Equilibrium Dynamics in Markets for Lemons*

Diego Moreno[†] John Wooders[‡]

December 24, 2011

Abstract

Akerlof (1970)'s discovery that competitive markets for lemons generate inefficient outcomes has important welfare implications, and rises fundamental questions about the role of time, frictions, and micro-infrastructure in market performance. We study the equilibria of *centralized* and *decentralized dynamic* markets for Lemons, and show that if markets are short lived and frictions are small, then decentralized markets perform better. If markets are long lived, the limiting equilibrium of a decentralized market generates the static competitive surplus, whereas a centralized market has competitive equilibria where low quality trades immediately and high quality trades with delay, which generate a greater surplus.

*We gratefully acknowledge financial support from Spanish Ministry of Science and Innovation, grants SEJ2007-67436 and ECO2011-29762.

[†]Departamento de Economía, Universidad Carlos III de Madrid, dmoreno@eco.uc3m.es.

[‡]Department of Economics, University of Technology Sydney, jwooders@gmail.com.

Notation Chart

The Market

τ: good quality, $\tau \in \{H, L\}$.

u^τ: value to buyers of a unit of τ-quality.

c^τ cost to sellers of τ-quality.

q^τ: fraction of sellers of τ-quality in the market.

t: a date at which the market is open, $t \in \{1, ..., T\}$.

δ: traders' discount factor.

$u(q)$ $= qu^H + (1-q)u^L$.

$\bar{S}$ $= (1-q^H)u^L$.

$\bar{q}$ $= \dfrac{c^H - u^L}{u^H - u^L}$, i.e., $u(\bar{q}) = c^H$.

Decentralized Market Equilibrium

r_t^τ: reservation price of sellers of τ-quality at date t.

λ_t^τ: probability that a seller of τ-quality who is matched at date t trades.

m_t^τ: stock of τ-quality sellers in the market at date t.

q_t^τ: fraction of τ-quality sellers in the market at date t.

V_t^τ: expected utility of a seller of τ-quality at date t.

V_t^B: expected utility of a buyer at date t.

S^{DE}: surplus in a decentralized market equilibrium – see equation (2).

ρ_t^τ: probability of a price offer of r_t^τ at date t.

$\hat{q}$ $= \dfrac{c^H - c^L}{u^H - c^L}$, i.e., $u(\hat{q}) - c^H = (1-\hat{q})(u^L - c^L)$.

Dynamic Competitive Equilibrium

s_t^τ: supply of τ-quality good at date t,

u_t: expected value to buyers of a unit supplied at date t.

d_t: demand at date t.

S^{CE}: surplus in a dynamic competitive equilibrium – see equation (3).

1 Introduction

Akerlof's finding that competitive markets for lemons generate inefficient outcomes is a cornerstone of the theory of markets with adverse selection. The prevalence of adverse selection in modern economies, from real good markets like housing or cars markets, to insurance markets or to markets for financial assets, warrants large welfare implications of this result, and calls for research on fundamental questions that remain open: How do dynamic markets perform? Does adverse selection improve or deteriorate over time? How illiquid are the different qualities in the market? What is the role of frictions in alleviating or aggravating adverse selection? Which market structures (e.g., centralized markets, or markets where trade is bilateral) perform better? Is there a role for government intervention? Our analysis attempts to provide an answer to these questions.

A partial solution to the adverse selection problem is the introduction of multiple markets for the good, differentiated by time. In this setting, if sellers are not too patient, then there is a dynamic competitive equilibrium in which both low and high quality units of the good trade: Low quality units trade immediately at a low price and high quality units trade with delay, but at a high price. Sellers of low quality prefer to trade immediately at the low price rather than suffer the delay necessary to obtain the high price. This dynamic competitive equilibrium yields more than surplus than obtained in the static competitive equilibrium and hence partially solves the Lemons problem. However, this solution fails (as we show) if players are patient relative to the horizon of the market: If the market is open for finite time and sellers are sufficiently patient, then only low quality units of the good trade in the dynamic competitive equilibrium, just as was the case in the static Akerlofian market.

The present paper studies decentralized trade, in which buyers and sellers match and then bargain bilaterally over the price, as a solution to the Lemons problem. We show that when the market is open for finite time, then decentralized trade yields more than the dynamic (and static) competitive surplus. We characterize the dynamics of prices and trading patterns over time in the unique decentralized market equilibrium. We also study the asymptotic properties of equilibrium as trading frictions vanish.

In the market we study there is an equal measure of buyers and seller initially in the market, and there is no further entry over time. Buyers are homogeneous, but sellers may have a unit of either high or low quality. A seller knows the quality of his good, but quality is unknown to buyers prior to purchase. Each period every agent remaining in the market is matched with positive probability with an agent of the opposite type. Once matched, a buyer makes a take-it-or-leave-it price offer to his partner. If the seller accepts, then they trade at the offered price and both agents exit the market. If the seller rejects the offer, then both agents remain in the market at the next period to look for a new match. Traders discount future gains. The possibility of not meeting a partner, the discounting of the future gains, and the finite duration of the market constitute trading "frictions."

We show that when traders are sufficiently patient, then there is unique decentralized market equilibrium: In the first period buyers make only "low" price offers (i.e., offers which are accepted only by low quality sellers) and non-serious offers which are rejected by both types of sellers. In the last period, buyers make only low price offers and "high" price offers (i.e., offers which are accepted by both types of sellers). In the intervening periods, buyer make all three types of price offers. Thus we provide a complete characterization of the trading patterns that may arise in equilibrium. Since low and high quality sellers trade at different rates, the average quality of the units remaining in the market changes over time: It rises quickly after the first period, it rises slowly in the intermediate periods, and it makes buyers indifferent between offering the price accepted only by low quality sellers and the price accepted by both type of sellers in the last period.

We also relate the decentralized market equilibrium to the competitive outcome. We show that when frictions are small, then low quality sellers obtain less than their competitive payoff, high quality sellers obtain their competitive payoff (of zero), and buyers obtain more than their competitive payoff. Total surplus exceeds the both the static and dynamic competitive surplus, and thus decentralized trade provides a partial solution to the Lemons problem when centralized trade does not. As frictions vanish (but holding the time horizon fixed), the payoff to low quality sellers increases, while the payoff to buyers decreases, but the total surplus remains asymptotically above the competitive surplus.

A property of equilibrium is that there is a long interval where most (but not all) price offers are non-serious, with the market for the good illiquid. As frictions vanish, in the limit equilibrium has a "bang-wait-bang" structure: There is trade in the first and last period, but the market is completely illiquid in the intervening periods.

Our final results concern the structure of equilibrium of infinitely-lived markets. We show that in this setting there are multiple dynamic competitive equilibria, and we characterize the equilibrium that maximizes the surplus. When players are patient, there is a decentralized market equilibrium in which in the first period buyers make both low and non-serious prices offers, and forever afterward they make both high, low, and non-serious price offers. As frictions vanish, in the limit all traders obtain their competitive payoff, only low quality units trade, and all of these units trade in the first period.

Related Literature

There is a large literature that examines the mini-micro foundations of competitive equilibrium. This literature has established that in markets for homogenous goods decentralized trade tends to yield competitive outcomes when trading frictions are small – see, e.g., Gale (1987) or Binmore and Herrero (1988) when bargaining is under complete information, and by Serrano and Yosha (1996) or Moreno and Wooders (1999) when bargaining is under incomplete information. Several papers by Wright and co-workers have studied decentralized markets with adverse selection motivated by questions from monetary economics – see, e.g., Velde, Weber and Wright (1999). More recently Blouin (2003) studies a decentralized market for lemons analogous to the one in the present paper. He assumes that the expected utility of a random unit is above the cost of high quality, and obtains results different from ours: he finds, for example, that each type of trader obtains a positive payoff (and therefore payoffs are not competitive) even as frictions vanish. (In our setting, for the parameter configurations considered in Blouin (2003) the equilibrium outcome approaches the competitive equilibrium in which all units trade at a price equal to the cost of high quality.) This discrepancy arises because in Blouin's setting only one of three exogenously given prices may emerge from bargaining.[1] (In our model, prices are determined endogenously without prior constraints.) Moreno and Wooders (2006) study the steady states of decentralized market for lemons with stationary entry, and

[1]Blouin (2003), however, obtains results for a market that operates over an infinite horizon, a case that seems intractable with fully endogenous prices.

finds that stationary equilibrium payoffs are competitive as frictions vanish.

The paper is organized as follows. Section 2 describes our market for lemons. Section 3 introduces a definition of dynamic competitive equilibrium and derives its properties. Section 4 describes a market where trade is decentralized, and introduces a notion of dynamic decentralized equilibrium. Section 5 presents results describing the properties of dynamic decentralized equilibria. Section 6 presents results for infinite lived markets for Lemons. Section 7 concludes with a discussion of static efficient mechanisms. Proofs are presented in the Appendix.

