as the basis of trade, it is an ideal set-up in which to isolate the roles of factor endowments and intensity differences that are unrelated to the basis of trade. For example, Matsuyama (2007a) uses a two-country Ricardian model to examine how factor intensity affects the extent of globalization and how globalization affects factor prices when certain factors are used more intensively in international trade than in domestic trade. The model is Ricardian in the sense that the patterns of comparative advantage are determined entirely by the exogenous technological differences. The factor proportions matter, however, because they determine the extent of globalization, as the effective trade costs vary with the relative endowments of the factor used intensively in international trade.

KIMINORI MATSUYAMA

See also **comparative advantage; globalization; international trade theory; terms of trade.**

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risk

The phenomenon of *risk* is one of the key determining factors in the formation of investment decisions, the operation of financial markets, and several other aspects of economic activity.

Risk versus uncertainty

The most fundamental distinction in this branch of economic theory, due to Knight (1921), is that of 'risk' versus 'uncertainty'. A situation is said to involve *risk* if the randomness facing an economic agent presents itself in the form of exogenously specified or scientifically calculable *objective probabilities*, as with gambles based on a roulette wheel or a pair of dice. A situation is said to involve *uncertainty* if the randomness presents itself in the form of alternative possible *events*, as with bets on a horse race, or decisions involving whether or not to buy earthquake insurance.

The standard approach to the modelling of preferences under uncertainty (as opposed to risk) has been the *state-preference approach* (for example, Arrow, 1964; Debreu, 1959, ch. 7; Hirshleifer, 1965; 1966; Karni, 1985; Yaari, 1969). Given the absence of exogenously specified objective probabilities, this approach represents the randomness facing the individual by a set of mutually exclusive and exhaustive *states of nature* or *states of the world* $\mathcal{S} = \{s_1, \ldots, s_n\}$. Depending upon the particular application, this partition of all conceivable futures may either be very coarse, as with the pair of states (it snows here tomorrow, it doesn't snow here tomorrow) or else very fine, so that the description of a single state might read 'it snows more than three inches here tomorrow *and* the temperature in Paris at noon is 73° *and* the price of gold in New York is over $900.00/ounce'. The objects of choice in this framework consist of *state-payoff bundles* of the form $(c_1, \ldots, c_n)$, which specify the payoff that the individual will receive in each of the respective states. As with regular commodity bundles, individuals are assumed to have preferences over state-payoff bundles which can be represented by indifference curves in the *state-payoff space* $\{(c_1, \ldots, c_n)\}$.

Even though the state-preference approach has led to important advances in the analysis of choice under uncertainty (see, for example, the above citations), the advantages of being able to draw on modern probability theory has led economists to hypothesize that an individual's *beliefs* in such settings can nevertheless still be represented by so-called *personal probabilities* or *subjective probabilities*, which take the form of an additive *subjective probability measure* $\mu(\cdot)$ over the state space $\mathcal{S}$. In such a case, a given state-payoff bundle $(c_1, \ldots, c_n)$ will be viewed as yielding outcome c_i with probability $\mu(s_i)$, so that the individual would evaluate the bundle $(c_1, \ldots, c_n)$ in the same manner as he or she would evaluate a casino gamble which yielded the payoffs $(c_1, \ldots, c_n)$ with respective objective probabilities $(\mu(s_1), \ldots, \mu(s_n))$. The hypothesis that individuals have such probabilistic beliefs and evaluate state-payoff bundles in such a manner is termed the *hypothesis of probabilistic sophistication*, and permits a unified application of probability theory to the analysis of decisions under both objective risk and subjective uncertainty. The joint hypothesis of probabilistic sophistication and expected utility risk preferences has been axiomatized by Ramsey (1926), Savage (1954), Anscombe and Aumann (1963), Pratt, Raiffa and Schlaifer (1964) and Raiffa (1968, ch.5), and probabilistic sophistication without expected utility has been axiomatized by Machina and Schmeidler (1992).

