \tag{A.18}$$

We thus have the following for all $\tau\in T$:

$$\begin{aligned} &\left|\int_{\underline{s}}^{\bar{s}} g(s;\tau)\cdot ds - \frac{\lambda_{\mathcal{S}}}{m}\cdot\sum_{i=0}^{m-1} g\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m};\tau\big)\right| \\ =\ &\left|\sum_{i=0}^{m-1}\int_{I_i}\Big[g(s;\tau)-g\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m};\tau\big)\Big]\cdot ds\right| \\ \leq\ &\sum_{i=0}^{m-1}\left|\int_{I_i}\Big[g(s;\tau)-g\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m};\tau\big)\Big]\cdot ds\right| \\ <\ & m\cdot\frac{\varepsilon}{2\cdot m} + (J-1)\cdot\frac{\varepsilon}{2\cdot J} < \varepsilon \end{aligned} \tag{A.19}$$

For the case $\alpha=1$, repeat the above argument with (A.16) replaced by the equal-length intervals

$$\begin{aligned} I_0=\big[\underline{s},\underline{s}+\tfrac{\lambda_{\mathcal{S}}}{m}\big]\ ,\ \ \ldots\ ,\ I_i=\big(\underline{s}+\tfrac{i\cdot\lambda_{\mathcal{S}}}{m},\underline{s}+\tfrac{(i+1)\cdot\lambda_{\mathcal{S}}}{m}\big]\ ,\ \ldots\ , \\ \ldots\ ,\ I_{m-1}=\big(\underline{s}+\tfrac{(m-1)\cdot\lambda_{\mathcal{S}}}{m},\bar{s}\big] \end{aligned} \tag{A.16$'$}$$

□

**Line Integral Approximation Theorem.** The following result for characterizing $W(\cdot)$'s evaluation of acts $f(\cdot)$ versus $f^*(\cdot)$ differs from the characterization formulas (A.11) and (A.14) in two respects: (i) it consists of the *limit* of a sequence of path integrals, and (ii) for each integral in this sequence, its integrand

$\sum_{x\in\mathcal{X}}[\Phi_x(f^{*-1}(x);f^m_\alpha)-\Phi_x(f^{-1}(x);f^m_\alpha)]$ evaluates the *same* events – namely the two acts' events $\{f^{-1}(x)\,|\,x\in\mathcal{X}\}$ and $\{f^{*-1}(x)\,|\,x\in\mathcal{X}\}$ – at every point along its path $\{f^m_\alpha(\cdot)\,|\,\alpha\in[0,1]\}$.

**Line Integral Approximation Theorem.** *If $W(\cdot)$ is event-smooth, then for any acts $f(\cdot), f^*(\cdot)$ and any $\varepsilon>0$ there exists $m_\varepsilon$ such that for each $m\geq m_\varepsilon$, $W(\cdot)$'s path derivative along the almost-objective mixture path $\{f^m_\alpha(\cdot)|\alpha\in[0,1]\}$ = $\{[f^*(\cdot)\text{ on }[0,\alpha]\underset{m}{\times}\mathcal{S}\,;f(\cdot)\text{ on }(\alpha,1]\underset{m}{\times}\mathcal{S}]\,|\,\alpha\in[0,1]\}$ from $f(\cdot)$ to $f^*(\cdot)$ exists and satisfies*

$$\left|\frac{dW\big(f^m_\alpha(\cdot)\big)}{d\alpha}-\sum_{x\in\mathcal{X}}\Big[\Phi_x\big(f^{*-1}(x);f^m_\alpha\big)-\Phi_x\big(f^{-1}(x);f^m_\alpha\big)\Big]\right| < \varepsilon \quad \text{(A.20)}$$

*at all but a finite number of values of $\alpha\in[0,1]$. This in turn implies the line integral approximation formula*

$$\begin{aligned}&W\big(f^*(\cdot)\big)-W\big(f(\cdot)\big)\\ =\;&\lim_{m\to\infty}\int_0^1\sum_{x\in\mathcal{X}}\Big[\Phi_x\big(f^{*-1}(x);f^m_\alpha\big)-\Phi_x\big(f^{-1}(x);f^m_\alpha\big)\Big]\cdot d\alpha\end{aligned} \quad \text{(A.21)}$$

*Proof.* Given arbitrary $\varepsilon>0$, represent the acts $f(\cdot)$ and $f^*(\cdot)$ as

$$f(\cdot)=[x_1\text{ on }E_1;\ldots;x_J\text{ on }E_J]\qquad f^*(\cdot)=[x_1^*\text{ on }E_1;\ldots;x_J^*\text{ on }E_J] \quad \text{(A.22)}$$

for some interval partition $\{E_1,\ldots,E_J\}$ of $[\underline{s},\bar{s}]$, and define the two families of functions $\{g(\cdot;\hat{f})\,|\,\hat{f}(\cdot)\in\mathcal{A}\}$ and $\{g^*(\cdot;\hat{f})\,|\,\hat{f}(\cdot)\in\mathcal{A}\}$ over $[\underline{s},\bar{s}]$ by

$$g(s;\hat{f})\underset{s,\hat{f}(\cdot)}{\equiv}\phi_{f(s)}(s;\hat{f})\qquad g^*(s;\hat{f})\underset{s,\hat{f}(\cdot)}{\equiv}\phi_{f^*(s)}(s;\hat{f}) \quad \text{(A.23)}$$

Since $g(\cdot;\hat{f})\equiv\phi_{x_j}(\cdot;\hat{f})$ and $g^*(\cdot;\hat{f})\equiv\phi_{x_j^*}(\cdot;\hat{f})$ over each interval $E_j$, we have that for all $\hat{f}(\cdot)\in\mathcal{A}$:

$$\begin{aligned}\int_{\underline{s}}^{\bar{s}}g(s;\hat{f})\cdot ds&=\sum\nolimits_{j=1}^{J}\int_{E_j}\phi_{x_j}(s;\hat{f})\cdot ds=\sum\nolimits_{j=1}^{J}\Phi_{x_j}\big(E_j;\hat{f}\big)\\&=\sum_{x\in\mathcal{X}}\Phi_x\big(f^{-1}(x);\hat{f}\big)\\ \int_{\underline{s}}^{\bar{s}}g^*(s;\hat{f})\cdot ds&=\sum\nolimits_{j=1}^{J}\int_{E_j}\phi_{x_j^*}(s;\hat{f})\cdot ds=\sum\nolimits_{j=1}^{J}\Phi_{x_j^*}\big(E_j;\hat{f}\big)\\&=\sum_{x\in\mathcal{X}}\Phi_x\big(f^{*-1}(x);\hat{f}\big)\end{aligned} \quad \text{(A.24)}$$

By event-smoothness, the families of functions $\{g(\cdot;\hat{f})\,|\,\hat{f}(\cdot)\in\mathcal{A}\}$ and $\{g^*(\cdot;\hat{f})\,|\,\hat{f}(\cdot)\in\mathcal{A}\}$ are uniformly bounded over each $E_j$. Similarly, $\{g(\cdot;\hat{f})\,|\,\hat{f}(\cdot)\in\mathcal{A}\}$ and

$\{g^*(\cdot;\hat{f})\,|\,\hat{f}(\cdot)\in\mathcal{A}\}$ are uniformly continuous over each $E_j$. A double application of the Lemma thus yields $m_\varepsilon$ such that

$$\begin{aligned}\left|\int_{\underline{s}}^{\bar{s}} g(s;\hat{f})\cdot ds - \frac{\lambda_{\mathcal{S}}}{m}\cdot\sum\nolimits_{i=0}^{m-1} g\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m};\hat{f}\big)\right| &< \frac{\varepsilon}{2}\\ \left|\int_{\underline{s}}^{\bar{s}} g^*(s;\hat{f})\cdot ds - \frac{\lambda_{\mathcal{S}}}{m}\cdot\sum\nolimits_{i=0}^{m-1} g^*\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m};\hat{f}\big)\right| &< \frac{\varepsilon}{2}\end{aligned} \tag{A.25}$$

and hence

$$\begin{aligned}&\bigg|\int_{\underline{s}}^{\bar{s}}\big[g^*(s;\hat{f})-g(s;\hat{f})\big]\cdot ds\\ &-\sum\nolimits_{i=0}^{m-1}\frac{\lambda_{\mathcal{S}}}{m}\cdot\Big[g^*\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m};\hat{f}\big)-g\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m};\hat{f}\big)\Big]\bigg| < \varepsilon\end{aligned} \tag{A.26}$$

for all $m\geq m_\varepsilon$, all $\alpha\in[0,1]$ and all $\hat{f}(\cdot)\in\mathcal{A}$.
Given arbitrary $m\geq m_\varepsilon$, let $\{f_\alpha^m(\cdot)|\alpha\in[0,1]\}=\{[f^*(\cdot)$ on $[0,\alpha]\underset{m}{\times}\mathcal{S};f(\cdot)$ on $(\alpha,1]\underset{m}{\times}\mathcal{S}]\,|\,\alpha\in[0,1]\}$ be the almost-objective mixture path from $f(\cdot)$ to $f^*(\cdot)$. Although $W(f_\alpha^m(\cdot))$ is *continuous* in $\alpha$ at all $\alpha\in[0,1]$, in general it will only be *differentiable* in $\alpha$ when each of the “front points” of its multiple sweep, that is, when each of the $m$ points

$$\underline{s}+\tfrac{\alpha\cdot\lambda_{\mathcal{S}}}{m}\ ,\ \ldots\ ,\ \underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m}\ ,\ \ldots\ ,\ \underline{s}+\tfrac{(m-1+\alpha)\cdot\lambda_{\mathcal{S}}}{m} \tag{A.27}$$

are continuity points of both $f(\cdot)$ and $f^*(\cdot)$. But this will be true at all but a finite number of $\alpha$ in $[0,1]$, so that except at that this finite set of values, $dW(f_\alpha^m(\cdot))/d\alpha$ exists and is given by

$$\begin{aligned}\frac{dW\big(f_\alpha^m(\cdot)\big)}{d\alpha} &= \sum\nolimits_{i=0}^{m-1}\frac{\lambda_{\mathcal{S}}}{m}\cdot\Big[\phi_{f^*(\underline{s}+\frac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m})}\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m};f_\alpha^m\big)\\ &\qquad\qquad -\phi_{f(\underline{s}+\frac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m})}\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m};f_\alpha^m\big)\Big]\\ &= \sum\nolimits_{i=0}^{m-1}\frac{\lambda_{\mathcal{S}}}{m}\cdot\Big[g^*\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m};f_\alpha^m\big)-g\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{m};f_\alpha^m\big)\Big]\end{aligned} \tag{A.28}$$