2 A Market for Lemons

Consider a market for an indivisible commodity whose quality can be either high or low. There is an equal measure of buyers and sellers present at the market open, which we normalize to one, and there is no further entry. A fraction $q^H \in (0,1)$ of the sellers are endowed with a unit of high-quality, whereas a fraction $q^L = 1 - q^H$ of the sellers are endowed with a unit of low-quality. A seller knows the quality of his good, but quality is unobservable to buyers. Preferences are characterized by values and costs: the cost to a seller of a unit of high (low) quality is c^H (c^L); the value to a buyer of a high (low) quality unit of the good is u^H (u^L). Thus, if a buyer and a seller trade at the price p, the buyer obtains a utility of $u - p$ and the seller obtains a utility of $p - c$, where $u = u^H$ and $c = c^H$ if the unit traded is of high quality, and $u = u^L$ and $c = c^L$ if it is of low quality. A buyer or a seller who does not trade obtains a utility of zero.

We assume that both buyers and sellers value high quality more than low quality (i.e., $u^H > u^L$ and $c^H > c^L$), and that each type of good is more valued by buyers than by sellers (i.e., $u^H > c^H$ and $u^L > c^L$). Also we restrict attention to markets in which the Lemons problem arises; that is, we assume that the expected value to a buyer of a randomly selected unit of the good, given by

$$u(q^H) := q^H u^H + q^L u^L < c^H,$$

is below the cost of high quality, c^H. Equivalently, we may state this assumption as

$$q^H < \bar{q} := \frac{c^H - u^L}{u^H - u^L}. \tag{1}$$

In this market, the Lemons problem arises since only low quality trades in the unique (static) competitive equilibrium, even though there are gains to trade for both qualities – see Figure 1. For future references, we describe this equilibrium in Remark 1 below.

Figure 1 goes here.

Remark 1. *In the unique static competitive equilibrium of the market all low quality units trade at the price u^L, and none of the high quality trade. Thus, the gains to trade to low quality sellers is $\bar{v}^L = u^L - c^L$, and the gains to trade to high quality sellers and to buyers are $\bar{v}^H = \bar{v}^B = 0$, and thus the surplus, $\bar{S} = q^L(u^L - c^L)$, is captured by low quality sellers.*

3 A Decentralized Market for Lemons

Consider a market for lemons as that described in Section 2 in which trade is bilateral. The market opens for T consecutive periods. Agents discount utility at a common rate $\delta \in (0,1]$, i.e., if a unit of quality τ trades at date t and price p, then the buyer obtains a utility of $\delta^{t-1}(u^\tau - p)$ and the seller obtains a utility of $\delta^{t-1}(p - c^\tau)$. Each period every buyer (seller) in the market meets a randomly selected seller (buyer) with probability $\alpha \in (0,1)$. A matched buyer proposes a price at which to trade. If the proposed price is accepted by the seller, then the agents trade at that price and both leave the market. If the proposed price is rejected by the seller, then the agents split and both remain in the market at the next period. A trader who is unmatched in the current period also remains in the market at the next period. An agent observes only the outcomes of his own matches.

In this market, a *pure strategy for a buyer* is a sequence of price offers $(p_1, ..., p_T) \in \mathbb{R}^T_+$. A *pure strategy for a seller* is a sequence of reservation prices $r = (r_1, ..., r_T) \in \mathbb{R}^T_+$, where r_t is the smallest price that the seller accepts at time $t \in \{1, ..., T\}$.[2]

A profile of buyers' strategies may be described by a sequence $\lambda = (\lambda_1, ..., \lambda_T)$, where λ_t is a *c.d.f.* with support on $\mathbb{R}_+$ specifying the probability distribution of

[2]Ignoring, as we do, that a trader may condition his actions on the history of his prior matches is inconsequential – see Osborne and Rubinstein (1990), pages 154-162.

price offers at date $t \in \{1, ..., T\}$. Given λ, the maximum expected utility at date t of a seller of quality $\tau \in \{H, L\}$ is $V_{T+1}^{\tau} = 0$, and for $t \leq T$ it is defined recursively as

$$V_t^{\tau} = \max_{x \in \mathbb{R}_+} \left\{ \alpha \int_x^{\infty} (p_t - c^{\tau})\, d\lambda_t(p_t) + \left(1 - \alpha \int_x^{\infty} d\lambda_t(p_t)\right) \delta V_{t+1}^{\tau} \right\}.$$

In this expression, the payoff to a seller of τ-quality who receives a price offer p_t is $p_t - c^{\tau}$ if p_t is at least his reservation price x, and it is δV_{t+1}^{τ}, his continuation utility, otherwise. Since all the seller of τ quality have the same maximum payoff, then their equilibrium reservation prices are identical. Therefore we restrict attention to strategy distributions in which all sellers of the type $\tau \in \{H, L\}$ use the same sequence of reservation prices $r^{\tau} \in \mathbb{R}_+^T$.

Let (λ, r^H, r^L) be a *strategy distribution* and let $t \in \{1, ..., T\}$. The probability that a seller of quality $\tau \in \{H, L\}$ who is matched at date t trades is

$$\lambda_t^{\tau} = \int_{r_t^{\tau}}^{\infty} d\lambda_t,$$

the stock of τ-quality sellers in the market is

$$m_{t+1}^{\tau} = (1 - \alpha \lambda_t^{\tau})\, m_t^{\tau},$$

with $m_1^{\tau} = q^{\tau}$, and the fraction of τ-quality sellers in the market is

$$q_t^{\tau} = \frac{m_t^{\tau}}{m_t^H + m_t^L}.$$

The maximum expected utility to buyer at date t is $V_{T+1}^B = 0$, and for $t \leq T$ it is defined recursively as

$$V_t^B = \max_{x \in \mathbb{R}_+} \left\{ \alpha \sum_{\tau \in \{H,L\}} q_t^{\tau} I(x, r_t^{\tau})(u^{\tau} - x) + \left(1 - \alpha \sum_{\tau \in \{H,L\}} q_t^{\tau} I(x, r_t^{\tau})\right) \delta V_{t+1}^B \right\},$$

where $I(x, y)$ is the indicator function whose value is 1 if $x \geq y$, and 0 otherwise. In this expression, the payoff to a buyer who offers the price x is $u^{\tau} - x$ when matched to a τ-quality seller who accepts the offer (i.e., $I(x, r_t^{\tau}) = 1$), and it is δV_{t+1}^B, her continuation utility, otherwise.

A strategy distribution (λ, r^H, r^L) is a *decentralized market equilibrium (DE)* if for each $t \in \{1, ..., T\}$:

$(DE.\tau)$ $r_t^\tau - c^\tau = \delta V_{t+1}^\tau$ for $\tau \in \{H, L\}$, and

$(DE.B)$ $\alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(p_t, r_t^\tau)(u^\tau - p_t) + \left(1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(p_t, r_t^\tau)\right) \delta V_{t+1}^B = V_t^B$ for every p_t in the support of λ_t.

Condition $DE.\tau$ ensures that each type τ seller is indifferent between accepting or rejecting an offer of his reservation price. Condition $DE.B$ ensures that price offers that are made with positive probability are optimal.

The *surplus* realized in a market equilibrium can be calculated as

$$S^{DE} = V_1^B + q^H V_1^H + q^L V_1^L. \tag{2}$$

4 Decentralized Market Equilibrium

In this section we study the equilibria of a decentralized market. Proposition 1 establishes basic properties of decentralized market equilibria.

Proposition 1. *If* (λ, r^H, r^L) *is a DE, then for all* $t \in \{1, ..., T\}$*:*

(1.1) $r_t^H = c^H > r_t^L$ *and* $q_{t+1}^H \geq q_t^H$.

(1.2) Only the high price $p_t = c^H$*, or the low price* $p_t = r_t^L$*, or negligible prices* $p_t < r_t^L$ *may be offered with positive probability.*

The intuition for these results is straightforward: Since buyers make price offers, they keep sellers at their reservation prices.[3] Sellers' reservation prices at T are equal to their costs, i.e., $r_T^\tau = c^\tau$, since agents who do not trade obtain a zero payoff. Thus, buyers never offer a price above c^H at T, and therefore the expected utility of high-quality sellers at T is zero, i.e., $V_T^H = 0$. Hence $r_{T-1}^H = c^H$. Also, since delay is costly (i.e., $\delta\alpha < 1$), low-quality sellers accept price offers below c^H, i.e., $r_{T-1}^L < c^H$. A simple induction argument shows that $r_t^H = c^H > r_t^L$ for all t. Obviously, prices $p > r_t^H$, accepted by both types of buyers, or prices in the interval (r_t^L, r_t^H), accepted only by low-quality sellers, are suboptimal, and are therefore made with probability zero. Moreover, since $r_t^H > r_t^L$ the proportion of high-quality sellers in the market (weakly) increases over time (i.e., $q_{t+1}^H \geq q_t^H$) as high-quality sellers (who only accept

[3] This is a version of the Diamond Paradox in our context.

offers of r_t^H) may exit the market at a slower rate than low-quality sellers (who accept offers of both r_t^H and r_t^L).

In a decentralized market equilibrium a buyer may offer: (i) a *high price*, $p = r_t^H = c^H$, which is accepted by both types of sellers, thus getting a unit of high quality with probability q_t^H and of low quality with probability $q_t^L = 1 - q_t^H$; or (ii) a *low price* $p = r_r^L$, which is accepted by low quality sellers and rejected by high quality sellers, thus trading only if the seller in the match has a unit of low quality (which occurs with probability q_t^L); or (iii) a *negligible price*, $p < r_t^L$, which is rejected by both types of sellers. In order to complete the description of a decentralized market equilibrium we need to determine the probabilities with which each of these three price offers are made.