Choice under risk: the expected utility model

For reasons of expositional ease, we consider a world with a single commodity (for example, wealth). An agent making a decision under either risk or probabilistic uncertainty can therefore be thought of as facing a choice set of alternative univariate probability distributions. In order to consider both discrete (for example, finite outcome) distributions as well as distributions with density functions, we represent each such probability distribution by means of its cumulative distribution function $F(\cdot)$, where $F(x) \equiv \text{prob}\ (\tilde{x} \succcurlyeq x)$ for the random variable $\tilde{x}$.

In such a case we can model the agent's preferences over alternative probability distributions in a manner completely analogous to the approach of standard (that is, non-stochastic) consumer theory: he or she is assumed to possess a ranking $\succcurlyeq$ over distributions which is complete, transitive and continuous (in an appropriate sense), and hence representable by a real-valued *preference* function $V(\cdot)$ over cumulative distribution functions, in the sense that $F^*(\cdot) \succcurlyeq F(\cdot)$ (that is, the distribution $F^*(\cdot)$ is weakly preferred to $F(\cdot)$) if and only if $V(F^*) \succcurlyeq V(F)$.

Of course, as in the non-stochastic case, the above set of assumptions implies nothing about the functional form of the preference functional $V(\cdot)$. For reasons of both normative appeal and analytic convenience, economists typically assume that $V(\cdot)$ is a *linear functional* of the distribution $F(\cdot)$, and hence takes the form

$$V(F) \equiv \int U(x)\ dF(x) \qquad (1)$$

for some function $U(\cdot)$ over wealth levels x, where $U(\cdot)$ is referred to as the individual's *von Neumann–Morgenstern utility function.* (For readers unfamiliar with the *Riemann–Stieltjes integral* $\int U(x)dF(x)$, it represents nothing more than the expected value of $U(\tilde{x})$ when $\tilde{x}$ possesses the cumulative distribution function $F(\cdot)$. Thus if $\tilde{x}$ took the values $x_1, \ldots, x_n$ with probabilities $p_1, \ldots, p_n$ then $\int U(x)dF(x)$ would equal $\sum U(x_i)p_i$, and if $\tilde{x}$ possessed the density function $f(\cdot) = F'(\cdot)$ then $\int U(x)dF(x)$ would equal $\int U(x)f(x)dx$.)

Since the right side of (1) may be thought of as the mathematical expectation of $U(\tilde{x})$, this specification is known as the expected utility model of preferences over random prospects (for a more complete statement of this model, see expected utility hypothesis). Within this framework, an individual's attitudes towards risk are reflected in the shape of his or her utility function $U(\tilde{x})$. Thus, for example, an individual would always prefer shifting probability mass from lower to higher outcome levels if and only if $U(x)$ were an increasing function of x, a condition which we shall henceforth always assume. Such a shift of probability mass is known as a *first order stochastically dominating shift*.

Risk aversion

The representation of an individual's preferences over distributions by the shape of his or her von Neumann–Morgenstern utility function provides the first step in the