Thus at all $\alpha\in[0,1]$ except this finite set of values, we can evaluate (A.26) at $\hat{f}(\cdot)=f_\alpha^m(\cdot)$ and substitute in (A.24) and (A.28) to obtain

$$\left|\frac{dW\big(f_\alpha^m(\cdot)\big)}{d\alpha}-\sum_{x\in\mathcal{X}}\Big[\varPhi_x\big(f^{*-1}(x);f_\alpha^m\big)-\varPhi_x\big(f^{-1}(x);f_\alpha^m\big)\Big]\right|<\varepsilon \tag{A.29}$$

To establish the line integral approximation formula (A.21), consider arbitrary $\varepsilon>0$ and observe that for each $m\geq m_\varepsilon$, the fact that (A.29) holds at all but a finite number of $\alpha$ in $[0,1]$ implies

$$\left|W\big(f^*(\cdot)\big)-W\big(f(\cdot)\big)-\int_0^1\sum_{x\in\mathcal{X}}\Big[\varPhi_x\big(f^{*-1}(x);f_\alpha^m\big)-\varPhi_x\big(f^{-1}(x);f_\alpha^m\big)\Big]\cdot d\alpha\right| =$$

$$\left| \int_0^1 \frac{dW\big(f_\alpha^m(\cdot)\big)}{d\alpha} \cdot d\alpha - \int_0^1 \sum_{x\in\mathcal{X}} \Big[ \varPhi_x\big(f^{*-1}(x); f_\alpha^m\big) - \varPhi_x\big(f^{-1}(x); f_\alpha^m\big)\Big] \cdot d\alpha \right| \tag{A.30}$$

$$\leq \int_0^1 \left| \frac{dW\big(f_\alpha^m(\cdot)\big)}{d\alpha} - \sum_{x\in\mathcal{X}} \Big[ \varPhi_x\big(f^{*-1}(x); f_\alpha^m\big) - \varPhi_x\big(f^{-1}(x); f_\alpha^m\big)\Big] \right| \cdot d\alpha \ < \ \varepsilon \qquad \square$$

*Proof of Theorem 0.* Given arbitrary family of signed measures $\{K(\cdot;\tau)\,|\,\tau \in T\}$ on $\mathcal{S}$ with uniformly bounded and uniformly continuous densities $\{k(\cdot;\tau)\,|\,\tau \in T\}$, finite interval union $\wp \subseteq [0,1]$, *interval* $E = [s_1, s_2] \subseteq \mathcal{S}$ and $\varepsilon > 0$, we will obtain $m_\varepsilon$ such that $|K(\wp \underset{m}{\times} E;\tau) - \lambda(\wp)\cdot K(E;\tau)| < \varepsilon$ for all $m > m_\varepsilon$ and all $\tau \in T$. All assertions of the theorem will follow from this, by additivity or as special cases.

Uniform boundedness implies some $k_{max} \in (0,\infty)$ such that $|k(s;\tau)| < k_{max}$ for all $s \in \mathcal{S}$ and $\tau \in T$, and uniform continuity implies some $\gamma > 0$ such that if $|s - s'| < \gamma$ then $|k(s;\tau) - k(s';\tau)| < \varepsilon/(2\cdot\lambda_{\mathcal{S}})$ for all $\tau \in T$. Select $m_\varepsilon$ so that $\lambda_{\mathcal{S}}/m_\varepsilon < \min\{\gamma, \varepsilon/(8\cdot k_{max})\}$, and consider arbitrary $m > m_\varepsilon$ and $\tau \in T$.

Of the $m$ successive equal-length intervals

$$\begin{aligned} I_0 = \big[\underline{s}, \underline{s} + \tfrac{\lambda_{\mathcal{S}}}{m}\big)\ ,\ \ \ldots\ ,\ I_i = \big[\underline{s} + i\cdot\tfrac{\lambda_{\mathcal{S}}}{m}, \underline{s} + (i+1)\cdot\tfrac{\lambda_{\mathcal{S}}}{m}\big)\ ,\ \ldots \\ \ldots\ ,\ \ I_{m-1} = \big[\underline{s} + (m-1)\cdot\tfrac{\lambda_{\mathcal{S}}}{m}, \bar{s}\big] \end{aligned} \tag{A.31}$$

exactly one (call it $I_{\underline{i}}$) will contain $s_1$, exactly one (call it $I_{\bar{i}}$) will contain $s_2$, and each of the remaining intervals $I_i$ will either be fully contained in $E$ or disjoint from $E$, which implies both

$$\begin{aligned} K\big(\wp \underset{m}{\times} E;\tau\big) = \int_{I_{\underline{i}}\cap(\wp \underset{m}{\times} E)} k(s;\tau)\cdot ds + \sum_{I_i \subseteq E} \int_{\{\underline{s} + \frac{(i+\omega)\cdot\lambda_{\mathcal{S}}}{m}\,|\,\omega\in\wp\}} k(s;\tau)\cdot ds \\ + \int_{I_{\bar{i}}\cap(\wp \underset{m}{\times} E)} k(s;\tau)\cdot ds \end{aligned} \tag{A.32}$$

$$\begin{aligned} \lambda(\wp)\cdot K\big(E;\tau\big) = \int_{I_{\underline{i}}\cap E} \lambda(\wp)\cdot k(s;\tau)\cdot ds + \sum_{I_i \subseteq E} \int_{I_i} \lambda(\wp)\cdot k(s;\tau)\cdot ds \\ + \int_{I_{\bar{i}}\cap E} \lambda(\wp)\cdot k(s;\tau)\cdot ds \end{aligned} \tag{A.33}$$

The first and third integrals in (A.32) and (A.33) will each be less than $k_{max}\cdot\lambda_{\mathcal{S}}/m$ in absolute value. For each interval $I_i \subseteq E$, all states in $I_i$ are within distance $\gamma$ of each other, so $|k(s;\tau) - k(s';\tau)| < \varepsilon/(2\cdot\lambda_{\mathcal{S}})$ for all $s$, $s' \in I_i$, which in turn implies

$$\left| k(s;\tau) - \frac{m}{\lambda_{\mathcal{S}}}\cdot\int_{I_i} k(s';\tau)\cdot ds' \right| \ < \ \frac{\varepsilon}{2\cdot\lambda_{\mathcal{S}}} \tag{A.34}$$

for all $s \in I_i$. Since $\lambda\{\underline{s} + (i+\omega)\cdot\lambda_{\mathcal{S}}/m \,|\, \omega \in \wp\} = \lambda(\wp)\cdot\lambda_{\mathcal{S}}/m$, for each $I_i \subseteq E$ we have

$$\left| \int_{\{\underline{s} + \frac{(i+\omega)\cdot\lambda_{\mathcal{S}}}{m}\,|\,\omega\in\wp\}} k(s;\tau)\cdot ds \ - \int_{I_i} \lambda(\wp)\cdot k(s';\tau)\cdot ds' \right| \ =$$

$$\begin{aligned}
&\left|\int_{\{\underline{s}+\frac{(i+\omega)\cdot\lambda_{\mathcal{S}}}{m}|\omega\in\wp\}} k(s;\tau)\cdot ds - \int_{\{\underline{s}+\frac{(i+\omega)\cdot\lambda_{\mathcal{S}}}{m}|\omega\in\wp\}} \frac{m}{\lambda_{\mathcal{S}}}\cdot ds \cdot \int_{I_i} k(s';\tau)\cdot ds'\right| \\
&= \left|\int_{\{\underline{s}+\frac{(i+\omega)\cdot\lambda_{\mathcal{S}}}{m}|\omega\in\wp\}} \left[k(s;\tau) - \frac{m}{\lambda_{\mathcal{S}}}\cdot\int_{I_i} k(s';\tau)\cdot ds'\right]\cdot ds\right| \qquad \text{(A.35)} \\
&< \int_{\{\underline{s}+\frac{(i+\omega)\cdot\lambda_{\mathcal{S}}}{m}|\omega\in\wp\}} \frac{\varepsilon}{2\cdot\lambda_{\mathcal{S}}}\cdot ds = \frac{\varepsilon}{2\cdot\lambda_{\mathcal{S}}}\cdot\lambda(\wp)\cdot\frac{\lambda_{\mathcal{S}}}{m} \le \frac{\varepsilon}{2\cdot m}
\end{aligned}$$

Subtracting (A.33) from (A.32) and invoking the above inequalities yields

$$\left|K\big(\wp \underset{m}{\times} E;\tau\big) - \lambda(\wp)\cdot K\big(E;\tau\big)\right| < 4\cdot\frac{k_{max}\cdot\lambda_{\mathcal{S}}}{m} + \sum_{I_i\subseteq E}\frac{\varepsilon}{2\cdot m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{(A.36)}$$

□

*Proof of Theorem 1.* This result follows from Theorem 2, by setting $E$ in that theorem equal to $\mathcal{S}$. □

*Proof of Theorem 2.* Given disjoint $\wp, \wp'$, nonnull event $E \in \mathcal{E}$, outcomes $x^* \succ x$ and act $f(\cdot) = [x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n]$, define $\wp_0 = [0,1] - (\wp \cup \wp')$. Since we can express (18)'s left-hand acts $f_m(\cdot) = [x^* \text{ on } \wp \underset{m}{\times} E; x \text{ on } \wp' \underset{m}{\times} E; f(\cdot) \text{ elsewhere}]$ as

$$\begin{aligned}
&\big[x^* \text{ on } \wp \underset{m}{\times} (E\cap E_1); x \text{ on } \wp' \underset{m}{\times} (E\cap E_1); x_1 \text{ on } \wp_0 \underset{m}{\times} (E\cap E_1); x_1 \text{ on } E_1 - E; \ldots; \\
&x^* \text{ on } \wp \underset{m}{\times} (E\cap E_n); x \text{ on } \wp' \underset{m}{\times} (E\cap E_n); x_n \text{ on } \wp_0 \underset{m}{\times} (E\cap E_n); x_n \text{ on } E_n - E\big] \qquad \text{(A.37)}
\end{aligned}$$

an argument identical to Step 1 of the Proof of Theorem 4 establishes that $\lim_{m\to\infty} W(f_m(\cdot))$ exists. A similar argument establishes the existence of $\lim_{m\to\infty} W(f'_m(\cdot))$ for (18)'s right-hand acts $f'_m(\cdot) = [x \text{ on } \wp \underset{m}{\times} E; x^* \text{ on } \wp' \underset{m}{\times} E; f(\cdot)$ elsewhere].