Let (λ, r^H, r^L) be a market equilibrium. Recall that λ_t^τ is the probability that a matched τ-quality seller trades at date t (i.e., gets an offer greater than or equal to r_t^τ). For $\tau \in \{H, L\}$ denote by ρ_t^τ the probability of a price offer equal to r_t^τ. Since the probability of offering a price greater than c^H is zero by Proposition 1, then the probability of a high price offer is $\rho_t^H = \lambda_t^H$. And since prices in the interval (r_t^L, r_t^H) are offered with probability zero, then the probability of a low price offer is $\rho_t^L = \lambda_t^L - \lambda_t^H$. Hence the probability of a negligible price offer is $1 - \left(\rho_t^H + \rho_t^L\right) = 1 - \lambda_t^L$. Thus, ignoring the inconsequential distribution of negligible price offers, henceforth we describe a *decentralized market equilibrium* by a collection $(\rho^H, \rho^L, r^H, r^L)$.

Our next remark follows immediately from Proposition 1 and the discussion above. It states that in a decentralized market that opens for a single period, only low price offers are made and only low quality trades. Thus, the basic properties of a static Lemons market are the same whether trade is centralized or decentralized – see Remark 1 above.

Remark 2. *If $T = 1$, then the unique DE is $(\rho^H, \rho^L, r^H, r^L) = (0, 1, c^H, c^L)$. Hence all matched low quality sellers trade at the price u^L, and none of the high quality sellers trade. Traders' expected utilities are $V_1^L = \alpha(u^L - c^L)$ and $V_1^H = V_1^B = 0$, and the surplus, $S = q^L \alpha(u^L - c^L)$, is captured by low quality sellers.*

In a market that opens for a single period, the sellers' reservation prices are their costs. Thus, a buyer's payoff is $u(q^H) - c^H$ if he offers c^H and is $(1 - q^H)(u^L - c^L)$ if he offers c^L. Let $\hat{q}$ be the fraction of high quality sellers in the market that makes a buyer indifferent between these two offers; i.e.,

$$\hat{q} := \frac{c^H - c^L}{u^H - c^L}.$$

It is easy to see that $\bar{q} < \hat{q}$. Since $q^H < \bar{q}$ by assumption, then $\bar{q} < \hat{q}$ implies $q^H < \hat{q}$, and therefore low price offers are optimal.

Proposition 2 below establishes that if *frictions are small*, then a market that opens for two or more periods has a unique decentralized market equilibrium, and it identifies which prices are offered at each date. (Precise expressions for the equilibrium reservation prices and mixtures over price offers are provided in the Appendix.) The following definition makes it precise what we mean by frictions being small.

We say that *frictions are small* when δ and α are sufficiently close to one that:

$(SF.1)$ $\delta\alpha(c^H - c^L) > u^L - c^L$, and

$(SF.2)$ $\delta\left[\alpha\left(1-q^H\right)\hat{q} - \hat{q} + q^H\right](c^H - c^L) > q^H(1-\hat{q})(u^L - c^L)$.

Since $c^H - c^L > u^L - c^L$, then $(SF.1)$ holds for δ and α sufficiently close to one. Also note that if $\alpha = 1$, then $(SF.2)$ reduces to $\delta(c^H - c^L) > u^L - c^L$, which holds for δ close to one. Hence $(SF.2)$ also holds for α and δ close to one.

Proposition 2. *If $T > 1$ and frictions are small, then the following properties uniquely determine a DE:*

(2.1) Only low and negligible prices are offered at date 1, i.e., $\rho_1^H = 0$, $\rho_1^L > 0$, and $1 - \rho_1^L - \rho_1^H > 0$.

(2.2) High, low and negligible prices are offered at intermediate dates, i.e., $\rho_t^H > 0$, $\rho_t^L > 0$, and $1 - \rho_t^H - \rho_t^L > 0$ for $t \in \{2, ..., T-1\}$.

(2.3) Only high and low prices are offered at the last date, i.e., $\rho_T^H > 0$, $\rho_T^L > 0$, and $1 - \rho_T^H - \rho_T^L = 0$.

Moreover, if $\delta^{T-1}\alpha(c^H - c^L) > u^L - c^L$, this is the unique DE.

Proposition 2 describes the trading patterns that arise in equilibrium: At the first date some matched low quality sellers trade and no high quality sellers trade. At the intermediate dates, some matched sellers of both types trade. At the last date all matched low quality sellers and some matched high quality sellers trade. This requires that some buyers make negligible price offers, i.e., offers which they know will be rejected, at every date except the last. And at every date but the first, there are transactions at different prices, since buyers offer with positive probability both $r_t^H = c^H$ and $r_t^L < c^H$.

Realizing that several different price offers must be made at each date is key to understanding the structure of equilibrium when frictions are small:

Suppose, for example, that all buyers made negligible offers at date t, i.e., $1 - \rho_t^H - \rho_t^L = 1$. Let t' be the first date following t where a buyer makes a non-negligible price offer. Since there is no trade between t' and t, then the distribution of qualities is the same at t' and t, i.e., $q_t^H = q_{t'}^H$. Thus, an impatient buyer is better off by offering at date t the price she offers at t', which implies that negligible prices are suboptimal at t. Hence $1 - \rho_t^H - \rho_t^L < 1$.

Suppose instead that all buyers offer the high price $r_t^H = c^H$ at some date t, i.e., $\rho_t^H = 1$. Then the reservation price of low-quality sellers will be near c^H, and above u^L, prior to t. Hence a low price offer (which if accepted buys a unit of low quality, whose value is u^L) is suboptimal prior to t. But if only high and negligible offers are made prior to t, then $q_t^H = q^H$, and a high price offer is suboptimal at t since $q_t^H < \bar{q}$. Hence $\rho_t^H < 1$.

Finally, suppose that all buyers offer the low price r_t^L at some date $t < T$, i.e., $\rho_t^L = 1$. Then all matched low quality sellers trade, and hence α near one implies $q_{t+1}^H > q^*$, and therefore $q_T^H > q^*$. But $q_T^H > q^*$ implies that $r_T^H = c^H$ is the only optimal price offer at date T, which contradicts that $\rho_T^H < 1$. Hence $\rho_t^L < 1$.

Since the expected utility of a random unit supplied at date 1 is less than c^H by assumption, then high price offers are suboptimal at date 1; i.e., $\rho_1^H = 0$. At date T the sellers' reservation prices are equal to their costs. Hence a buyer obtains a positive payoff by offering the low price. Since a buyer who does not trade obtains zero, then negligible price offers are suboptimal at date T, i.e., $\rho_T^H + \rho_T^L = 1$.

More involved arguments establish that all three types of price offers – high, low, and negligible – are made in every date except the first and last (i.e., $\rho_t^H > 0$, $\rho_t^L > 0$, and $1 - \rho_t^H - \rho_t^L > 0$ for $t \in \{2, \ldots, T-1\}$). Identifying the probabilities of the different price offers is delicate: Their past values determine the current market composition, and their future values determine the sellers' reservation prices. In equilibrium, the market composition and the sellers' reservation prices make buyers indifferent between all three price offers at each intermediate date. In the proof of Lemma 3 in the Appendix we derive closed form expressions for these probabilities.

Proposition 3 below shows that the surplus generated by a decentralized market equilibrium is greater than the (static) competitive equilibrium surplus. Of course, an implication of adverse selection in our setting is that the competitive equilibrium is inefficient since only low quality units trade. Units of both qualities trade in the DE, although with delay. The loss that results from delay in trading low quality units is more than offset by the gains realized from trade of high quality units. (In the next section we study the outcomes of dynamic competitive equilibria.)

Proposition 3. *In the equilibrium described in Proposition 2 the traders' payoffs are* $V_1^H = 0$,

$$V_1^L = \left(1 - \delta^{T-1}\alpha\left(1-\hat{q}\right)\right)\left(u^L - c^L\right),$$

and

$$V_1^B = \delta^{T-1}\alpha\left(1-\hat{q}\right)\left(u^L - c^L\right),$$

and the surplus is

$$S^{DE} = \left[q^L + \delta^{T-1}\alpha q^H\left(1-\hat{q}\right)\right]\left(u^L - c^L\right) > \bar{S}.$$

Thus, the payoff to buyers (low quality sellers) is above (below) their competitive payoff, and decreases (increases) with T and increases (decreases) with δ and α. Also, the surplus is above the competitive surplus, and decreases with T and increases with δ and α.

The comparative statics for buyer payoff are intuitive: In equilibrium negligible price offers are optimal for buyers at every date except the last. In other words, only at the last date does a buyer capture any gains to trade. Hence buyer payoff is increasing in α. Also decreasing T or increasing in δ reduces delay costs and therefore increases buyer payoff. Low quality sellers capture surplus whenever high price offers are made, i.e., at every date except the first. The probability of a high price offer decreases with both α and δ, and thus their surplus also decreases.

Surplus is increasing in δ and α, and it is decreasing in T. Thus shortening the horizon over which the market operates is advantageous. Indeed, surplus is maximized when $T = 2$.