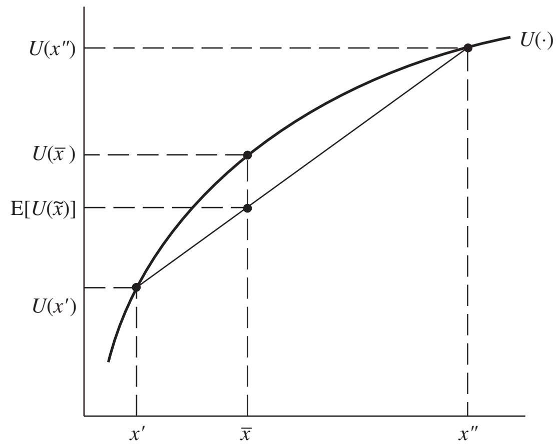

Figure 1 Von Neumann–Morgenstern utility function of a risk-averse individual

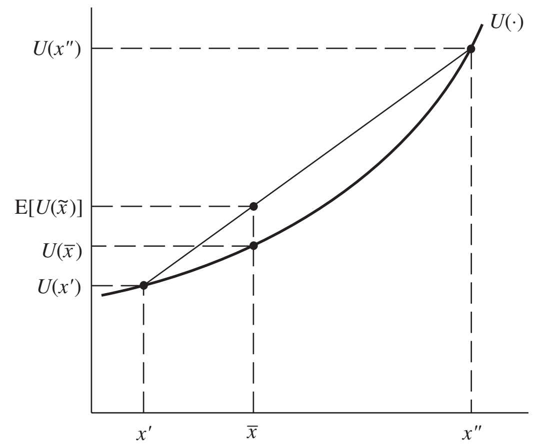

Figure 2 Von Neumann–Morgenstern utility function of a risk-loving individual

modern economic characterization of risk. After all, whatever the notion of 'riskier' means, it is clear that bearing a random wealth $\tilde{x}$ is riskier than receiving a certain payment of $\bar{x} = E[\tilde{x}]$ (the expected value of the random variable $\tilde{x}$). We therefore have from Jensen's inequality that an individual would be *risk averse*, that is, would always prefer a payment of $E[\tilde{x}]$ (and obtaining utility $U(E[\tilde{x}])$) to bearing the risk $\tilde{x}$ (and obtaining expected utility $E[U(\tilde{x})]$) if and only if his or her utility function were concave. This condition is illustrated in Figure 1, where the random variable $\tilde{x}$ is assumed to take on the values x' and x'' with respective probabilities 2/3 and 1/3.

Of course, not all individuals need be risk averse in the sense of the previous paragraph. Another type of individual is a *risk lover*. Such an individual would have a *convex* utility function, and would accordingly prefer receiving a random wealth $\tilde{x}$ to receiving its mean $E[\tilde{x}]$ with certainty. An example of such a utility function is given in Figure 2.

Standard deviation as a measure of risk

While the above characterizations of risk aversion and risk preference allow for the derivation of many results in the theory of choice under risk, they say nothing about which of a pair of non-degenerate random variables $\tilde{x}$ and $\tilde{y}$ is the more risky. Since real-world choices are almost never between risky and riskless situations but rather over alternative risky situations, such a means of comparison is necessary.

The earliest and best-known univariate measure of the riskiness of a random variable $\tilde{x}$ is its *variance* $\sigma^2 = E[(\tilde{x} - E[\tilde{x}])^2]$ or alternatively its *standard deviation* $\sigma = E[(\tilde{x} - E[\tilde{x}])^2]^{1/2}$. The tractability of these measures, as well as their well-known statistical properties, led to the widespread use of mean-standard deviation analysis in the 1950s and 1960s, and in particular to the

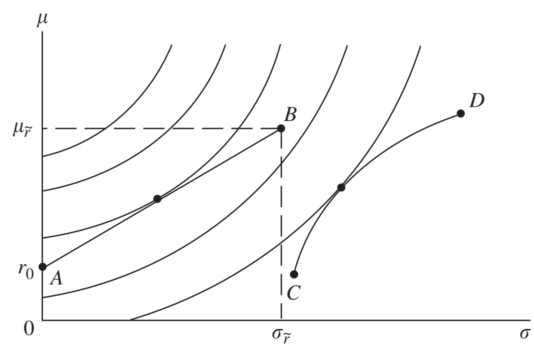

Figure 3 Portfolio analysis in the mean-standard deviation model

development of modern portfolio theory by Markowitz (1952; 1959), Tobin (1958) and others. As an example of this, consider Figure 3. Points A and B correspond to the distributions of a riskless asset with (per dollar) gross return r_0 and a risky asset with random return $\tilde{r}$ with mean $\mu_{\tilde{r}}$ and standard deviation $\sigma_{\tilde{r}}$. An investor dividing a dollar between the two assets in proportions α:$(1-\alpha)$ will possess a portfolio whose return has a mean of $\alpha \cdot r_0 + (1-\alpha) \cdot \mu_{\tilde{r}}$ and standard deviation $(1-\alpha) \cdot \sigma_{\tilde{r}}$, so that the set of attainable (μ,σ) combinations consists of the line segment connecting the points A and B in the figure. It is straightforward to show that, if the individual were also allowed to *borrow* at rate r_0 in order to finance purchase of the risky asset (that is, could sell the riskless asset short), then the set of attainable (μ,σ) combinations would be the ray emanating from A and passing through B and beyond.