For $\lambda(\wp) = \lambda(\wp')$, we establish $\lim_{m\to\infty} W(f_m(\cdot)) = \lim_{m\to\infty} W(f'_m(\cdot))$ by taking arbitrary $\varepsilon > 0$ and obtaining $m_\varepsilon$ such that $|W(f'_m(\cdot)) - W(f_m(\cdot))| < \varepsilon$ for all $m > m_\varepsilon$. Applying event-smoothness and Theorem 0 to the families of signed measures $\{\Phi_{x^*}(\cdot;f)\,|\,f(\cdot)\in\mathcal{A}\}$ and $\{\Phi_x(\cdot;f)\,|\,f(\cdot)\in\mathcal{A}\}$ yields $m_\varepsilon$ such that

$$\begin{aligned}
&\left|\Phi_{x^*}(\wp \underset{m}{\times} E;f) - \lambda(\wp)\cdot\Phi_{x^*}(E;f)\right|, \left|\Phi_x(\wp \underset{m}{\times} E;f) - \lambda(\wp)\cdot\Phi_x(E;f)\right| \\
&\left|\Phi_{x^*}(\wp' \underset{m}{\times} E;f) - \lambda(\wp')\cdot\Phi_{x^*}(E;f)\right|, \left|\Phi_x(\wp' \underset{m}{\times} E;f) - \lambda(\wp')\cdot\Phi_x(E;f)\right|
\end{aligned} \qquad \text{(A.38)}$$

are all less than $\varepsilon/8$ for all $m > m_\varepsilon$ and all $f(\cdot) \in \mathcal{A}$. Select arbitrary $m > m_\varepsilon$ and consider the acts $f_m(\cdot)$ and $f'_m(\cdot)$. By (A.38) and $\lambda(\wp') = \lambda(\wp)$, we have

$$\left|\Phi_{x^*}(\wp \underset{m}{\times} E;f) - \Phi_{x^*}(\wp' \underset{m}{\times} E;f) + \Phi_x(\wp' \underset{m}{\times} E;f) - \Phi_x(\wp \underset{m}{\times} E;f)\right| \underset{\text{all} f(\cdot)\in\mathcal{A}}{<} \frac{\varepsilon}{2} \qquad \text{(A.39)}$$

Defining $\{f^k_\alpha(\cdot)\,|\,\alpha \in [0,1]\}_{k=1}^\infty = \{[f'_m(\cdot) \text{ on } [0,\alpha] \underset{k}{\times} \mathcal{S}; f_m(\cdot) \text{ on } (\alpha,1] \underset{k}{\times} \mathcal{S}]\,|\,\alpha \in [0,1]\}_{k=1}^\infty$ as the almost-objective mixture paths from $f'_m(\cdot)$ to $f_m(\cdot)$, (A.21) implies

$$W\big(f_m(\cdot)\big) - W\big(f'_m(\cdot)\big)$$

$$= \lim_{k\to\infty} \int_0^1 \sum_{x\in\mathcal{X}} \Big[ \varPhi_x\big(f_m^{-1}(x); f_\alpha^k\big) - \varPhi_x\big(f_m^{\prime -1}(x); f_\alpha^k\big) \Big] \cdot d\alpha$$

$$= \lim_{k\to\infty} \int_0^1 \Big[ \varPhi_{x^*}\big(\wp \underset{m}{\times} E; f_\alpha^k\big) - \varPhi_{x^*}\big(\wp' \underset{m}{\times} E; f_\alpha^k\big) + \varPhi_x\big(\wp' \underset{m}{\times} E; f_\alpha^k\big) - \varPhi_x\big(\wp \underset{m}{\times} E; f_\alpha^k\big) \Big] \cdot d\alpha \qquad \text{(A.40)}$$

By (A.39), the integrand in (A.40) is less than $\varepsilon/2$ in absolute value for all $\alpha \in [0,1]$ and all $k$, which implies $|W(f'_m(\cdot)) - W(f_m(\cdot))| < \varepsilon$.

For $\lambda(\wp) > \lambda(\wp')$, we establish $\lim_{m\to\infty} W(f_m(\cdot)) > \lim_{m\to\infty} W(f'_m(\cdot))$ by obtaining obtain $\hat{m}$ and $\delta > 0$ such that $W(f_m(\cdot)) - W(f'_m(\cdot)) \geq \delta/2$ for all $m > \hat{m}$. Recall that $\varPhi_{x*}(E; f) - \varPhi_x(E; f)$ is bounded above 0 by some positive $\underline{\varPhi}_{x^*,x,E}$ for all $f(\cdot) \in \mathcal{A}$, and define $\delta = [\lambda(\wp) - \lambda(\wp')] \cdot \underline{\varPhi}_{x^*,x,E}$. Applying event-smoothness and Theorem 0 to $\{\varPhi_{x^*}(\cdot; f) \,|\, f(\cdot) \in \mathcal{A}\}$ and $\{\varPhi_x(\cdot; f) \,|\, f(\cdot) \in \mathcal{A}\}$ yields an $\hat{m}$ such that

$$\Big|\varPhi_{x^*}(\wp \underset{m}{\times} E; f) - \lambda(\wp)\cdot\varPhi_{x^*}(E; f)\Big| \;,\; \Big|\varPhi_x(\wp \underset{m}{\times} E; f) - \lambda(\wp)\cdot\varPhi_x(E; f)\Big|$$
$$\Big|\varPhi_{x^*}(\wp' \underset{m}{\times} E; f) - \lambda(\wp')\cdot\varPhi_{x^*}(E; f)\Big| \;,\; \Big|\varPhi_x(\wp' \underset{m}{\times} E; f) - \lambda(\wp')\cdot\varPhi_x(E; f)\Big| \qquad \text{(A.41)}$$

are all less than $\delta/8$ for all $m > \hat{m}$ and all $f(\cdot) \in \mathcal{A}$, which in turn implies

$$\varPhi_{x^*}(\wp \underset{m}{\times} E; f) - \varPhi_{x^*}(\wp' \underset{m}{\times} E; f) + \varPhi_x(\wp' \underset{m}{\times} E; f) - \varPhi_x(\wp \underset{m}{\times} E; f)$$
$$> \; [\lambda(\wp) - \lambda(\wp')] \cdot \big(\varPhi_{x^*}(E; f) - \varPhi_x(E; f)\big) - \delta/2 \;\; \geq \;\; \delta/2 \qquad \text{(A.42)}$$

for all $m > \hat{m}$ and all $f(\cdot) \in \mathcal{A}$. Select arbitrary $m > \hat{m}$ and consider the acts $f_m(\cdot)$ and $f'_m(\cdot)$. Defining $\{f_\alpha^k(\cdot) \,|\, \alpha \in [0,1]\}_{k=1}^\infty$ as the almost-objective mixture paths from $f'_m(\cdot)$ to $f_m(\cdot)$ as before, (A.21) again implies

$$W\big(f_m(\cdot)\big) - W\big(f'_m(\cdot)\big)$$
$$= \lim_{k\to\infty} \int_0^1 \sum_{x\in\mathcal{X}} \Big[ \varPhi_x\big(f_m^{-1}(x); f_\alpha^k\big) - \varPhi_x\big(f_m^{\prime -1}(x); f_\alpha^k\big) \Big] \cdot d\alpha \qquad \text{(A.43)}$$
$$= \lim_{k\to\infty} \int_0^1 \Big[ \varPhi_{x^*}\big(\wp \underset{m}{\times} E; f_\alpha^k\big) - \varPhi_{x^*}\big(\wp' \underset{m}{\times} E; f_\alpha^k\big) + \varPhi_x\big(\wp' \underset{m}{\times} E; f_\alpha^k\big) - \varPhi_x\big(\wp \underset{m}{\times} E; f_\alpha^k\big) \Big] \cdot d\alpha$$

By (A.42), the integrand in (A.43) is greater than or equal to $\delta/2$ in absolute value for all $\alpha \in [0,1]$ and all $k$, which implies $W(f_m(\cdot)) - W(f'_m(\cdot)) \geq \delta/2 > 0$. □

*Proof of Theorem 3.* For given outcomes $x_1, \ldots, x_n$ and partition $\{\wp_1, \ldots, \wp_n\}$ define the acts $f_m(\cdot) = [x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}]$. Step 1 shows that $\{W(f_m(\cdot))\}_{m=1}^\infty$ is a Cauchy sequence in $R^1$ and hence converges to some $\bar{W}$. Step 2 shows that for any other partition $\{\hat{\wp}_1, \ldots, \hat{\wp}_n\}$ with $\lambda(\hat{\wp}_i) = \lambda(\wp_i)$ for $i = 1, \ldots, n$, the acts $\hat{f}_m(\cdot) = [x_1 \text{ on } \hat{\wp}_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \hat{\wp}_n \underset{m}{\times} \mathcal{S}]$ satisfy $W(\hat{f}_m(\cdot)) -$

$W(f_m(\cdot)) \to 0$, so $W(\hat{f}_m(\cdot))$ also converges to $\bar{W}$, which we can accordingly express as $V_W(x_1, \lambda(\wp_1); \ldots; x_n, \lambda(\wp_n))$. (That $V_W(\cdot)$ is a preference function over objective lotteries follows from the preservation properties (11).) Step 3 shows that $V_W(\cdot)$ exhibits strict first order stochastic dominance preference.