Proposition 4 below identifies the probabilities of high, low, and negligible price offers as frictions vanish. A remarkable feature of equilibrium is that at every intermediate date all price offers are negligible; that is, all trade concentrates at the first and last date. Thus, the market freezes, and *both* qualities become completely illiquid. And since the market is active for only two dates (the first and the last), not surprisingly the equilibrium is independent of T (so long as it is at least two and finite).

Proposition 4. *If* $T > 1$, *as* α *and* δ *approach one the probabilities of price offers approach* $(\tilde{\rho}^H, \tilde{\rho}^L)$ *given by*

(4.1) $\tilde{\rho}_1^H = 0 < \tilde{\rho}_1^L = \dfrac{\hat{q} - q^H}{\hat{q} - \hat{q}q^H} < 1.$

(4.2) $\tilde{\rho}_1^H = \tilde{\rho}_t^L = \tilde{\rho}_t^H = 0 \text{ for } 1 < t < T.$

(4.3) $\tilde{\rho}_T^H = \dfrac{\hat{q}(u^L - c^L)}{c^H - c^L} > 0$, *and* $\tilde{\rho}_T^L = 1 - \tilde{\rho}_T^H > 0.$

Hence trade concentrates in the first and last dates. Moreover, the payoff to buyers remains above their competitive payoff and approaches

$$\tilde{V}^B = (1 - \hat{q}) \left(u^L - c^L\right),$$

the payoff to low quality sellers remains below their competitive payoff and approaches

$$\tilde{V}^L = \hat{q} \left(u^L - c^L\right),$$

and the surplus remains above the competitive surplus and approaches

$$\tilde{S}^{DE} = \left[q^L + q^H (1 - \hat{q})\right] \left(u^L - c^L\right),$$

independently of T.

We consider now decentralized markets that open for infinitely many periods, i.e., such that $T = \infty$. In these markets, given a strategy distribution (λ, r^H, r^L) one calculates the sequence of traders' expected utilities by solving a dynamic optimization problem. The definition of decentralized market equilibrium, however, remains the same.

Proposition 5 identifies a DE when frictions are small. This equilibrium is the limit, as T approaches infinity, of the equilibrium described in Proposition 2. Although there are multiple equilibria when $T = \infty$,[4] this limiting equilibrium is a natural selection since for *every* finite T the DE identified in Proposition 2 is the unique equilibrium for sufficiently large α and δ.

Proposition 5. *If $T = \infty$ and frictions are small, then the limit of the sequence of the DE identified in Proposition 2, which is given by*

(5.1) $r_t^H = c^H$, $r_t^L = u^L$ *for all* t,

(5.2) $\rho_1^H = 0$, $\rho_1^L = \dfrac{\bar{q} - q^H}{\alpha\,(1 - q^H)\,\bar{q}}$, *and*

(5.3) $\rho_t^L = 0$, $\rho_t^H = \dfrac{1-\delta}{\delta\alpha}\dfrac{u^L - c^L}{c^H - u^L}$ *for* $t > 1$,

is a DE. In this equilibrium the traders' payoffs are $V_1^B = V_1^H = 0$ *and* $V_1^L = u^L - c^L$, *and the surplus is*

$$S_\infty^{DE} = q^L(u^L - c^L) = \bar{S},$$

independently of the values of α *and* δ.

In equilibrium all units trade eventually. At the first date only some matched low quality seller trade. At subsequent dates, matched sellers of both types trade with the same constant probability. The traders' payoffs are competitive independently of α and δ and hence so is the surplus. This holds even if frictions are non-negligible, provided they are sufficiently small.[5]

The examples in Table 1 illustrate our results for a market with $u^H = 1$, $c^H = .6$, $u^L = .4$, $c^L = .2$, and $q^L = .2$. It is easy to verify that frictions are small (i.e., both $SF.1$ and $SF.2$ hold) for these examples, although the sufficient condition in Proposition 2 for uniqueness does not hold when $\delta = \alpha = .9$. When the market is of finite duration (e.g., $T = 10$) buyers make low and negligible offers at the market open,

[4] For example, there are DE similar to the one identified in Proposition 5, except that there in no trade in a single period.

[5] In a market with stationary entry, Moreno and Wooders (2010)'s show that the surplus are competitive as frictions vanish, but are above the competitive surplus when frictions are non-negligible. In a continuous time version of the same model, Kim (2011) finds the surplus to be competitive even if frictions are non-negligible.

they make high and low price offers at the market close, and they primarily make negligible price offers at intermediate dates. As frictions vanish, the market freezes at intermediate dates as all price offers are negligible. The surplus realized with decentralized trading exceeds the static competitive surplus (of $q^L(u^L - c^L) = .16$), and it is does so even in the limit as frictions vanish. Even as frictions vanish, not all units trade.

In contrast, when the market is open indefinitely, then decentralized trading yields exactly the competitive surplus *independently of the magnitude of frictions*, so long as they are small. At the market open, only low and negligible price offers are made; at every subsequent date only high and negligible price offers are made, although most offers are negligible. As δ approaches one, the probability of a high price offer approaches zero (the market freezes). However, so long as δ is less then one, the probability of trading at each date is positive and constant, and thus all units trade eventually.[6]

[6] When $T = \infty$ and δ approaches one, then the expected delay for a high quality seller before he trades approaches infinity. Nonetheless, the gains to trade realized by trading high quality units is asymptotically positive since the players are becoming perfectly patient.

	$T=10$				$T=\infty$			
	$\delta=\alpha=.9$		$\delta=\alpha=1$		$\delta=\alpha=.9$		$\delta=\alpha=.99$	
t	ρ_t^H	ρ_t^L	ρ_t^H	ρ_t^L	ρ_t^H	ρ_t^L	ρ_t^H	ρ_t^L
1	0.0000	0.6908	0.0000	0.7500	0.0000	0.5556	0.0000	0.5051
2	0.1034	0.0296	$\vdots$	0.0000	0.1235	0.0000	0.0102	0.0000
3	0.1016	0.0327	$\vdots$	$\vdots$	0.1235	0.0000	0.0102	0.0000
4	0.0996	0.0362	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
5	0.0975	0.0400	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
6	0.0953	0.0442	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
7	0.0930	0.0488	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
8	0.0905	0.0540	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	0.0879	0.1219	0.0000	0.0000	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	0.3673	0.6327	0.2500	0.7500	$\vdots$	$\vdots$	$\vdots$	$\vdots$
V^L	0.1651		0.1000		0.2000		0.2000	
V^B	0.0349		0.1000		0.0000		0.0000	
S^{DE}	0.1670		0.1800		0.1600		0.1600	

Figure 2 below shows the evolution of the market composition for several different specifications of market frictions when $T = 10$. It illustrates several features of equilibrium: (i) high quality trades more slowly as frictions are smaller, (ii) low quality initially trades more slowly as frictions are smaller, but the total measure of high quality that trades by the market close are larger as frictions are smaller, (iii) the fraction of sellers in the market of high quality increases more quickly when frictions are smaller but equals .5 at the market close, regardless of the level of frictions.

Figure 2 goes here.

5 A Dynamic Competitive Market for Lemons

In this section we consider a competitive market that opens for T consecutive periods, and as in Section 3 we assume that agents discount utility at a common rate $\delta \in (0, 1]$.

The supply and demand schedules are defined as follows. Let $p = (p_1, ..., p_T) \in \mathbb{R}^T_+$ be a sequence of prices. The gains to trade to sellers of quality $\tau \in \{H, L\}$ is

$$v^\tau(p) = \max_{t \in \{1,...,T\}} \{0, \delta^{t-1}(p_t - c^\tau)\},$$

where $\delta^{t-1}(p_t - c^\tau)$ is the gain to trade to a τ-quality seller who supplies at t, and zero is the utility of not trading. The *supply of τ-quality good*, $S^\tau(p)$, is the set of sequences $s^\tau = (s^\tau_1, ..., s^\tau_T) \in \mathbb{R}^T_+$ satisfying:

$(S.1)$ $\sum_{t=1}^T s^\tau_t \leq q^\tau$,

$(S.2)$ $s^\tau_t > 0$ implies $\delta^{t-1}(p_t - c^\tau) = v^\tau(p)$, and

$(S.3)$ $\left(\sum_{t=1}^T s^\tau_t - q^\tau\right) v^\tau(p) = 0$.

Condition $S.1$ requires that no more of good τ than is available, q^τ, is supplied. Condition $S.2$ requires that supply is positive only in periods where it is optimal to supply. Condition $S.3$ requires that supply be equal to the total amount available, q^τ, if the gain to trade for a τ-quality seller is positive (i.e., when $v^\tau(p) > 0$).

Denote by $u_t \in [u^L, u^H]$ the expected value to buyers of a unit drawn at random from those supplied at date t. If $u = (u_1, ..., u_T)$ is a sequence of buyers' expected values, then the gains to trade to a buyer is

$$v^B(p, u) = \max_{t \in \{1,...,T\}} \{0, \delta^{t-1}(u_t - p_t)\},$$

where $\delta^{t-1}(u_t - p_t)$ is the gain to trade to a buyer who demands a unit of the good at t, and zero is the utility to not trading. The *market demand*, $D(p, u)$, is the set of sequences $d = (d_1, ..., d_T) \in \mathbb{R}^T_+$ satisfying:

$(D.1)$ $\sum_{t=1}^T d_t \leq 1$,

$(D.2)$ $d_t > 0$ implies $\delta^{t-1}(u_t - p_t) = v^B(p, u)$, and

$(D.3)$ $\left(\sum_{t=1}^T d_t - 1\right) v^B(p, u) = 0$.