If we then represent the individual's risk preferences by means of indifference curves in this diagram, we obtain his or her optimal portfolio (the example in the figure implies an equal division of funds between the two assets). In the more general case of choice between a pair of risky assets, the set of (μ,σ) combinations generated by alternative divisions of wealth between them will trace out a possibly nonlinear locus such as the one between points C and D in the diagram, with the curvature of this locus determined by the degree of statistical dependence (that is, covariance) between the two random returns.

As mentioned, the representation and analysis of risk and risk-taking by means of the variance or standard deviation of a distribution proved tremendously useful in the theory of finance, culminating in the mean-standard deviation-based *capital asset pricing model* of Sharpe (1964), Lintner (1965), Mossin (1966) and Treynor (1999). However, by the late 1960s the mean-standard deviation approach was under attack for two reasons.

The first reason (known since the 1950s) was the fact that an expected utility maximizer would evaluate all distributions solely on the basis of his or her means and standard deviations if and only if their von Neumann–Morgenstern utility function took the quadratic form $U(x) \equiv ax + bx^2$ for $b \Leftarrow 0$. The sufficiency of this condition is established by noting that $E[U(\tilde{x})] = E[a\tilde{x} + b\tilde{x}^2] = a \cdot E[\tilde{x}] + b \cdot (E[\tilde{x}]^2 + \sigma^2)$. To prove necessity, note that the distributions that yield a 2/3:1/3 chance of the outcomes $(x - \delta) : (x + 2\delta)$ and a 1/3:2/3 chance of the outcomes $(x - 2\delta) : (x + \delta)$ both possess the same mean and variance for each x and δ, so that $(2/3) \cdot U(x - \delta) + (1/3) \cdot U(x + 2\delta) \equiv (1/3) \cdot U(x - 2\delta) + (1/3) \cdot U(x + \delta)$ for all x and δ. Differentiating with respect to δ and simplifying yields $U'(x + 2\delta) + U'(x - 2\delta) \equiv U'(x + \delta) + U'(x - \delta)$ for all x and δ. This implies that $U'(\cdot)$ must be linear and hence that $U(\cdot)$ must be quadratic.

The assumption of quadratic utility is objectionable. If an individual with such a utility function is risk averse (that is, if $b<0$), then (*a*) utility will decrease as wealth increases beyond $1/(2b)$, and (*b*) the individual will be more averse to constant additive risks about high wealth levels than about low wealth levels – in contrast to the observation that those with greater wealth take greater risks (see for example Hicks, 1962, or Pratt, 1964).

Borch (1969) struck the second and strongest blow to the mean-standard deviation approach. He showed that, for any two points (μ_1,σ_1) and (μ_2,σ_2) in the (μ,σ) plane which a mean-standard deviation preference ordering would rank as indifferent, it is possible to find random variables $\tilde{x}_1$ and $\tilde{x}_2$ which possess these respective (μ,σ) values and where $\tilde{x}_2$ first order stochastically dominates $\tilde{x}_1$. However, *any* person with an increasing von Neumann–Morgenstern utility function would strictly prefer $\tilde{x}_2$ to $\tilde{x}_1$. In response to these arguments and the additional criticisms of Feldstein (1969), Samuelson (1967) and others, the use of mean-standard deviation analysis in economic theory waned. See, however, the work of Meyer (1987) for a partial rehabilitation of such two-moment models of preferences.

Besides the variance or standard deviation of a distribution, several other univariate measures of risk have been proposed. Examples include the *mean absolute deviation* $E[|\tilde{x} - E[\tilde{x}]|]$, the *interquartile range* $F^{-1}(.75) - F^{-1}(.25)$, and the classical statistical measures of *entropy* $\sum \ln(p_i) \cdot p_i$ or $\int \ln(f(x)) \cdot f(x) \cdot dx$. Although they provide the convenience of a single numerical index, each of these measures is subject to problems of the sort encountered with the variance or standard deviation. In particular, the entropy measure is based exclusively on the *probability levels* of a random variable, and is particularly unresponsive to its *outcome values* – for example, the 50:50 gambles over the values \$49:\$51 and \$0:\$100 possess identical entropy levels.