*Step 1.* Given $\varepsilon > 0$, we need $m_\varepsilon$ such that $|W(f_{m'}(\cdot)) - W(f_{m''}(\cdot))| < \varepsilon$ for all $m', m'' > m_\varepsilon$. Applying event-smoothness and Theorem 0 to each of the $n$ families of signed measures $\{\Phi_{x_1}(\cdot; f) \,|\, f(\cdot) \in \mathcal{A}\}, \ldots, \{\Phi_{x_n}(\cdot; f) \,|\, f(\cdot) \in \mathcal{A}\}$ yields an $m_\varepsilon$ such that, for $i = 1, \ldots, n$

$$\left|\, \Phi_{x_i}(\wp_i \underset{m}{\times} \mathcal{S}; f) - \lambda(\wp_i) \cdot \Phi_{x_i}(\mathcal{S}; f) \,\right| \;<\; \varepsilon/(4n) \qquad \begin{array}{l}\text{all } m > m_\varepsilon \\ \text{all } f(\cdot) \in \mathcal{A}\end{array} \tag{A.44}$$

Select arbitrary $m'$, $m'' > m_\varepsilon$ and consider the pair of acts $f_{m'}(\cdot)$ and $f_{m''}(\cdot)$. By (A.44) we have

$$\left|\, \Phi_{x_i}(\wp_i \underset{m''}{\times} \mathcal{S}; f) - \Phi_{x_i}(\wp_i \underset{m'}{\times} \mathcal{S}; f) \,\right| \;<\; \varepsilon/(2n) \qquad \begin{array}{l} i = 1, \ldots, n \\ \text{all } f(\cdot) \in \mathcal{A}\end{array} \tag{A.45}$$

Defining $\{f_\alpha^k(\cdot) \,|\, \alpha \in [0,1]\}_{k=1}^\infty$ as the almost-objective mixture paths from $f_{m'}(\cdot)$ to $f_{m''}(\cdot)$, (A.21) yields

$$\begin{aligned} & W\big(f_{m''}(\cdot)\big) - W\big(f_{m'}(\cdot)\big) \\ &= \lim_{k\to\infty} \int_0^1 \sum_{x \in \mathcal{X}} \Big[\Phi_x\big(f_{m''}^{-1}(x); f_\alpha^k\big) - \Phi_x\big(f_{m'}^{-1}(x); f_\alpha^k\big)\Big] \cdot d\alpha \\ &= \lim_{k\to\infty} \int_0^1 \sum\nolimits_{i=1}^n \Big[\Phi_{x_i}\big(\wp_i \underset{m''}{\times} \mathcal{S}; f_\alpha^k\big) - \Phi_{x_i}\big(\wp_i \underset{m'}{\times} \mathcal{S}; f_\alpha^k\big)\Big] \cdot d\alpha \end{aligned} \tag{A.46}$$

By (A.45), the integrand in (A.46) is less than $\varepsilon/2$ in absolute value for all $\alpha \in [0,1]$ and all $k$, which implies $|W(f_{m''}(\cdot)) - W(f_{m'}(\cdot))| < \varepsilon$. But since $m'$ and $m''$ were arbitrary integers greater than $m_\varepsilon$, this implies $\{W(f_m(\cdot))\}_{m=1}^\infty$ is a Cauchy sequence, and hence converges to some $\bar{W}$.

*Step 2.* Take any other sequence of acts $\hat{f}_m(\cdot) = [x_1 \text{ on } \hat{\wp}_1 \underset{m}{\times} \mathcal{S}; \ldots; x_n \text{ on } \hat{\wp}_n \underset{m}{\times} \mathcal{S}]$ where $\lambda(\hat{\wp}_i) = \lambda(\wp_i)$ for $i = 1, \ldots, n$. Given arbitrary $\varepsilon > 0$, we need $\hat{m}_\varepsilon$ such that $|W(\hat{f}_m(\cdot)) - W(f_m(\cdot))| < \varepsilon$ for all $m > \hat{m}_\varepsilon$. Applying Theorem 0 again yields an $\hat{m}_\varepsilon$ such that, for $i = 1, \ldots, n$, both

$$\begin{array}{ll} \left|\Phi_{x_i}(\wp_i \underset{m}{\times} \mathcal{S}; f) - \lambda(\wp_i) \cdot \Phi_{x_i}(\mathcal{S}; f)\right| < \varepsilon/(4n) & \text{all } m > \hat{m}_\varepsilon \\ \left|\Phi_{x_i}(\hat{\wp}_i \underset{m}{\times} \mathcal{S}; f) - \lambda(\hat{\wp}_i) \cdot \Phi_{x_i}(\mathcal{S}; f)\right| < \varepsilon/(4n) & \text{all } f(\cdot) \in \mathcal{A} \end{array} \tag{A.47}$$

Select arbitrary $m > \hat{m}_\varepsilon$ and consider the acts $f_m(\cdot)$ and $\hat{f}_m(\cdot)$. By (A.47) and $\lambda(\hat{\wp}_i) = \lambda(\wp_i)$ we have

$$\left|\Phi_{x_i}(\hat{\wp}_i \underset{m}{\times} \mathcal{S}; f) - \Phi_{x_i}(\wp_i \underset{m}{\times} \mathcal{S}; f)\right| < \varepsilon/(2n) \qquad \begin{array}{l} i = 1, \ldots, n \\ \text{all } f(\cdot) \in \mathcal{A}\end{array} \tag{A.48}$$

Defining $\{\hat{f}^k_\alpha(\cdot)\,|\,\alpha \in [0,1]\}_{k=1}^\infty$ as the almost-objective mixture paths from $f_m(\cdot)$ to $\hat{f}_m(\cdot)$, (A.21) implies

$$\begin{aligned} &W\big(\hat{f}_m(\cdot)\big) - W\big(f_m(\cdot)\big) \\ &= \lim_{k\to\infty}\int_0^1 \sum_{x\in\mathcal{X}} \Big[\varPhi_x\big(\hat{f}_m^{-1}(x);\hat{f}^k_\alpha\big) - \varPhi_x\big(f_m^{-1}(x);\hat{f}^k_\alpha\big)\Big]\cdot d\alpha \\ &= \lim_{k\to\infty}\int_0^1 \sum\nolimits_{i=1}^n \Big[\varPhi_{x_i}\big(\hat{\wp}_i \underset{m}{\times} \mathcal{S};\hat{f}^k_\alpha\big) - \varPhi_{x_i}\big(\wp_i \underset{m}{\times} \mathcal{S};\hat{f}^k_\alpha\big)\Big]\cdot d\alpha \end{aligned} \tag{A.49}$$

By (A.48), the integrand in (A.49) is less than $\varepsilon/2$ in absolute value for all $\alpha \in [0,1]$ and all $k$, which implies $|W(\hat{f}_m(\cdot)) - W(f_m(\cdot))| < \varepsilon$. But since $m$ was an arbitrary integer greater than $\hat{m}_\varepsilon$, this implies $W(\hat{f}_m(\cdot)) - W(f_m(\cdot))$ converges to 0, and hence that $W(\hat{f}_m(\cdot))$ converges to $\bar{W}$.

*Step 3.* To establish strict first order stochastic dominance preference, it suffices to show $V_W(\hat{\mathbf{P}}) > V_W(\mathbf{P})$ for any $\hat{\mathbf{P}} = (\hat{x}_1,p_1;x_2,p_2;\ldots;x_n,p_n)$ and $\mathbf{P} = (x_1,p_1;x_2,p_2;\ldots;x_n,p_n)$ with $\hat{x}_1 \succ x_1$ and $p_1 > 0$. Select interval partition $\{\wp_1,\ldots,\wp_n\}$ of $[0,1]$ with $\lambda(\wp_i) = p_i$ for $i = 1,\ldots,n$, and define $\hat{f}_m(\cdot) = [\hat{x}_1$ on $\wp_1 \underset{m}{\times} \mathcal{S};\, x_2$ on $\wp_2 \underset{m}{\times} \mathcal{S};\ldots;x_n$ on $\wp_n \underset{m}{\times} \mathcal{S}]$ and $f_m(\cdot) = [x_1$ on $\wp_1 \underset{m}{\times} \mathcal{S}; x_2$ on $\wp_2 \underset{m}{\times} \mathcal{S};\ldots;x_n$ on $\wp_n \underset{m}{\times} \mathcal{S}]$. Event-smoothness yields $\varPhi_{\hat{x}_1}(S;f) - \varPhi_{x_1}(\mathcal{S};f) \geq \underline{\varPhi}_{\hat{x}_1,x_1,\mathcal{S}} > 0$ for all $f(\cdot) \in \mathcal{A}$, and event-smoothness and Theorem 0 yield some $\hat{m}$ such that

$$\Big|\varPhi_{\hat{x}_1}(\wp_1 \underset{m}{\times} \mathcal{S};f) - \lambda(\wp_1)\cdot\varPhi_{\hat{x}_1}(\mathcal{S};f)\Big| \quad \text{and} \quad \Big|\varPhi_{x_1}(\wp_1 \underset{m}{\times} \mathcal{S};f) - \lambda(\wp_1)\cdot\varPhi_{x_1}(\mathcal{S};f)\Big| \tag{A.50}$$

are both less than $\delta = \lambda(\wp_1)\cdot\underline{\varPhi}_{\hat{x}_1,x_1,\mathcal{S}}/4$ for all $m > \hat{m}$ and all $f(\cdot) \in \mathcal{A}$. Select arbitrary $m > \hat{m}$ and consider the acts $\hat{f}_m(\cdot)$ and $f_m(\cdot)$. By (A.50) we have

$$\begin{aligned} \varPhi_{\hat{x}_1}(\wp_1 \underset{m}{\times} \mathcal{S};f) - \varPhi_{x_1}(\wp_1 \underset{m}{\times} \mathcal{S};f) &> \lambda(\wp_1)\cdot\big[\varPhi_{\hat{x}_1}(\mathcal{S};f) - \varPhi_{x_1}(\mathcal{S};f)\big] - 2\cdot\delta \\ &\geq \lambda(\wp_1)\cdot\underline{\varPhi}_{\hat{x}_1,x_1,\mathcal{S}} - 2\cdot\delta \;=\; 2\cdot\delta \;>\; 0 \end{aligned} \tag{A.51}$$

for all $f(\cdot) \in \mathcal{A}$. Defining $\{f^k_\alpha(\cdot)|\alpha \in [0,1]\}_{k=1}^\infty$ as the almost-objective mixture paths from $f_m(\cdot)$ to $\hat{f}_m(\cdot)$, (A.21) yields

$$\begin{aligned} &W\big(\hat{f}_m(\cdot)\big) - W\big(f_m(\cdot)\big) \;= \\ &\lim_{k\to\infty}\int_0^1 \Big[\varPhi_{\hat{x}_1}\big(\wp_1 \underset{m}{\times} \mathcal{S};f^k_\alpha\big) - \varPhi_{x_1}\big(\wp_1 \underset{m}{\times} \mathcal{S};f^k_\alpha\big)\Big]\cdot d\alpha \;\geq\; 2\cdot\delta \;>\; 0 \end{aligned} \tag{A.52}$$

Since $m > \hat{m}$ was arbitrary, we thus have $V_W(\hat{\mathbf{P}}) = \lim_{m\to\infty} W(\hat{f}_m(\cdot)) > \lim_{m\to\infty} W(f_m(\cdot)) = V_W(\mathbf{P})$. □