Condition $D.1$ requires that the total demand of good does not exceed the measure of buyers, which we normalized to one. Condition $D.2$ requires that demand be positive only at dates where buying is optimal. Condition $D.3$ requires that demand be equal to the measure of buyers when buyers have positive gains to trade (i.e., when $v^B(p, u) > 0$).

With this notation in hand we introduce a notion of dynamic competitive equilibrium along the lines in the literature – see e.g., Wooders (1998), Janssen and Roy (2004), and Moreno and Wooders (2001).

A *dynamic competitive equilibrium (CE)* is a profile (p, u, s^H, s^L, d) such that $s^H \in S^H(p)$, $s^L \in S^L(p)$, and $d \in D(p, u)$, and for each $t \in \{1, ..., T\}$:

$(CE.1)$ $s_t^H + s_t^L = d_t$, and

$(CE.2)$ $s_t^H + s_t^L = d_t > 0$ implies $u_t = \dfrac{u^H s_t^H + u^L s_t^L}{s_t^H + s_t^L}$.

Condition $CE.1$ requires that the market clear at each date, and condition $CE.2$ requires that the expectations described by the vector u are correct whenever there is trade. For a market that opens for a single date (i.e., if $T = 1$), our definition reduces to Akerlof's.

The surplus generated in a CE, (p, u, s^H, s^L, d), may be calculated as

$$S^{CE} = \sum_{\tau \in \{H,L\}} \sum_{t=1}^{T} s_t^\tau \delta^{t-1}(u^\tau - c^\tau). \tag{3}$$

As our next proposition shows, there a CE where all low quality units trade at date 1 at the price u^L, and none of the high quality units trade. Every CE has these properties if traders are sufficiently patient.

Proposition 6. *There is a CE in which all low quality units trade immediately at the price u^L and none of the high quality units trade, e.g., (p, u, s^H, s^L, d) given by $p_t = u_t = u^L$ for all t, $s_1^L = d_1 = q^L$, and $s_1^H = s_t^H = s_t^L = d_t = 0$ for $t > 1$ is a CE. In these equilibria the traders' gains to trade to low quality sellers is $u^L - c^L$, the gains to trade to high quality sellers and buyers is zero, and the surplus is*

$$S^{CE} = q^L(u^L - c^L) = \bar{S}.$$

Moreover, if $\delta^{T-1}(c^H - c^L) > u^L - c^L$, then every CE has these properties.

The intuition for why high quality does not trade when traders are patient is clear: If high quality were to trade at $t \leq T$, then p_t must be at least c^H. Hence the gains to trade to low quality sellers is at least $\delta^{T-1}(c^H - c^L) > u^L - c^L > 0$, and therefore all low quality sellers trade at prices greater than u^L. But at a price $p \in (u^L, c^H)$

only low quality sellers supply, and therefore the demand is zero. Hence all trade is at prices of at least c^H. Since $u(q^H) < c^H$ by assumption, and all low quality is supplied, there must be a date at which there is trade and the expected utility of a random unit supplied is below c^H. This contradicts that there is demand at such a date. Given that there is not trade of high quality, the low quality sellers are the short side of the market and therefore capture the entire surplus, i.e., the price is u^L.

Proposition 7 below establishes that if traders are sufficiently impatient, then there are dynamic competitive equilibria where high quality trades. Thus, the market eventually recovers from adverse selection, e.g., in long-lived competitive markets the adverse selection problem is less severe.

Proposition 7. *If $\delta^{T-1}\left(u^H - c^L\right) \leq u^L - c^L$, then there are CE in which all units trade.*

The inequality of Proposition 7 holds for any discount factor when the market remains open for infinitely many periods. In this case, there are dynamic competitive equilibria where all qualities trade. Our constructions in the proof of Proposition 7 suggest the high quality may have to trade with an increasingly long delay as the discount factor approaches one. Thus, the question arises whether the surplus realized from trading high quality units is positive, and how large it is, as δ approaches one. Proposition 8 provides an answer to these questions.

Proposition 8. *If $T = \infty$, then as δ approaches one the maximum surplus that can be realized in a CE, $\tilde{S}^{CE}$, is at least the surplus that can be realized in a DE, and is greater that the competitive surplus, i.e.,*

$$\tilde{S}^{CE} \geq \tilde{S}^{DE} > \bar{S}.$$

Even though high quality units trade with an increasingly long delay as δ approaches one, there are competitive equilibria that realize a surplus above the static competitive surplus $\bar{S}$. Interestingly, as frictions vanish a market that opens for an infinite number of periods has dynamic competitive equilibria that generate the same surplus as that of a decentralized market that opens for finitely many periods. In contrast, the CE of a market that opens for finitely many periods generates the static competitive surplus for discount factors sufficiently close to one.

6 Discussion

As propositions 1 to 7 show, the performance of dynamic market for lemons differs depending on the horizon over which they remain open and on the market infrastructure. When friction are small, a decentralized market that operates over a finite horizon is able to recover partially from adverse selection: some high quality units and most low quality units trade, and the surplus is above the static competitive surplus. As friction vanish some high quality units continue to trade, and all low quality units trade, although some of these units trade with delay. Interestingly, trade tends to concentrate in the first and last date, and the traders payoffs and surplus does not depend on the market duration; i.e., the surplus and payoffs are the same whether the market opens for just two periods, or a large but finite number of periods, as in the intermediate periods buyers make negligible price offers; the waiting time is necessary for low quality sellers to have a reservation price sufficiently low.

Dynamic competitive (centralized) markets that open for a finite number of periods do not perform well when frictions are small as in equilibrium only low quality trades – the equilibrium outcomes of these markets are the same as those of a static competitive market. Dynamic competitive markets that open for an infinite number of periods, however, have more efficient equilibria where all low quality units and some high quality units trade, and the surplus is above the static competitive surplus.

It is remarkable that the surplus realized in the most efficient dynamic competitive equilibrium of a market that open for infinitely many periods is the same as that generated in a decentralized market that open for finitely many periods. Thus, as friction vanish (i.e., as α and δ approach one) an infinitely (finitely) lived centralized markets generates the same surplus as a finitely (infinitely) lived decentralized markets. Table 2 below summarizes these results.

$\tilde{S}$	$\tilde{S}^{DE}$	$\tilde{S}^{CE}$
$T < \infty$	$\left[q^L + q^H(1-\hat{q})\right](u^L - c^L)$	$q^L(u^L - c^L)$
$T = \infty$	$q^L(u^L - c^L)$	$\left[q^L + q^H(1-\hat{q})\right](u^L - c^L)$

Table 2: Surplus as friction vanish.

It is worth noting that neither a decentralized market, nor a dynamic competitive market is able to yield the surplus that may be realized by a (static) *efficient* *mechanism* (i.e., a mechanism that maximizes the surplus over all incentive compatible and individually rational mechanisms). In our context, a mechanism is defined by a collection $[(p^H, z^H), (p^L, z^L)]$, specifying for each quality report $\tau \in \{H, L\}$ a money transfer from the buyer to the seller, $p^\tau \in \mathbb{R}_+$, and a probability that the seller transfers the good to the buyer, $z^\tau \in [0, 1]$.[7]

An efficient mechanism is a solution to the problem

$$\max_{(p,z)\in\mathbb{R}^2_+\times[0,1]^2} q^H z^H (u^H - c^H) + q^L z^L (u^L - c^L)$$

subject to

$$p^\tau - z^\tau c^\tau \geq p^\sigma - z^\sigma c^\tau \text{ for each } \tau, \sigma \in \{H, L\}, \tag{$IC.\tau$}$$

$$q^H z^H u^H + q^L z^L u^L - (q^H p^H + q^L p^L) \geq 0, \tag{$IR.B$}$$

$$p^\tau - z^\tau c^\tau \geq 0 \text{ for each } \tau \in \{H, L\}. \tag{$IR.\tau$}$$

The constraint $IC.\tau$ guarantees that the mechanism is incentive compatible, i.e., it is optimal for a type τ seller to report his type truthfully. The constraints $IR.B$ and $IR.\tau$ guarantee that participating in the mechanism is individually rational for buyers and sellers; i.e., that no trader obtains a negative expected payoff.

It is straightforward to show that the efficient mechanism satisfies $z^L = 1 > z^H = q^L(u^L - c^L)/\left(c^H - c^L - q^H(u^H - c^L)\right)$, and generates a surplus of

$$S^* = q^L(u^L - c^L) + \frac{q^H(u^H - c^H)}{c^H - q^H u^H - (1 - q^H)c^L} q^L(u^L - c^L).$$

Obviously, $S^* > q^L(u^L - c^L)$, since $q^H u^H + (1 - q^H)c^L < u(q^H) < c^H$ by assumption.

By Proposition 2 the surplus in a decentralized market increases with α and δ. Hence using the limiting surplus provided in Proposition 3 we have

$$\begin{aligned} S^* - S^{DE} &> S^* - \tilde{S}^{DE} \\ &= \frac{q^H(u^H - c^H)^2(u^L - c^L)}{(u^H - c^L)\left(c^H - (1 - q^H)c^L - q^H u^H\right)} \\ &> 0. \end{aligned}$$

[7] By the Revelation Principle, we can restrict attention to "direct" mechanisms. Also note that there is no need for buyers to report to the mechanism since they have no private information.