Increasing risk

By the late 1960s, the failure to find a satisfactory univariate measure of risk led to another approach to this problem. Working independently, several researchers (Hadar and Russell, 1969; Hanoch and Levy, 1969; and Rothschild and Stiglitz, 1970; 1971) developed an alternative characterization of increasing risk. The appeal of this approach is twofold. First, it formalizes three different intuitive notions of increasing risk. Second, it allows for the straightforward derivation of comparative statics results in a wide variety of economic situations. Unlike the univariate measures described above, however, this approach provides only a partial ordering of random variables. In other words, not all pairs of random variables can be compared with respect to their riskiness.

We now state three alternative formalizations of the notion that a cumulative distribution function $F^*(\cdot)$ is riskier than another distribution $F(\cdot)$ with the same mean. In the following, all distributions are assumed to be over the outcome interval $[0, M]$ unless otherwise indicated.

The first definition of increasing risk captures the notion that 'risk is what all risk averters hate'. Thus an increase in risk must lower the expected utility of all risk averters. Formally:

(A) *$F^*(\cdot)$ and $F(\cdot)$ have the same mean and $\int U(x)dF^*(x) \leq \int U(x)dF(x)$ for every concave utility function $U(\cdot)$.*

Note that this relationship will *not* be satisfied by every pair of distributions with the same mean. That is to say, there exist pairs $F(\cdot)$ and $F^*(\cdot)$, with the same mean, but such that some risk-averse utility functions prefer $F(\cdot)$ to $F^*(\cdot)$ but other risk-averse utility functions prefer $F^*(\cdot)$ to $F(\cdot)$. This reflects the above-stated fact that comparative risk is a *partial* rather than a *complete* order over the family of probability distributions, even over families of distributions with a common mean. (Although comparative risk is not a complete order, it is a *transitive* order, in the sense that, if the pair $F^*(\cdot)$ and $F(\cdot)$ satisfy condition (A), and the pair $F^{**}(\cdot)$ and $F^*(\cdot)$ satisfy condition (A), then the pair $F^{**}(\cdot)$ and $F(\cdot)$ will also satisfy condition (A).)

The second characterization of the notion that a random variable $\tilde{y}$ with distribution $F^*(\cdot)$ is riskier than a variable $\tilde{x}$ with distribution $F(\cdot)$ is that $\tilde{y}$ consists of the variable $\tilde{x}$ plus an additional zero-mean noise term $\tilde{\varepsilon}$. One possible specification of this is that $\tilde{\varepsilon}$ statistically independent of $\tilde{x}$. However, this condition is too strong in the sense that it does not allow the variance of $\tilde{\varepsilon}$ to depend upon the magnitude of $\tilde{x}$, as in the case of heteroskedastic noise. Instead, Rothschild and Stiglitz (1970) modelled the addition of noise by the condition:

(B) *$F(\cdot)$ and $F^*(\cdot)$ are the respective cumulative distribution functions of the random variables $\tilde{x}$ and $\tilde{x}+\tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|x] \equiv 0$ for all values of x.*

The third notion of increasing risk involves the concept, due to Rothschild and Stiglitz (1970), of a *mean preserving spread*. Intuitively, such a spread consists of moving probability mass from some region in the centre of a probability distribution out to its tails in a manner that preserves the expected value of the distribution, as seen in the top panels of Figures 4 and 5. In the discrete case of Figure 4, probability mass is moved from the pair of outcome values *b* and *c* out to the outcome values *a* and *d*. In the continuous density case of Figure 5, probability mass is moved from the interval (*b*, *c*) out to the intervals (*a*, *b*) and (*c*, *d*). We can unify, generalize and formalize this condition by saying that $F^*(\cdot)$ differs from $F(\cdot)$ by a 'mean preserving spread' if they have the same mean and there exists a single crossing point x_0 such that $F^*(x) \geq F(x)$ for all $x \leq x_0$ and $F^*(x) \leq F(x)$ for all $x \geq x_0$ (see the middle panels of Figures 4 and 5). Since it is clear that *sequences* of such spreads will also lead to riskier distributions, the third characterization of increasing risk is:

(C) *$F^*(\cdot)$ may be obtained from $F(\cdot)$ by a finite sequence, or as the limit of an infinite sequence, of mean preserving spread.*

Although the single crossing property of the previous paragraph serves to characterize cumulative distribution functions that differ by a single mean preserving spread, distributions that differ by a sequence of such spreads will typically not satisfy the single crossing condition. However, if we consider the integrals of these cumulative distribution functions, we see from the bottom panels of Figures 4 and 5 that a mean preserving spread will always serve to raise or preserve the value of this integral for each *x*, and (since $F^*(\cdot)$ and $F(\cdot)$ have the same mean) will exactly preserve it for *x*=*M*. In contrast to the single crossing property, this so-called 'integral condition' *will* continue to be satisfied by distributions which differ by a sequence of one or more mean preserving spreads. Accordingly, we may rewrite condition (C) above by the analytically more convenient:

(C′) *The integral $\int_0^x [F^*(\xi) - F(\xi)] \cdot d\xi$ is non-negative for all $x > 0$, and is equal to 0 at x=M.*

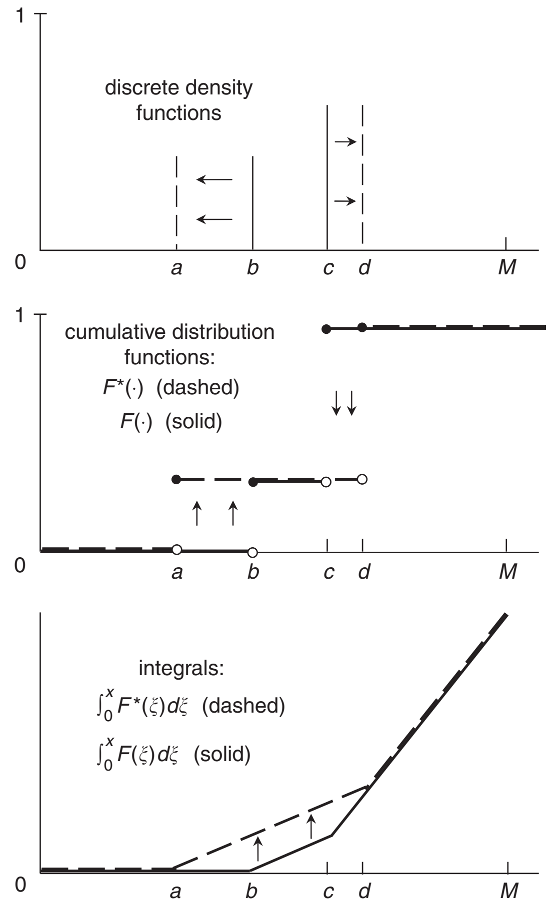

Figure 4 Mean preserving spread of a discrete distribution

Rothschild and Stiglitz (1970) showed that these three concepts of increasing risk are the same by proving that conditions (A), (B) and (C/C′) are equivalent. Thus, a single partial ordering of distribution functions corresponds simultaneously to the notion that risk is what risk averters hate, to the notion that adding noise to a random variable increases its risk, and to the notion that moving probability mass from the centre of a probability distribution to its tails increases the riskiness of the distribution. The original Rothschild–Stiglitz formulation and proofs have since been further strengthened and extended by Machina and Pratt (1997).