*Proof of Theorem 4.* Since $f^*_m(\cdot) = [f_1(\cdot)$ on $\wp_1 \underset{m}{\times} \mathcal{S};\ldots;f_n(\cdot)$ on $\wp_n \underset{m}{\times} \mathcal{S}]$ and $\hat{f}^*_m(\cdot) = [\hat{f}_1(\cdot)$ on $\hat{\wp}_1 \underset{m}{\times} \mathcal{S};\ldots;\hat{f}_{\hat{n}}(\cdot)$ on $\hat{\wp}_{\hat{n}} \underset{m}{\times} \mathcal{S}]$ imply probabilistically equivalent almost-objective acts over each event in their common refinement $\{E_1^*, \ldots, E_K^*\}$, we can define $\{x_1^*, \ldots, x_J^*\}$ as the union of their payoffs, and write

$$\begin{aligned} f_m^*(\cdot) = \big[&x_1^* \text{ on } \wp_{1,1} \underset{m}{\times} E_1^*; \ldots; x_J^* \text{ on } \wp_{1,J} \underset{m}{\times} E_1^*; \ldots \\ &\ldots; x_1^* \text{ on } \wp_{K,1} \underset{m}{\times} E_K^*; \ldots; x_J^* \text{ on } \wp_{K,J} \underset{m}{\times} E_K^*\big] \\ \hat{f}_m^*(\cdot) = \big[&x_1^* \text{ on } \hat{\wp}_{1,1} \underset{m}{\times} E_1^*; \ldots; x_J^* \text{ on } \hat{\wp}_{1,J} \underset{m}{\times} E_1^*; \ldots \\ &\ldots; x_1^* \text{ on } \hat{\wp}_{K,1} \underset{m}{\times} E_K^*; \ldots; x_J^* \text{ on } \hat{\wp}_{K,J} \underset{m}{\times} E_K^*\big] \end{aligned} \tag{A.53}$$

where for each $k = 1, \ldots, K$, the partitions $\{\wp_{k,1}, \ldots, \wp_{k,J}\}$ and $\{\hat{\wp}_{k,1}, \ldots, \hat{\wp}_{k,J}\}$ of $[0,1]$ satisfy $\lambda(\wp_{k,j}) = \lambda(\hat{\wp}_{k,j})$ for $j = 1, \ldots, J$. As in the proof of Theorem 3, we show $\lim_{m\to\infty} W(f_m^*(\cdot)) = \lim_{m\to\infty} W(\hat{f}_m^*(\cdot))$ by showing that $\{W(f_m^*(\cdot))\}_{m=1}^{\infty}$ is a Cauchy sequence in $R^1$, and then that $W(f_m^*(\cdot)) - W(\hat{f}_m^*(\cdot)) \to 0$.

*Step 1.* Given $\varepsilon > 0$, we need $m_\varepsilon$ such that $|W(f_{m'}^*(\cdot)) - W(f_{m''}^*(\cdot))| < \varepsilon$ for all $m', m'' > m_\varepsilon$. Applying Theorem 0 to each of the $K\cdot J$ combinations of $\wp_{k,j}$, $E_k^*$, $\{\Phi_{x_j^*}(\cdot; f) | f(\cdot) \in \mathcal{A}\}$ yields an $m_\varepsilon$ such that

$$\left|\Phi_{x_j^*}(\wp_{k,j} \underset{m}{\times} E_k^*; f) - \lambda(\wp_{k,j})\cdot\Phi_{x_j^*}(E_k^*; f)\right| \;<\; \varepsilon/(4\cdot K\cdot J) \tag{A.54}$$

for all $m > m_\varepsilon$, all $k = 1, \ldots, K$, all $j = 1, \ldots, J$, and all $f(\cdot) \in \mathcal{A}$.

Select arbitrary $m', m'' > m_\varepsilon$ and consider the acts $f_{m'}^*(\cdot)$ and $f_{m''}^*(\cdot)$. By (A.54) we have

$$\left|\Phi_{x_j^*}(\wp_{k,j} \underset{m''}{\times} E_k^*; f) - \Phi_{x_j^*}(\wp_{k,j} \underset{m'}{\times} E_k^*; f)\right| \;<\; \varepsilon/(2\cdot K\cdot J) \tag{A.55}$$

for all $j = 1, \ldots, J$, all $k = 1, \ldots, K$, and all $f(\cdot) \in \mathcal{A}$. Defining $\{f_\alpha^{*\ell}(\cdot) | \alpha \in [0,1]\}_{\ell=1}^{\infty} = \{[f_m'(\cdot) \text{ on } [0,\alpha] \underset{\ell}{\times} \mathcal{S}; f_m(\cdot) \text{ on } (\alpha,1] \underset{\ell}{\times} \mathcal{S}] \,|\, \alpha \in [0,1]\}_{\ell=1}^{\infty}$ as the almost-objective mixture paths from $f_{m'}^*(\cdot)$ to $f_{m''}^*(\cdot)$, (A.21) implies

$$\begin{aligned} &W\big(f_{m''}^*(\cdot)\big) - W\big(f_{m'}^*(\cdot)\big) \;= \\ &\lim_{\ell\to\infty} \int_0^1 \sum_{k=1}^K \sum_{j=1}^J \left[\Phi_{x_j^*}(\wp_{k,j} \underset{m''}{\times} E_k^*; f_\alpha^{*\ell}) - \Phi_{x_j^*}(\wp_{k,j} \underset{m'}{\times} E_k^*; f_\alpha^{*\ell})\right]\cdot d\alpha \end{aligned} \tag{A.56}$$

By (A.55), the integrand in (A.56) is less than $\varepsilon/2$ in absolute value for all $\alpha \in [0,1]$ and all $\ell$, which implies $|W(f_{m''}^*(\cdot)) - W(f_{m'}^*(\cdot))| < \varepsilon$. But since $m'$ and $m''$ were arbitrary integers greater than $m_\varepsilon$, this implies that $\{W(f_m^*(\cdot))\}_{m=1}^{\infty}$ is a Cauchy sequence, and hence converges to some $\bar{W}^*$.

*Step 2.* Given arbitrary $\varepsilon > 0$, we need $\hat{m}_\varepsilon$ such that $|W(\hat{f}_m^*(\cdot)) - W(f_m^*(\cdot))| < \varepsilon$ for all $m > \hat{m}_\varepsilon$. Applying Theorem 0 again yields an $\hat{m}_\varepsilon$ such that both

$$\begin{aligned} &\left|\Phi_{x_j^*}(\wp_{k,j} \underset{m}{\times} E_k^*; f) - \lambda(\wp_{k,j})\cdot\Phi_{x_j^*}(E_k^*; f)\right| < \varepsilon/(4\cdot K\cdot J) \\ \text{and} \quad & \\ &\left|\Phi_{x_j^*}(\hat{\wp}_{k,j} \underset{m}{\times} E_k^*; f) - \lambda(\hat{\wp}_{k,j})\cdot\Phi_{x_j^*}(E_k^*; f)\right| < \varepsilon/(4\cdot K\cdot J) \end{aligned} \tag{A.57}$$

for all $m > \hat{m}_\varepsilon$, all $k = 1, \ldots, K$, all $j = 1, \ldots, J$, and all $f(\cdot) \in \mathcal{A}$.

Select arbitrary $m > \hat{m}_\varepsilon$ and consider the acts $f_m^*(\cdot)$ and $\hat{f}_m^*(\cdot)$. By (A.57) and $\lambda(\wp_{k,j}) = \lambda(\hat{\wp}_{k,j})$ we have

$$\left|\varPhi_{x_j^*}(\hat{\wp}_{k,j} \underset{m}{\times} E_k^*; f) - \varPhi_{x_j^*}(\wp_{k,j} \underset{m}{\times} E_k^*; f)\right| < \varepsilon/(2\cdot K\cdot J) \tag{A.58}$$

for all $k = 1, \ldots, K$, all $j = 1, \ldots, J$, and all $f(\cdot) \in \mathcal{A}$. Defining $\{\hat{f}_\alpha^{*\ell}(\cdot)\,|\,\alpha \in [0,1]\}_{\ell=1}^\infty$ as the almost-objective mixture paths from $f_m^*(\cdot)$ to $\hat{f}_m^*(\cdot)$, (A.21) again implies

$$\begin{aligned} &W\big(\hat{f}_m^*(\cdot)\big) - W\big(f_m^*(\cdot)\big) = \\ &\lim_{\ell\to\infty}\int_0^1 \sum_{k=1}^K\sum_{j=1}^J \Big[\varPhi_{x_j^*}\big(\hat{\wp}_{k,j} \underset{m}{\times} E_k^*; \hat{f}_\alpha^{*\ell}\big) - \varPhi_{x_j^*}\big(\wp_{k,j} \underset{m}{\times} E_k^*; \hat{f}_\alpha^{*\ell}\big)\Big]\cdot d\alpha \end{aligned} \tag{A.59}$$

By (A.58), the integrand in (A.59) is less than $\varepsilon/2$ in absolute value for all $\alpha \in [0,1]$ and all $\ell$, which implies $|W(\hat{f}_m^*(\cdot)) - W(f_m^*(\cdot))| < \varepsilon$. But since $m$ was an arbitrary integer greater than $\hat{m}_\varepsilon$, this implies that $W(\hat{f}_m^*(\cdot)) - W(f_m^*(\cdot))$ converges to 0. □

*Proof of Theorem 5.* Given $\wp, \wp'$, disjoint $E$, $E'$, $x^* \succ x$ and $f(\cdot) = [x_1$ on $E_1; \ldots; x_n$ on $E_n]$, probabilistic sophistication of $W_{PS}(\cdot)$ implies

$$\begin{aligned} &W_{PS}\big(x^* \text{ on } (\wp \underset{m}{\times} \mathcal{S}) \cap E; x \text{ on } (\wp' \underset{m}{\times} \mathcal{S}) \cap E'; f(\cdot) \text{ elsewhere}\big) \;= \\ &V\big(x^*, \mu(\wp \underset{m}{\times} E); x, \mu(\wp' \underset{m}{\times} E'); x_1, \mu(E_1 - \wp \underset{m}{\times} E - \wp' \underset{m}{\times} E'); \ldots \\ &\quad \ldots; x_n, \mu(E_n - \wp \underset{m}{\times} E - \wp' \underset{m}{\times} E')\big) \end{aligned} \tag{A.60}$$