Hence, a decentralized market is not able to generate the surplus of a static efficient mechanism.[8]

As for the relation between the surplus generated in a long lived competitive market and that a the static efficient mechanism, we have

$$S^{*}-S^{CE}>S^{*}-\tilde{S}^{CE}=S^{*}-\tilde{S}^{DE}>0.$$

Figure 2 below provides graphs of the mappings S^*, S and $\bar{S}$????

Discuss: For markets for lemons with stationary entry, Janssen and Roy (2002 and 2000?) ...have shown that the only stationary dynamic competitive equilibrium is the repetition of the static competitive equilibrium.[9] Thus, time alone does not explain the difference in surplus realized under centralized and decentralized trade.

Discuss: Camargo and Lester

7 Appendix: Proofs

We begin by establishing a number of lemmas.

Lemma 1. *Assume that $T>1$, and let (λ, r^H, r^L) be a DE. Then for each $t \in \{1,...,T\}$:*

$(L1.1)$ $\lambda_t(\max\{r_t^H, r_t^L\})=1$.

$(L1.2)$ $q_t^{\tau}>0$ for $\tau \in \{H, L\}$.

$(L1.3)$ $r_t^H=c^H>r_t^L$, $V_t^B>0=V_t^H$, *and* $V_t^L \leq \alpha(c^H-c^L)$.

$(L1.4)$ $q_{t+1}^H \geq q_t^H$.

$(L1.5)$ $\lambda_t(c^H)=1$.

$(L1.6)$ $\lambda_t(p)=\lambda_t(r_t^L)$ *for all* $p \in [r_t^L, c^H)$.

$(L1.7)$ $\lambda_T^L=1$.

[8] Gale (1996) studies the properties of the competitive equilibria of markets with adverse selection where agents exchange contracts specifying a price and a probability of trade, and shows that even with a complete contract structure, equilibria are not typically incentive-efficient.

[9] They also find non-stationary equilibria, however, where all qualities trade although with delay. The authors do not evaluate the surplus realized at these equilibria – they focus on the issue of price volatility.

($L1.8$) *If* $\lambda_t^L = \lambda_t^H$, *then* $q_{t+1}^{\tau} = q_{t+1}^{\tau}$ for $\tau \in \{H, L\}$.

Proof: Let $t \in \{1, ..., T\}$.

We prove $L1.1$. Write $\bar{p} = \max\{r_t^H, r_t^L\}$, and suppose that $\lambda_t(\bar{p}) < 1$. Then there is $\hat{p} > \bar{p}$ in the support of λ_t. Since $I(\bar{p}, r_t^{\tau}) = I(\hat{p}, r_t^{\tau}) = 1$ for $\tau \in \{H, L\}$, we have

$$\begin{aligned} V_t^B &\geq \alpha \sum_{\tau \in \{H,L\}} q_t^{\tau} I(\bar{p}, r_t^{\tau})(u^{\tau} - \hat{p}) + \left[1 - \alpha \sum_{\tau \in \{H,L\}} q_t^{\tau} I(\bar{p}, r_t^{\tau})\right] \delta V_{t+1}^B \\ &= \alpha \sum_{\tau \in \{H,L\}} q_t^{\tau} \left(u^{\tau} - \bar{p}\right) + (1-\alpha)\, \delta V_{t+1}^B \\ &> \alpha \sum_{\tau \in \{H,L\}} q_t^{\tau} \left(u^{\tau} - \hat{p}\right) + (1-\alpha)\, \delta V_{t+1}^B \\ &= \alpha \sum_{\tau \in \{H,L\}} q_t^{\tau} I(\hat{p}, r_t^{\tau})(u^{\tau} - \hat{p}) + \left[1 - \alpha \sum_{\tau \in \{H,L\}} q_t^{\tau} I(\hat{p}, r_t^{\tau})\right] \delta V_{t+1}^B, \end{aligned}$$

which contradicts $DE.B$.

We prove $L1.2$ by induction: Let $\tau \in \{H, L\}$. We have $q_1^{\tau} = q^{\tau} > 0$. Assume that $q_k^{\tau} > 0$ for some $k \geq 1$; $q_{k+1}^{\tau} > 0$. Since $\alpha \in (0, 1)$, we have $(1 - \alpha\lambda_k^{\tau})q_k^{\tau} > 0$. Hence

$$q_{k+1}^{\tau} = \frac{(1 - \alpha\lambda_k^{\tau})q_k^{\tau}}{q_k^L + q_k^L} > 0.$$

We prove $L1.3$ by induction. Because $V_{T+1}^{\tau} = 0$ for $\tau \in \{B, H, L\}$, then $DE.H$ and $DE.L$ imply

$$r_T^H = c^H + \delta V_{T+1}^H = c^H > c^L = r_T^L = c^L + \delta V_{T+1}^L.$$

Hence $\lambda_T(c^H) = 1$ by $L1.1$, and therefore $V_T^H = 0$ and $V_T^L \leq \alpha(c^H - c^L)$. Moreover, $0 < q_T^L \left(u^L - c^L\right) \leq V_T^B$. Assume that $L1.3$ holds for $k \leq T$; we show that it holds for $k - 1$. Since $V_k^H = 0$, $DE.H$ implies $r_{k-1}^H = c^H + \delta V_k^H = c^H$. Since $V_k^L = \alpha(c^H - c^L)$, then $DE.L$ implies $r_{k-1}^L = c^L + \delta V_k^L \leq (1 - \delta\alpha)c^L + \delta\alpha c^H < c^H$. Hence $\lambda_k(c^H) = 1$ by $L1.1$, and therefore $V_{k-1}^H = 0$ and $V_{k-1}^L \leq \alpha(c^H - c^L)$. Also $V_{k+1}^B \geq \delta V_k^B > 0$.

In order to prove $L1.4$, note that $L1.2$ implies $\lambda_t^H \leq \lambda_t^L$. Hence

$$q_{t+1}^H = \frac{\left(1 - \alpha\lambda_t^H\right) q_t^H}{\left(1 - \alpha\lambda_t^H\right) q_t^H + \left(1 - \alpha\lambda_t^L\right) q_t^L} \geq q_t^H.$$

As for $L1.5$, it is a direct implication of $L1.1$ and $L1.2$.

We prove $L1.6$. Suppose that $\lambda_t(p) > \lambda_t(r_t^L)$ for some $p \in (r_t^L, r_t^H)$. Then there is $\hat{p}$ in the support of λ_t such that $r_t^L < \hat{p} < r_t^H$. Since $I(\hat{p}, r_t^L) = 1$ and $I(\hat{p}, r_t^H) = 0$, we have

$$\begin{aligned} V_t^B &\geq \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(r_t^L, r_t^\tau)(u^\tau - r_t^L) + \left[1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(r_t^L, r_t^\tau)\right] \delta V_{t+1}^B \\ &= \alpha q_t^L \left(u^L - r_t^L\right) + \left(1 - \alpha q_t^L\right) \delta V_{t+1}^B \\ &> \alpha q_t^L \left(u^L - \hat{p}\right) + \left(1 - \alpha q_t^L\right) \delta V_{t+1}^B \\ &= \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\hat{p}, r_t^\tau)(u^\tau - \hat{p}) + \left[1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\hat{p}, r_t^\tau)\right] \delta V_{t+1}^B, \end{aligned}$$

which contradicts $DE.B$.

We prove $\lambda_T^L = 1$. Suppose by way of contradiction that $\lambda_T^L < 1$. Then there is $\hat{p} < c^L$ in the support of λ_T. Since $I(\hat{p}, r_t^H) = 0$, and $V_{T+1}^B = 0$, we have $V_T(\hat{p}) = 0$. However, $V_T(c^L) = q_T^L \left(u^L - c^L\right) > 0$ by $L1.3$, which contradicts $DE.B$.

We prove $L1.8$. We have Hence $\lambda_t^L = \lambda_t^H$ implies

$$q_{t+1}^\tau = \frac{\left(1 - \alpha \lambda_t^\tau\right) q_t^\tau}{\left(1 - \alpha \lambda_t^H\right) q_t^H + \left(1 - \alpha \lambda_t^L\right) q_t^L} = \frac{q_t^\tau}{q_t^H + q_t^L} = q_t^\tau. \ \square$$

Proof of Proposition 1. Follows from lemmas $L1.3$, $L1.5$ and $L1.6$ above. $\square$

As argued above, $L1.5$ and $L1.6$ imply that in a market equilibrium the only price offers that may be made with positive probability each date t are c^H, r_t^L, and prices below r_t^L. Therefore the distribution of transaction prices is determined by the probabilities of offering these prices, given by $\rho_t^H = \lambda_t^H$, $\rho_t^L = \lambda_t^L - \lambda_t^H$, and $1 - \rho_t^H - \rho_t^L$, respectively. Lemma 2 establishes some properties that these probabilities have in a DE.

Lemma 2. *Assume that $T > 1$, and let $(\rho^H, \rho^L, r^H, r^L)$ be a DE. Then:*

($L2.1$) $\rho_T^H + \rho_T^L = 1$.