This characterization permits the derivation of general and powerful comparative statics theorems concerning economic agents' responses to increases in risk. The general framework for these results is that of an individual with a von Neumann–Morgenstern utility function $U(x, \alpha)$ which depends upon both the outcome of some random variable $\tilde{x}$ as well as a *control variable* α which the individual chooses so as to maximize expected utility $\int U(x,\alpha)dF(x;r)$, where the distribution function $F(\cdot\,;r)$ depends upon

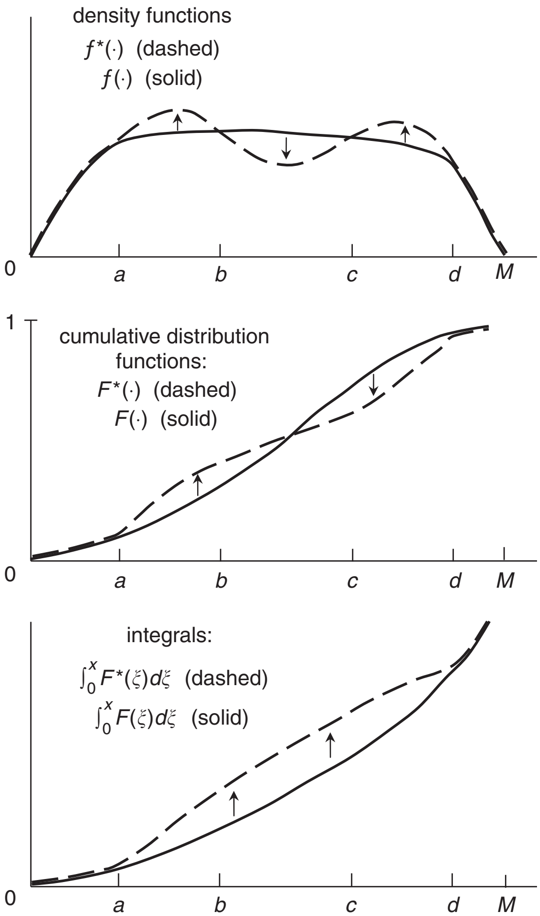

Figure 5 Mean preserving spread of a density function

some exogenous parameter r (x for example might be the return on a risky asset, and α the amount invested in it). For convenience, we assume that $F(0;r) \equiv prob(\tilde{x} \leq 0) \geq 0$ for all r. The first order condition for this problem is then:

$$\int U_\alpha(x,\alpha)dF(x;r) = 0 \qquad (2)$$

where $U_\alpha(x,\alpha) = \partial U(x,\alpha)/\partial\alpha$, and we assume that the second derivative $U_{\alpha\alpha}(x,\alpha) = \partial^2 U(x,\alpha)/\partial\alpha^2$ is always negative to insure we have a maximum. Implicit differentiation of (2) then yields the comparative statics derivative:

$$d\alpha/dr = -\int U_\alpha(x,\alpha)dF_r(x;r) \Big/ \int U_{\alpha\alpha}(x,\alpha)dF(x;r) \qquad (3)$$

where $F_r(x;r) = \partial F(x;r)/\partial r$. Since the denominator of this expression is negative by assumption, the sign of $d\alpha/dr$ is given by the sign of the numerator $\int U_\alpha(x,\alpha)dF_r(x;r)$. Integrating by parts twice yields:

$$\begin{aligned}\int U_\alpha(x,\alpha)dF_r(x;r) &= \int U_{xx\alpha}(x,\alpha)\cdot\left[\int_0^x F_r(\xi;r)d\xi\right]dx \\ &= \int U_{xx\alpha}(x,\alpha)\cdot\left[\frac{d}{dr}\int_0^x F(\xi;r)d\xi\right]dx \qquad (4)\end{aligned}$$