Theorem 0 implies $\lim_{m\to\infty} \mu(\wp \underset{m}{\times} E) = \lambda(\wp)\cdot\mu(E)$ and $\lim_{m\to\infty} \mu(\wp' \underset{m}{\times} E') = \lambda(\wp')\cdot\mu(E')$. Disjointness of $E$ and $E'$ implies

$$\begin{aligned} \mu\big(E_i - \wp \underset{m}{\times} E - \wp' \underset{m}{\times} E'\big) &= \mu(E_i) - \mu\big(E_i \cap (\wp \underset{m}{\times} E)\big) - \mu\big(E_i \cap (\wp' \underset{m}{\times} E')\big) \\ &= \mu(E_i) - \mu\big(\wp \underset{m}{\times} (E_i \cap E)\big) - \mu\big(\wp' \underset{m}{\times} (E_i \cap E')\big) \quad i = 1, \ldots, n \end{aligned} \tag{A.61}$$

so that Theorem 0 also implies

$$\begin{aligned} &\lim_{m\to\infty} \mu\big(E_i - \wp \underset{m}{\times} E - \wp' \underset{m}{\times} E'\big) \;= \\ &\mu(E_i) - \lambda(\wp)\cdot\mu(E_i \cap E) - \lambda(\wp')\cdot\mu(E_i \cap E') \quad i = 1, \ldots, n \end{aligned} \tag{A.62}$$

Defining $\rho_i = \mu(E_i) - \lambda(\wp)\cdot\mu(E_i \cap E) - \lambda(\wp')\cdot\mu(E_i \cap E')$, continuity of $V(\cdot)$ in each probability yields

$$\begin{aligned} &\lim_{m\to\infty} W_{PS}\big(x^* \text{ on } (\wp \underset{m}{\times} \mathcal{S}) \cap E; x \text{ on } (\wp' \underset{m}{\times} \mathcal{S}) \cap E'; f(\cdot) \text{ elsewhere}\big) \\ &= \; V\big(x^*, \lambda(\wp)\cdot\mu(E); x, \lambda(\wp')\cdot\mu(E'); x_1, \rho_1; \ldots; x_n, \rho_n\big) \end{aligned} \tag{A.63}$$

Since we similarly have

$$\begin{aligned}&\lim_{m\to\infty} W_{PS}\big(x \text{ on } (\wp \underset{m}{\times} \mathcal{S})\cap E\,; x^* \text{ on } (\wp' \underset{m}{\times} \mathcal{S})\cap E'\,; f(\cdot) \text{ elsewhere}\big)\\ &= \ V\big(x, \lambda(\wp)\cdot\mu(E)\,; x^*, \lambda(\wp')\cdot\mu(E')\,; x_1, \rho_1; \ldots; x_n, \rho_n\big)\end{aligned} \tag{A.64}$$

inequality (equality) (25) follows from first order stochastic dominance preference of $V(\cdot)$. □

*Proof of Theorem 6.* We establish the bottom equation in (29), which implies each of the other assertions of the theorem. Given the subjective acts $f_1(\cdot), \ldots, f_n(\cdot) \in \mathcal{A}$ and partition $\{\wp_1, \ldots, \wp_n\}$ of $[0, 1]$, we can express $f_1(\cdot), \ldots, f_n(\cdot)$ in the form

$$\begin{aligned} f_1(\cdot) &= [x_{1,1} \text{ on } \hat{E}_1; \ldots; x_{1,J} \text{ on } \hat{E}_J]\\ &\ \ \vdots\\ f_n(\cdot) &= [x_{n,1} \text{ on } \hat{E}_1; \ldots; x_{n,J} \text{ on } \hat{E}_J]\end{aligned} \tag{A.65}$$

for some partition $\{\hat{E}_1, \ldots, \hat{E}_J\}$ of $\mathcal{S}$, so that we can express $\big(f_1(\cdot)$ on $\wp_1 \underset{m}{\times} \mathcal{S}$; $\ldots$; $f_n(\cdot)$ on $\wp_n \underset{m}{\times} \mathcal{S}\big)$ as

$$\begin{aligned}&\big(x_{1,1} \text{ on } \wp_1 \underset{m}{\times} \hat{E}_1; \ldots; x_{1,J} \text{ on } \wp_1 \underset{m}{\times} \hat{E}_J; \ldots\\ &\quad \ldots; x_{n,1} \text{ on } \wp_n \underset{m}{\times} \hat{E}_1; \ldots; x_{n,J} \text{ on } \wp_n \underset{m}{\times} \hat{E}_J\big)\end{aligned} \tag{A.66}$$

Recall that every event-smooth state-dependent expected utility preference function can be expressed in form $W_{SDEU}(f(\cdot)) \equiv \int_S U(f(s)|s)\cdot d\mu(s) \equiv \sum_{x\in X}\int_{f^{-1}(x)} U(x|s)\cdot\nu(s)\cdot ds \equiv \sum_{x\in X} \varPhi_x(f^{-1}(x))$, where each $\varPhi_x(\cdot)$ has a bounded and continuous density $\phi_x(s) \equiv U(x|s)\cdot\nu(s)$ on $\mathcal{S}$. Thus we can write

$$W_{SDEU}\big(f_1(\cdot) \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; f_n(\cdot) \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\big) \equiv \sum_{i=1}^{n}\sum_{j=1}^{J} \varPhi_{x_{i,j}}(\wp_i \underset{m}{\times} \hat{E}_j) \tag{A.67}$$

and Theorem 0 yields

$$\begin{aligned}&\lim_{m\to\infty} W_{SDEU}\ \big(f_1(\cdot) \text{ on } \wp_1 \underset{m}{\times} \mathcal{S}; \ldots; f_n(\cdot) \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\big)\\ &= \lim_{m\to\infty} \sum\nolimits_{i=1}^{n}\sum\nolimits_{j=1}^{J} \varPhi_{x_{i,j}}\big(\wp_i \underset{m}{\times} \hat{E}_j\big)\\ &= \sum\nolimits_{i=1}^{n} \lambda(\wp_i)\cdot\sum\nolimits_{j=1}^{J} \varPhi_{x_{i,j}}(\hat{E}_j)\\ &= \sum\nolimits_{i=1}^{n} \lambda(\wp_i)\cdot\sum\nolimits_{j=1}^{J} \int_{\hat{E}_j} U(x_{i,j}|s)\cdot d\mu(s)\\ &= \sum\nolimits_{i=1}^{n} \lambda(\wp_i)\cdot\sum\nolimits_{j=1}^{J} \int_{\hat{E}_j} U(f_i(s)|s)\cdot d\mu(s)\\ &= \sum\nolimits_{i=1}^{n} \lambda(\wp_i)\cdot\int_{\mathcal{S}} U(f_i(s)|s)\cdot d\mu(s)\end{aligned} \tag{A.68}$$

□

*Proof of Theorem 7.* Define $V_W(\cdot)$ as from Theorem 3, and for each $E \in \mathcal{E}$ define $\mu(E) \in [0,1]$ as the unique solution to $W(x^* \text{ on } E; x \text{ on } {\sim}E) = \lim_{m\to\infty} W(x^* \text{ on } [0,\mu(E)] \underset{m}{\times} \mathcal{S}; x \text{ on } (\mu(E),1] \underset{m}{\times} \mathcal{S})$, or equivalently, to $W(x^* \text{ on } E; x \text{ on } {\sim}E) = V_W(x^*, \mu(E); x, 1-\mu(E))$, so that $\mu(\emptyset) = 0$ and $\mu(\mathcal{S}) = 1$.

Picking an arbitrary act $f(\cdot) \in \mathcal{A}$ is equivalent to picking an arbitrary partition $\{E_1, \ldots, E_n\}$ of $\mathcal{S}$ and then assigning arbitrary outcomes $x_1, \ldots, x_n$ to these events. Consider an arbitrary partition $\{E_1, \ldots, E_n\}$, labeled so that $E_1$ is nonnull. For $i = 1, \ldots, n-1$, let $\alpha_i \in [0,1]$ solve

$$W\begin{pmatrix} x^* \text{ on } E_1 \cup \cdots \cup E_i \\ x \text{ on } E_{i+1} \\ x \text{ on } E_k \quad k > i+1 \end{pmatrix} = \lim_{m\to\infty} W\begin{pmatrix} \left[x^* \text{ on } [0,\alpha_i] \underset{m}{\times} \mathcal{S}; x \text{ on } (\alpha_i,1] \underset{m}{\times} \mathcal{S}\right] & \text{on } E_1 \cup \cdots \cup E_i \\ \left[x^* \text{ on } [0,\alpha_i] \underset{m}{\times} \mathcal{S}; x \text{ on } (\alpha_i,1] \underset{m}{\times} \mathcal{S}\right] & \text{on } E_{i+1} \\ x & \text{on } E_k \quad k > i+1 \end{pmatrix} \tag{A.69}$$

and define

$$\begin{aligned} \tau_1 &= \alpha_{n-1} \cdot \alpha_{n-2} \cdot \ldots \cdot \alpha_3 \cdot \alpha_2 \cdot \alpha_1 \\ \tau_2 &= \alpha_{n-1} \cdot \alpha_{n-2} \cdot \ldots \cdot \alpha_3 \cdot \alpha_2 \cdot (1-\alpha_1) \\ \tau_3 &= \alpha_{n-1} \cdot \alpha_{n-2} \cdot \ldots \cdot \alpha_3 \cdot (1-\alpha_2) \\ \tau_4 &= \alpha_{n-1} \cdot \alpha_{n-2} \cdot \ldots \cdot (1-\alpha_3) \\ &\vdots \\ \tau_{n-1} &= \alpha_{n-1} \cdot (1-\alpha_{n-2}) \\ \tau_n &= (1-\alpha_{n-1}) \end{aligned} \tag{A.70}$$

which satisfy $\tau_1 + \ldots + \tau_n = 1$. For arbitrary $x_1, \ldots, x_n$, repeated application of the Almost-Objective/Subjective Replacement Axiom, the definition of the mixture operation $\alpha \cdot f_m^i(\cdot) \oplus (1-\alpha) \cdot f_m^j(\cdot)$ (see Note 11), and Theorem 4 (Reduction of Almost-Objective×Subjective Uncertainty) yields:

$$W\begin{pmatrix} x_1 \text{ on } E_1 \\ x_2 \text{ on } E_2 \\ x_3 \text{ on } E_3 \\ \vdots \\ x_n \text{ on } E_n \end{pmatrix} = \lim_{m\to\infty} W\begin{pmatrix} [x_1 \text{ on } [0,\alpha_1] \underset{m}{\times} \mathcal{S}; x_2 \text{ on } (\alpha_1,1] \underset{m}{\times} \mathcal{S}] & \text{on } E_1 \\ [x_1 \text{ on } [0,\alpha_1] \underset{m}{\times} \mathcal{S}; x_2 \text{ on } (\alpha_1,1] \underset{m}{\times} \mathcal{S}] & \text{on } E_2 \\ x_3 & \text{on } E_3 \\ \vdots & \vdots \\ x_n & \text{on } E_n \end{pmatrix}$$