($L2.2$) $\rho_t^H + \rho_t^L > 0$ *for each* $t \in \{1, ..., T\}$.

($L2.3$) $\rho_1^H = 0 < \rho_1^L$.

Proof: Since $\rho_T^H + \rho_T^L = \lambda_T^L$, then $L2.1$ follows from $L1.7$.

We proof $L2.2$. Since $\rho_T^H + \rho_T^L = 1 > 0$ by $L2.1$, let $k < T$ be the largest date such that $\rho_k^H + \rho_k^L = 0$ and $\rho_{k+1}^H + \rho_{k+1}^L > 0$. Then $q_{k+1}^\tau = q_k^\tau$ for $\tau \in \{H, L\}$. If $\rho_{k+1}^H > 0$, then offering r_{k+1}^H is optimal, i.e.,

$$V_{k+1}^B = \alpha(q_{k+1}^H u^H + q_{k+1}^L u^L - c^H) + (1-\alpha)\,\delta V_{k+2}^B.$$

Moreover, we have

$$q_{k+1}^H u^H + q_{k+1}^L u^L - c^H \geq \delta V_{k+2}^B,$$

for otherwise the payoff to offering a price less than r_{k+1}^L dominates offering of c^H. Hence

$$V_{k+1}^B \leq q_{k+1}^H u^H + q_{k+1}^L u^L - c^H.$$

But then

$$q_k^H u^H + q_k^L u^L - c^H = q_{k+1}^H u^H + q_{k+1}^L u^L - c^H \geq V_{k+1}^B > \delta V_{k+1}^B$$

and therefore making a negligible price offer at k is not optimal, contrary to the assumption that $\rho_k^H + \rho_k^L = 0$ (i.e., that all buyers' price offers are rejected). Hence $\rho_{k+1}^H = 0$, and thus $\rho_{k+1}^L > 0$. Since $V_k^L \geq 0$, then $r_{k+1}^L \geq r_k^L$. The payoff to offering r_k^L at period k is

$$q_k^H \delta V_{k+1}^B + q_k^L (u^L - r_k^L) \leq \delta V_{k+1}^B.$$

where the inequality follows since negligible price offers are optimal at date k. Since $1 - q_k^H = q_k^L$, then

$$u^L - r_k^L \leq \delta V_{k+1}^B.$$

Now since $\rho_{k+1}^L > 0$, i.e., price offers of r_{k+1}^L are optimal at date $k+1$, we have

$$q_{k+1}^L (u^L - r_{k+1}^L) + q_{k+1}^H \delta V_{k+2}^B \geq \delta V_{k+2}^B.$$

Hence

$$\delta V_{k+2}^B \leq u^L - r_{k+1}^L,$$

Also

$$V_{k+1}^B = \alpha q_{k+1}^L (u^L - r_{k+1}^L) + \left(1 - \alpha q_{k+1}^L\right) \delta V_{k+2}^B \leq u^L - r_{k+1}^L.$$

Summing up

$$u^L - r_k^L \leq \delta V_{k+1}^B < V_{k+1}^B \leq u^L - r_{k+1}^L;$$

i.e., $r^L_{k+1} < r^L_k$, which is a contradiction.

We prove $L2.3$. Since $q^H_1 = q^H < \bar{q}$ by assumption, we have

$$\begin{aligned} q^H_1 u^H + q^L_1 u^L - c^H &< 0 \\ (\text{by } L1.3) &< \delta V^B_2. \end{aligned}$$

Hence offering c^H is not optimal; i.e., $\rho^H_1 = 0$. Therefore $\rho^L_1 > 0$ by $L1.2.\square$

Lemma 3. *If $T > 1$ and frictions are small, then the properties (2.1), (2.2) and (2.3) of Proposition 2 uniquely determine a DE. In this equilibrium the payoffs and surplus are those given in Proposition 3.*

Proof. Properties (2.1), (2.2) and (2.3) together with the equilibrium conditions provide a system of equations that DE must satisfy. We show that this system has a unique solution, which we calculate. This solution provides the strategy distribution, $(\rho^H, \rho^L, r^H, r^L)$, as well as the sequences of traders' expected utilities, and the sequences of stocks and fractions of sellers of each type. We then calculate the surplus.

Since $\rho^H_T > 0$ and $\rho^L_T > 0$, then

$$(1 - q^H_T)(u^L - c^L) = q^H_T u^H + (1 - q^H_T)u^L - c^H.$$

Hence $q^H_T = \hat{q}$, and the buyers' expected utility at T is

$$V^B_T = \alpha(1 - \hat{q})(u^L - c^L).$$

Since $1 - \rho^H_t - \rho^L_t > 0$ for all $t < T$ by (2.2), then $V^B_t = \delta V^B_{t+1}$ for $t < T$, and therefore

$$V^B_t = \delta^{T-1} \alpha \left(1 - \hat{q}\right) \left(u^L - c^L\right) \tag{4}$$

for all t.

Since $\rho^H_t > 0$ and $\rho^L_t > 0$ for $1 < t < T$ by (2.2), then

$$q^H_t \left(u^H - c^H\right) + (1 - q^H_t) \left(u^L - c^H\right) = \delta V^B_{t+1}.$$

Hence

$$q^H_t = \frac{c^H - u^L + \delta^{T-t} \alpha(1 - \hat{q})(u^L - c^L)}{u^H - u^L}, \tag{5}$$

for $1 < t < T$, and $q_T^H = \hat{q}$ by $L4.3$.

Since $\rho_t^L > 0$ and $1 - \rho_t^H - \rho_t^L > 0$ for $t < T$ by (2.2), then

$$\alpha q_t^L \left(u^L - r_t^L\right) + (1 - \alpha q_t^L)\delta V_{t+1}^B = \delta V_{t+1}^B,$$

i.e.,

$$\delta V_{t+1}^B = u^L - r_t^L.$$

Hence for $t < T$ we have

$$r_t^L = u^L - \delta^{T-t}\alpha(1 - \hat{q})(u^L - c^L), \tag{6}$$

and $r_T^L = c^L$.

Since $r_t^L - c^L = \delta V_{t+1}^L$ for all t by $DE.L$, then

$$u^L - c^L - \delta^{T-t}\alpha(1 - \hat{q})(u^L - c^L) = \delta V_{t+1}^L.$$

Reindexing we get

$$V_t^L = \frac{u^L - c^L}{\delta} - \delta^{T-t}\alpha(1 - \hat{q})(u^L - c^L), \tag{7}$$

for $t > 1$. And since $\rho_1^H = 0$ by (2.1), then

$$V_1^L = \delta V_2^L = \left(1 - \delta^{T-1}\alpha\left(1 - \hat{q}\right)\right)\left(u^L - c^L\right). \tag{8}$$

Again since $r_t^L - c^L = \delta V_{t+1}^L$ for all t, then the expected utility of a low-quality seller is

$$V_t^L = \alpha\rho_t^H(c^H - c^L) + (1 - \alpha\rho_t^H)\delta V_{t+1}^L,$$

i.e.,

$$V_t^L - \delta V_{t+1}^L = \alpha\rho_t^H(c^H - c^L - \delta V_{t+1}^L).$$

Using equation (7), then for $1 < t < T$ we have

$$V_t^L - \delta V_{t+1}^L = \left(\frac{1-\delta}{\delta}\right)\left(u^L - c^L\right).$$

Hence

$$\left(\frac{1-\delta}{\delta}\right)\left(u^L - c^L\right) = \alpha\rho_t^H(c^H - c^L - \delta V_{t+1}^L).$$

Solving for ρ_t^H yields

$$\rho_t^H = \frac{1-\delta}{\delta\alpha}\frac{u^L - c^L}{c^H - u^L + \delta^{T-t}\alpha(1 - \hat{q})(u^L - c^L)} \tag{9}$$

for $1 < t < T$. Recall that $\rho_1^H = 0$. Clearly $\rho_t^H > 0$. Further, since $\delta\alpha\left(c^H - u^L\right) > u^L - c^L$, then

$$\rho_t^H < (1-\delta)\,\frac{u^L - c^L}{\delta\alpha\,(c^H - u^L)} < 1.$$

Since low $\rho_{T-1}^L > 0$ and $1 - \rho_{T-1}^L - \rho_{T-1}^H > 0$ by (2.2), then $u^L - r_{T-1}^L = \delta V_T^B$. Hence

$$V_T^L = \alpha\rho_T^H(c^H - c^L),$$

implies

$$\delta\alpha(1-\hat{q})(u^L - c^L) = u^L - c^L - \delta\alpha\rho_T^H(c^H - c^L).$$

Solving for ρ_T^H yields

$$\rho_T^H = \frac{u^L - c^L - \delta\alpha(1-\hat{q})(u^L - c^L)}{\delta\alpha\,(c^H - c^L)} = (1 - \delta\alpha(1-\hat{q}))\,\frac{(u^L - c^L)}{\delta\alpha\,(c^H - c^L)}. \tag{10}$$

Since $0 < 1 - \delta\alpha\,(1-\hat{q}) < 1$, and $\delta\alpha\left(c^H - c^L\right) > (u^L - c^L)$, then $0 < \rho_T^H < 1$.