Thus, if increases in the parameter r imply increases in the riskiness of the distribution $F(\cdot, r)$, it follows from condition (C′) that the signs of the square-bracketed terms in (4) will be non-negative, so that the effect of r upon α depends upon the sign of $U_{xx\alpha}(x,\alpha) = \partial^3 U(x,\alpha)/\partial x^2\partial\alpha$. Thus, if $U_{xx\alpha}(x,\alpha)$ is uniformly negative a mean preserving increase in risk in the distribution of x will lead to a fall in the optimal value of the control variable α, and vice versa. Another way to see this is to note that if $U_\alpha(x,\alpha)$ is concave in x then a mean preserving increase in risk will lower the left side of the first order condition (2), which (since $U_{\alpha\alpha}(x,\alpha) \leq 0$) will require a drop in α to re-establish the equality. Economists, mathematicians and scientists routinely use this technique when analysing models involving risk; see for example Rothschild and Stiglitz (1971), Dionne, Eeckhoudt and Gollier (1993), Eeckhoudt, Gollier and Schlesinger (1996), Jewitt (1987), Ormiston (1992), Tzeng (2001), Nowak (2004), Chateauneuf, Cohen and Meilijson (2004), Baker (2006), and Beladi, de la Vina and Firoozi (2006).

Related topics

The characterization of risk outlined in the previous section has been extended along several lines. Diamond and Stiglitz (1974), for example, have replaced the notion of a mean preserving spread with that of a mean *utility* preserving spread to obtain a general characterization of a *compensated increase in risk*. They relate this notion to the well known Arrow–Pratt characterization of comparative risk aversion (see EXPECTED UTILITY HYPOTHESIS).

In addition, researchers such as Ekern (1980), Fishburn (1982), Fishburn and Vickson (1978), Hansen, Holt, and Peled (1978), Tesfatsion (1976), and Whitmore (1970) have extended the above work to the development of a general theory of *stochastic dominance*, which provides a whole sequence of similarly characterized partial orders on distributions, each presenting a corresponding set of equivalent conditions involving algebraic conditions on the distributions, types of spreads, and classes of utility functions which prefer (or are averse to) such spreads. The comparative statics analysis presented above may be similarly extended to such characterizations. An extensive bibliography of the stochastic dominance literature is given in Bawa (1982). Finally, various extensions of the notions of increasing risk and stochastic dominance to the case of multivariate distributions may be found in Epstein and Tanny (1980), Fishburn and Vickson (1978), Huang, Kira and Vertinsky (1978), Lehmann (1955), Levhari, Parousch and Peleg (1975), Levy and Parousch (1974), Russell and Seo (1978), Sherman (1951), and Strassen (1965); see also the mathematical results in Marshall and Okun (1979).

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See also **expected utility hypothesis; risk aversion; uncertainty.**

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risk aversion

Arrow–Pratt theory of risk aversion

The classical theory of risk aversion, due to Pratt (1964) and Arrow (1965), is rooted in the expected utility theory of decision making. An agent's preferences are assumed to have an expected utility representation. The objects of choice are real valued random variables defined either on a finite or infinite set of states of the world with probabilities of states that may be either objective or subjective. The intended interpretation of a random variable is as an agent's risky wealth.

An agent whose expected utility representation of preferences is written $E[u(\tilde{x})]$, where u is the von Neumann–Morgenstern utility function and E denotes the expectation (expected value), is *risk averse* if

$$E[u(\tilde{x})] \leq u(E(\tilde{x})) \tag{1}$$

for every risky wealth $\tilde{x}$. If (1) holds with strict inequality for every non-deterministic $\tilde{x}$, the agent is *strictly risk averse.* Jensen's inequality implies that, if utility function u is concave, the agent is risk averse. The converse is also true. Thus, the concavity of u is a necessary and sufficient condition for risk aversion. Moreover, strict concavity of u is a necessary and sufficient condition for strict risk aversion. Examples of strictly concave von Neumann–Morgenstern utility functions, commonly used in applied work, include the negative exponential utility $u(w) = e^{-\alpha w}$ with $\alpha > 0$, the logarithmic utility $u(w) = \ln(w)$, and the power utility $u(w) = \frac{1}{1-\alpha} w^{1-\alpha}$ with $\alpha > 0$, $\alpha \neq 1$.

It is useful to have a measure of the intensity of risk aversion. The most natural measure is *risk compensation.* It is by definition the amount $\rho(w, \tilde{z})$ of deterministic wealth one could extract from an agent in exchange for relieving her of zero-expectation risk $\tilde{z}$ at an initial