$$= \lim_{m\to\infty} W\begin{pmatrix} [x_1 \text{ on } [0,\alpha_1{\cdot}\alpha_2] \underset{m}{\times} \mathcal{S}; x_2 \text{ on } (\alpha_1{\cdot}\alpha_2, \alpha_2] \underset{m}{\times} \mathcal{S}; x_3 \text{ on } (\alpha_2,1] \underset{m}{\times} \mathcal{S}] & \text{on } E_1 \\ [x_1 \text{ on } [0,\alpha_1{\cdot}\alpha_2] \underset{m}{\times} \mathcal{S}; x_2 \text{ on } (\alpha_1{\cdot}\alpha_2, \alpha_2] \underset{m}{\times} \mathcal{S}; x_3 \text{ on } (\alpha_2,1] \underset{m}{\times} \mathcal{S}] & \text{on } E_2 \\ [x_1 \text{ on } [0,\alpha_1{\cdot}\alpha_2] \underset{m}{\times} \mathcal{S}; x_2 \text{ on } (\alpha_1{\cdot}\alpha_2, \alpha_2] \underset{m}{\times} \mathcal{S}; x_3 \text{ on } (\alpha_2,1] \underset{m}{\times} \mathcal{S}] & \text{on } E_3 \\ x_4 & \text{on } E_4 \\ \vdots & \vdots \\ x_n & \text{on } E_n \end{pmatrix}$$

$$= \lim_{m\to\infty} W\begin{pmatrix} [x_1 \text{ on } [0,\alpha_1{\cdot}\alpha_2{\cdot}\alpha_3] \underset{m}{\times} \mathcal{S}; x_2 \text{ on } (\alpha_1{\cdot}\alpha_2{\cdot}\alpha_3, \alpha_2{\cdot}\alpha_3] \underset{m}{\times} \mathcal{S}; & \text{on } E_1 \\ x_3 \text{ on } (\alpha_2{\cdot}\alpha_3, \alpha_3] \underset{m}{\times} \mathcal{S}; x_4 \text{ on } (\alpha_3,1] \underset{m}{\times} \mathcal{S}] & \\ [x_1 \text{ on } [0,\alpha_1{\cdot}\alpha_2{\cdot}\alpha_3] \underset{m}{\times} \mathcal{S}; x_2 \text{ on } (\alpha_1{\cdot}\alpha_2{\cdot}\alpha_3, \alpha_2{\cdot}\alpha_3] \underset{m}{\times} \mathcal{S}; & \text{on } E_2 \\ x_3 \text{ on } (\alpha_2{\cdot}\alpha_3, \alpha_3] \underset{m}{\times} \mathcal{S}; x_4 \text{ on } (\alpha_3,1] \underset{m}{\times} \mathcal{S}] & \\ [x_1 \text{ on } [0,\alpha_1{\cdot}\alpha_2{\cdot}\alpha_3] \underset{m}{\times} \mathcal{S}; x_2 \text{ on } (\alpha_1{\cdot}\alpha_2{\cdot}\alpha_3, \alpha_2{\cdot}\alpha_3] \underset{m}{\times} \mathcal{S}; & \text{on } E_3 \\ x_3 \text{ on } (\alpha_2{\cdot}\alpha_3, \alpha_3] \underset{m}{\times} \mathcal{S}; x_4 \text{ on } (\alpha_3,1] \underset{m}{\times} \mathcal{S}] & \\ [x_1 \text{ on } [0,\alpha_1{\cdot}\alpha_2{\cdot}\alpha_3] \underset{m}{\times} \mathcal{S}; x_2 \text{ on } (\alpha_1{\cdot}\alpha_2{\cdot}\alpha_3, \alpha_2{\cdot}\alpha_3] \underset{m}{\times} \mathcal{S}; & \text{on } E_4 \\ x_3 \text{ on } (\alpha_2{\cdot}\alpha_3, \alpha_3] \underset{m}{\times} \mathcal{S}; x_4 \text{ on } (\alpha_3,1] \underset{m}{\times} \mathcal{S}] & \\ x_5 & \text{on } E_5 \\ \vdots & \vdots \\ x_n & \text{on } E_n \end{pmatrix} \quad \text{(A.71)}$$

$\ldots$

$$= \lim_{m\to\infty} W\begin{pmatrix} [x_1 \text{ on } [0,\tau_1] \underset{m}{\times} \mathcal{S}; \ldots; x_i \text{ on } (\sum_{j=1}^{i-1}\tau_j, \sum_{j=1}^{i}\tau_j] \underset{m}{\times} \mathcal{S}; & \text{on } E_1 \\ \ldots; x_n \text{ on } (\sum_{j=1}^{n-1}\tau_j, 1] \underset{m}{\times} \mathcal{S}] & \\ \vdots & \vdots \\ [x_1 \text{ on } [0,\tau_1] \underset{m}{\times} \mathcal{S}; \ldots; x_i \text{ on } (\sum_{j=1}^{i-1}\tau_j, \sum_{j=1}^{i}\tau_j] \underset{m}{\times} \mathcal{S}; & \text{on } E_n \\ \ldots; x_n \text{ on } (\sum_{j=1}^{n-1}\tau_j, 1] \underset{m}{\times} \mathcal{S}] & \end{pmatrix}$$

$$= \lim_{m\to\infty} W\begin{pmatrix} x_1 \text{ on } [0,\tau_1] \underset{m}{\times} \mathcal{S}; \ldots; x_i \text{ on } (\sum_{j=1}^{i-1}\tau_j, \sum_{j=1}^{i}\tau_j] \underset{m}{\times} \mathcal{S}; \\ \ldots; x_n \text{ on } (\sum_{j=1}^{n-1}\tau_j, 1] \underset{m}{\times} \mathcal{S} \end{pmatrix}$$

Since $\lambda(\,(\sum_{j=1}^{i-1}\tau_j, \sum_{j=1}^{i}\tau_j]\,) = \tau_i$, Theorem 3 implies $W(x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n) = V_W(x_1, \tau_1; \ldots; x_n, \tau_n)$. Since the values $\tau_1, \ldots, \tau_n$ were obtained from $E_1, \ldots, E_n$ independently of the outcomes $x_1, \ldots, x_n$, for each $i = 1, \ldots, n$ we can set $x_i = x^*$ and $x_j = x$ for $j \neq i$, to obtain $W(x^* \text{ on } E_i; x \text{ on } {\sim}E_i) = V_W(x^*, \tau_i; x, 1-\tau_i)$. This implies $\tau_i = \mu(E_i)$ for each $i$, so $W(x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n) = V_W(x_1, \mu(E_1); \ldots; x_n, \mu(E_n))$.

To see that $\mu(\cdot)$ is finitely additive, observe that for arbitrary disjoint events $E^*$ and $E$ we have $V_W(x^*, \mu(E^* \cup E); x, \mu({\sim}(E^* \cup E))) = W(x^* \text{ on } E^* \cup E; x$

on $\sim(E^* \cup E)) = W(x^*$ on $E^*; x^*$ on $E$; $x$ on $\sim(E^* \cup E)) = V_W(x^*, \mu(E^*);$ $x^*, \mu(E); x, \mu(\sim(E^* \cup E))) = V_W(x^*, \mu(E^*) + \mu(E); x, \mu(\sim(E^* \cup E)))$. $\square$

For Theorem 8, we extend the analytical notions of this paper to a bivariate state space $\mathcal{S} \times \mathcal{T} = [\underline{s}, \bar{s}] \times [\underline{t}, \bar{t}]$ as follows: Define $\mathcal{A}_{\mathcal{S}\times\mathcal{T}}$ as the family of all finite-outcome acts $f(\cdot,\cdot)$ whose events are finite unions of rectangles in $\mathcal{S} \times \mathcal{T}$; define $\lambda(\cdot)$ and $\lambda(\cdot,\cdot)$ as univariate and bivariate uniform Lebesgue measure; define event-continuity with respect to the distance function $\delta(f(\cdot,\cdot), f_0(\cdot,\cdot)) \equiv \lambda\{(s,t) \in \mathcal{S} \times \mathcal{T} \,|\, f(s,t) \neq f_0(s,t)\}$; and adopt the corresponding bivariate versions of the change sets $\Delta E_x^+$ and $\Delta E_x^-$, of the event-differentiability formula (A.6), and of the event-smoothness conditions (A.7)$'$ for the local evaluation measures $\{\Phi_x(\cdot; f) | x \in \mathcal{X}\}$ over bivariate events in $\mathcal{S} \times \mathcal{T}$ and local evaluation densities $\{\phi_x(\cdot,\cdot; f) | x \in \mathcal{X}\}$ over $\mathcal{S} \times \mathcal{T}$.

*Proof of Theorem 8.* Define $W(\cdot)$'s restriction to acts on $\mathcal{S}$, and to acts on $\mathcal{T}$, by

$$W_S\big(x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n\big) \equiv W\big(x_1 \text{ on } E_1 \times \mathcal{T}; \ldots; x_n \text{ on } E_n \times \mathcal{T}\big)$$
$$W_T\big(x_1 \text{ on } \hat{E}_1; \ldots; x_n \text{ on } \hat{E}_n\big) \equiv W\big(x_1 \text{ on } \mathcal{S} \times \hat{E}_1; \ldots; x_n \text{ on } \mathcal{S} \times \hat{E}_n\big) \tag{A.72}$$

for arbitrary $x_1, \ldots, x_n$ and finite-interval partitions $\{E_1, \ldots, E_n\}$ of $\mathcal{S}$ and $\{\hat{E}_1, \ldots, \hat{E}_n\}$ of $\mathcal{T}$. Since $W_{\mathcal{S}}(\cdot)$ and $W_{\mathcal{T}}(\cdot)$ inherit event-smoothness and outcome-monotonicity, Theorem 3 implies preference functions $V_{W_{\mathcal{S}}}(\cdot)$ and $V_{W_{\mathcal{T}}}(\cdot)$, satisfying strict first order stochastic dominance preference, such that

$$\lim_{m\to\infty} W_{\mathcal{S}}\big(x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{S};\ \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{S}\big) \equiv V_{W_{\mathcal{S}}}\big(x_1, \lambda(\wp_1); \ldots; x_n, \lambda(\wp_n)\big) \tag{A.73}$$

$$\lim_{m\to\infty} W_{\mathcal{T}}\big(x_1 \text{ on } \wp_1 \underset{m}{\times} \mathcal{T};\ \ldots; x_n \text{ on } \wp_n \underset{m}{\times} \mathcal{T}\big) \equiv V_{W_{\mathcal{T}}}\big(x_1, \lambda(\wp_1); \ldots; x_n, \lambda(\wp_n)\big) \tag{A.74}$$

for arbitrary $x_1, \ldots, x_n$ and finite-interval partitions $\{\wp_1, \ldots, \wp_n\}$ of $[0,1]$.

Define the acts $f_m^{\mathcal{S}}(\cdot,\cdot) = [x_1 \text{ on } (\wp_1 \underset{m}{\times} \mathcal{S}) \times \mathcal{T}; \ldots; x_n \text{ on } (\wp_n \underset{m}{\times} \mathcal{S}) \times \mathcal{T}]$ and $f_m^{\mathcal{T}}(\cdot,\cdot) = [x_1 \text{ on } \mathcal{S} \times (\wp_1 \underset{m}{\times} \mathcal{T}); \ldots; x_n \text{ on } \mathcal{S} \times (\wp_n \underset{m}{\times} \mathcal{T})]$ from (34). To establish the proof we will show $\lim_{m\to\infty} W(f_m^{\mathcal{S}}(\cdot,\cdot)) = \lim_{m\to\infty} W(f_m^{\mathcal{T}}(\cdot,\cdot))$, which by (A.72) implies that (A.73) equals (A.74), and hence that $V_{W_{\mathcal{S}}}(\cdot) \equiv V_{W_{\mathcal{T}}}(\cdot)$.

Given arbitrary $\varepsilon > 0$, we need $m_\varepsilon$ such that $|W(f_m^{\mathcal{S}}(\cdot,\cdot)) - W(f_m^{\mathcal{T}}(\cdot,\cdot))| < \varepsilon$ for all $m > m_\varepsilon$. Applying bivariate event-smoothness and Theorem 0 to the families of univariate measures $\{\Phi_{x_j}(\cdot \times \mathcal{T}; f) \,|\, f(\cdot,\cdot) \in \mathcal{A}_{\mathcal{S}\times\mathcal{T}}\}_{j=1}^n$ and $\{\Phi_{x_j}(\mathcal{S} \times \cdot\,; f) \,|\, f(\cdot,\cdot) \in \mathcal{A}_{\mathcal{S}\times\mathcal{T}}\}_{j=1}^n$ yields an $m_\varepsilon$ such that

$$\big|\Phi_{x_j}\big((\wp_j \underset{m}{\times} \mathcal{S}) \times \mathcal{T}; f\big) - \lambda(\wp_j) \cdot \Phi_{x_j}(\mathcal{S} \times \mathcal{T}; f)\big| < \varepsilon/(4n)$$
$$\big|\Phi_{x_j}\big(\mathcal{S} \times (\wp_j \underset{m}{\times} \mathcal{T}); f\big) - \lambda(\wp_j) \cdot \Phi_{x_j}(\mathcal{S} \times \mathcal{T}; f)\big| < \varepsilon/(4n) \tag{A.75}$$

and hence

$$\left|\Phi_{x_j}\big(\mathcal{S}\times(\wp_j \underset{m}{\times}\mathcal{T}); f\big) - \Phi_{x_j}\big((\wp_j \underset{m}{\times}\mathcal{S})\times\mathcal{T}; f\big)\right| < \varepsilon/(2n) \qquad \text{(A.76)}$$

for all $j = 1,\ldots,n$, all $m > m_\varepsilon$ and all $f(\cdot,\cdot) \in \mathcal{A}_{\mathcal{S}\times\mathcal{T}}$.

Select arbitrary $m > m_\varepsilon$ and consider the acts $f_m^{\mathcal{S}}(\cdot,\cdot)$ and $f_m^{\mathcal{T}}(\cdot,\cdot)$. By bivariate event-smoothness and the lemma, there exists large $k$ such that each of the expressions

$$\begin{aligned}&\left|\int_{\underline{s}}^{\bar{s}}\phi_{f_m^{\mathcal{S}}(s,t)}(s,t;f)\cdot ds - \frac{\lambda_{\mathcal{S}}}{k}\cdot\sum_{i=0}^{k-1}\phi_{f_m^{\mathcal{S}}(\underline{s}+\frac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t)}\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t;f\big)\right|\\ &\left|\int_{\underline{s}}^{\bar{s}}\phi_{f_m^{\mathcal{T}}(s,t)}(s,t;f)\cdot ds - \frac{\lambda_{\mathcal{S}}}{k}\cdot\sum_{i=0}^{k-1}\phi_{f_m^{\mathcal{T}}(\underline{s}+\frac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t)}\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t;f\big)\right|\end{aligned} \qquad \text{(A.77)}$$

is less than $\varepsilon/(4\cdot\lambda_{\mathcal{T}})$ for all $\alpha\in[0,1]$, all $t\in\mathcal{T}$, and all $f(\cdot,\cdot)\in\mathcal{A}_{\mathcal{S}\times\mathcal{T}}$, where $\lambda_{\mathcal{S}} = \lambda(\mathcal{S})$ and $\lambda_{\mathcal{T}} = \lambda(\mathcal{T})$, so that each of the expressions

$$\begin{aligned}&\Bigg|\int_{\underline{t}}^{\bar{t}}\int_{\underline{s}}^{\bar{s}}\phi_{f_m^{\mathcal{S}}(s,t)}(s,t;f)\cdot ds\cdot dt\\ &\quad - \frac{\lambda_{\mathcal{S}}}{k}\cdot\sum_{i=0}^{k-1}\int_{\underline{t}}^{\bar{t}}\phi_{f_m^{\mathcal{S}}(\underline{s}+\frac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t)}\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t;f\big)\cdot dt\Bigg|\\ &\Bigg|\int_{\underline{t}}^{\bar{t}}\int_{\underline{s}}^{\bar{s}}\phi_{f_m^{\mathcal{T}}(s,t)}(s,t;f)\cdot ds\cdot dt\\ &\quad - \frac{\lambda_{\mathcal{S}}}{k}\cdot\sum_{i=0}^{k-1}\int_{\underline{t}}^{\bar{t}}\phi_{f_m^{\mathcal{T}}(\underline{s}+\frac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t)}\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t;f\big)\cdot dt\Bigg|\end{aligned} \qquad \text{(A.78)}$$

is less that $\varepsilon/4$ for all $\alpha\in[0,1]$ and all $f(\cdot,\cdot)\in\mathcal{A}_{\mathcal{S}\times\mathcal{T}}$. This and (A.76) imply

$$\begin{aligned}&\Bigg|\frac{\lambda_{\mathcal{S}}}{k}\cdot\sum_{i=0}^{k-1}\int_{\underline{t}}^{\bar{t}}\phi_{f_m^{\mathcal{T}}(\underline{s}+\frac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t)}\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t;f\big)\cdot dt\\ &\quad-\frac{\lambda_{\mathcal{S}}}{k}\cdot\sum_{i=0}^{k-1}\int_{\underline{t}}^{\bar{t}}\phi_{f_m^{\mathcal{S}}(\underline{s}+\frac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t)}\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t;f\big)\cdot dt\Bigg| <\\ &\left|\int_{\underline{t}}^{\bar{t}}\int_{\underline{s}}^{\bar{s}}\phi_{f_m^{\mathcal{T}}(s,t)}(s,t;f)\cdot ds\cdot dt - \int_{\underline{t}}^{\bar{t}}\int_{\underline{s}}^{\bar{s}}\phi_{f_m^{\mathcal{S}}(s,t)}(s,t;f)\cdot ds\cdot dt\right| + \frac{\varepsilon}{2} =\\ &\left|\sum_{j=1}^{n}\Phi_{x_j}\big(\mathcal{S}\times(\wp_j\underset{m}{\times}\mathcal{T});f\big) - \sum\nolimits_{j=1}^{n}\Phi_{x_j}\big((\wp_j\underset{m}{\times}\mathcal{S})\times\mathcal{T};f\big)\right| + \frac{\varepsilon}{2} < \varepsilon\end{aligned} \qquad \text{(A.79)}$$

for all $\alpha\in[0,1]$ and all $f(\cdot,\cdot)\in\mathcal{A}_{\mathcal{S}\times\mathcal{T}}$. Define the path $\{\hat{f}_\alpha^k(\cdot,\cdot)\,|\,\alpha\in[0,1]\}$ from $f_m^{\mathcal{S}}(\cdot,\cdot)$ to $f_m^{\mathcal{T}}(\cdot,\cdot)$ by

$$\hat{f}_\alpha^k(\cdot,\cdot) \equiv \big[f_m^{\mathcal{T}}(\cdot,\cdot)\text{ on }\big([0,\alpha]\underset{k}{\times}\mathcal{S}\big)\times\mathcal{T};\ f_m^{\mathcal{S}}(\cdot,\cdot)\text{ on }\big((\alpha,1]\underset{k}{\times}\mathcal{S}\big)\times\mathcal{T}\big] \qquad \text{(A.80)}$$

From bivariate event-differentiability, $W(\cdot)$'s derivative at all but a finite number of points along the path $\{\hat{f}_\alpha^k(\cdot,\cdot)\,|\,\alpha\in[0,1]\}$ is given by

$$\frac{dW\big(\hat{f}_\alpha^k(\cdot,\cdot)\big)}{d\alpha} = \sum_{i=0}^{k-1}\frac{\lambda_{\mathcal{S}}}{k}\cdot\Bigg[\int_{\underline{t}}^{\bar{t}}\phi_{f_m^{\mathcal{T}}(\underline{s}+\frac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t)}\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t;\hat{f}_\alpha^k\big)\cdot dt - \int_{\underline{t}}^{\bar{t}}\phi_{f_m^{\mathcal{S}}(\underline{s}+\frac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t)}\big(\underline{s}+\tfrac{(i+\alpha)\cdot\lambda_{\mathcal{S}}}{k},t;\hat{f}_\alpha^k\big)\cdot dt\Bigg] \tag{A.81}$$

Since (A.79) implies $|dW(\hat{f}_\alpha^k(\cdot,\cdot))/d\alpha|<\varepsilon$ except at this finite set of points, we have

$$\Big|W\big(f_m^{\mathcal{S}}(\cdot,\cdot)\big)-W\big(f_m^{\mathcal{T}}(\cdot,\cdot)\big)\Big| = \left|\int_0^1\frac{dW\big(\hat{f}_\alpha^k(\cdot,\cdot)\big)}{d\alpha}\cdot d\alpha\right| < \varepsilon \tag{A.82}$$

But since $m$ was an arbitrary value greater than $m_\varepsilon$, we have $\lim_{m\to\infty}W(f_m^{\mathcal{S}}(\cdot,\cdot))$ $=\lim_{m\to\infty}W(f_m^{\mathcal{T}}(\cdot,\cdot))$. □

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