Finally, we compute ρ_t^L. For each $t \in \{1, \ldots, T-1\}$, we have

$$q_{t+1}^H = \frac{(1-\alpha\rho_t^H)q_t^H}{(1-\alpha\rho_t^H)q_t^H + (1-\alpha(\rho_t^L + \rho_t^H))q_t^L}.$$

Solving for ρ_t^L we obtain

$$\rho_t^L = (1-\alpha\rho_t^H)\frac{q_{t+1}^H - q_t^H}{\alpha q_{t+1}^H\,(1-q_t^H)} \tag{11}$$

for $t < T$. Since $1 \geq q_{t+1}^H \geq q_t^H$ and $\rho_t^H < 1$, then $\rho_t^L > 0$. And since $\delta\left(c^H - u^L\right) > \delta\alpha\left(c^H - u^L\right) > u^L - c^L$, then

$$\frac{u^L - c^L}{c^H - u^L} < \delta < 1,$$

and

$$(1-\delta)\,\frac{u^L - c^L}{c^H - u^L} < 1.$$

Using (5), for $t > 1$ we have

$$\begin{aligned}
\frac{q_{t+1}^H - q_t^H}{\alpha q_{t+1}^H\,(1-q_t^H)} &= \frac{(1-\delta)\delta^{T-t-1}(1-\hat{q})(u^L - c^L)}{c^H - u^L + \delta^{T-t-1}\alpha(1-\hat{q})(u^L - c^L)}\left(\frac{u^H - u^L}{u^H - c^H - \delta^{T-t}\alpha(1-\hat{q})(u^L - c^L)}\right) \\
&< (1-\delta)\frac{(1-\hat{q})(u^L - c^L)}{c^H - u^L}\left(\frac{u^H - u^L}{u^H - c^H - (1-\hat{q})(u^L - c^L)}\right) \\
&= (1-\delta)\frac{u^L - c^L}{c^H - u^L} \\
&< 1.
\end{aligned}$$

Hence $\rho_t^L < 1$. Since

$$\alpha\left(1-q^H\right) > 1 - \left(1-(1-\hat{q})\frac{u^L-c^L}{\delta\left(c^H-u^L\right)}\right)\frac{q^H}{\hat{q}} > 1 - \frac{q^H}{\hat{q}}$$

by $SF.2$, using (5) again and noticing that $\rho_1^H = 0$, and $q_2^H \leq \hat{q}$ as shown above we have

$$\rho_1^L = \frac{q_2^H - q^H}{q_2^H \alpha\left(1-q^H\right)} < \frac{q_2^H - q^H}{q_2^H\left(1-\frac{q^H}{\hat{q}}\right)} = \frac{q_2^H - q^H}{q_2^H - q^H\frac{q_2^H}{\hat{q}}} < 1.$$

Finally, $\rho_T^L + \rho_T^H = 1$ implies

$$\rho_T^L = 1 - \rho_T^H = 1 - \frac{u^L - c^L}{\delta\alpha\left(c^H-u^L\right)}\left(1-\delta\alpha(1-\hat{q})\right). \tag{12}$$

Since $0 < \rho_T^H < 1$ as shown above, we have $0 < \rho_T^L < 1$.

Using equations (4) and (8), noticing that $q^H + q^L = 1$, we can calculate the surplus as

$$S = \left[q^L + \delta^{T-1}\alpha q^H\left(1-\hat{q}\right)\right]\left(u^L-c^L\right). \ \square \tag{13}$$

Recall $q^H < \bar{q}$ by assumption, and note that our assumptions $\bar{q} < \hat{q} < 1$. Lemmas 4 and 5 establish properties that DE has when frictions are small and traders are patient.

Lemma 4. *Assume that $T > 1$ and frictions are small, and let $\left(\rho^H, \rho^L, r^H, r^L\right)$ be a market equilibrium. If $\delta^{T-1}\alpha\left(c^H-c^L\right) > u^L - c^L$, then for all $t \in \{1, ..., T\}$:*

(*L4.1*) $\rho_t^H < 1$.

(*L4.2*) $\rho_t^L < 1$.

(*L4.3*) $\rho_T^H > 0$, $\rho_T^L > 0$, *and* $q_T^H = \hat{q}$.

(*L4.4*) $V_t^L > 0$.

(*L4.5*) $\rho_t^L > 0$.

(*L4.6*) $\rho_t^H < \dfrac{u^L-c^L}{\delta\alpha\left(c^H-u^L\right)}$.

Proof: We prove $L4.1$. Assume that $\rho_t^H = 1$ for some t. Then $V_t^L = \alpha(c^H - c^L) + (1-\alpha)\,\delta V_{t+1}^L$. Since

$$\delta^{t-1}\alpha\left(c^H-c^L\right) \geq \delta^{T-1}\alpha\left(c^H-c^L\right) > u^L - c^L$$

we have

$$r_1^L = c^L + \delta V_2^L \geq c^L + \delta^{t-1} V_t^L > c^L + u^L - c^L = u^L,$$

and therefore offering r_1^L at date 1 is suboptimal, contradicting that $\rho_1^L > 0$ ($L2.3$). Hence $\rho_t^H < 1$.

We prove $L4.2$. We first show that $\rho_t^L < 1$ for $t < T$. Assume that $\rho_t^L = 1$ for some $t < T$. By $SF.1$, $\delta\left(c^H - c^L\right) > \delta\alpha\left(c^H - c^L\right) > u^L - c^L$; hence $\hat{q} < 1$ implies

$$0 < 1 - (1 - \hat{q}) \frac{u^L - c^L}{\delta\left(c^H - u^L\right)} < 1,$$

Since

$$\delta\left[\alpha\left(1 - q^H\right)\hat{q} - \hat{q} + q^H\right]\left(c^H - c^L\right) > q^H (1 - \hat{q})\left(u^L - c^L\right)$$

by $SF.2$, then

$$\begin{aligned} q^H + (1-\alpha)(1-q^H) &= 1 - \alpha(1 - q^H) \\ &< \left(1 - (1-\hat{q}) \frac{u^L - c^L}{\delta\left(c^H - u^L\right)}\right) \frac{q^H}{\hat{q}} \\ &< \frac{q^H}{\hat{q}}. \end{aligned}$$

Therefore

$$q_T^H \geq q_{t+1}^H = \frac{q_t^H}{q_t^H + (1-\alpha) q_t^L} \geq \frac{q^H}{q^H + (1-\alpha)(1-q^H)} > \hat{q}.$$

Hence

$$\begin{aligned} q_T^H u^H + q_T^L u^L - c^H &> \hat{q} u^H + (1 - \hat{q}) u^L - c^H \\ &> (1 - \hat{q})\left(u^L - c^L\right) \\ &> q_T^L (u^L - c^L), \end{aligned}$$

i.e., offering $r_T^L = c^L$ at date T is not optimal. Hence $\rho_T^L = 0$, and therefore $\rho_T^H = 1$ by $L2.1$, which contradicts $L4.1$. Hence $\rho_t^H < 1$ for all $t < T$.

We show that $\rho_T^L < 1$. Assume that $\rho_T^L = 1$. Then $q_T^H \leq \hat{q}$ (since otherwise an offer of r_T^L is suboptimal), $V_T^L = 0$ and $V_T^B = \alpha q_T^L\left(u^L - c^L\right)$. Hence $r_{T-1}^L = c^L$ by $DE.L$, and the payoff to offering at date $T-1$ a price below c^L is

$$\delta V_T^B = \delta\alpha q_T^L (u^L - c^L).$$

Hence

$$q^L_{T-1}(u^L - c^L) + q^H_{T-1}\delta V^B_T - \delta V^B_T = q^L_{T-1}(u^L - c^L)\left(1 - \delta\alpha q^L_T\right) > 0,$$

i.e., the payoff to offering c^L at date $T-1$ is greater than that of offering less than c^L. Therefore $\rho^L_{T-1} + \rho^H_{T-1} = 1$. Since $q^H_{T-1} \leq q^H_T$ by $L1.4$ and $q^H_T \leq \hat{q}$, then the payoff to offering c^H at $T-1$ is

$$\begin{aligned} q^H_{T-1}u^H + q^L_{T-1}u^L - c^H &\leq q^H_T u^H + q^L_T u^L - c^H \\ &\leq q^L_T(u^L - c^L) \\ &\leq q^L_{T-1}(u^L - c^L) \\ &< q^L_{T-1}(u^L - c^L) + q^H_{T-1}\delta V^B_T, \end{aligned}$$

where the last term is the payoff to offering c^L at $T-1$. Hence $\rho^H_{T-1} = 0$, and therefore $\rho^L_{T-1} = 1$, which contradicts that $\rho^L_{T-1} < 1$ as shown above. Hence $\rho^L_T < 1$.

We prove $L4.3$. We have $\rho^H_T < 1$ by $L4.1$, and therefore $L2.1$ implies $\rho^L_T > 0$. Since $\rho^L_T < 1$ by $L4.2$, then $\rho^H_T > 0$ by $L2.1$. Now, since both high price offers and low price offers are optimal at date T, and reservation prices are $r^H_T = c^H$ and $r^L_T = c^L$, we have

$$q^H_T u^H + q^L_T u^L - c^H = q^L_T(u^L - c^L),$$

i.e.,

$$q^H_T u^H + (1 - q^H_T)u^L - c^H = (1 - q^H_T)(u^L - c^L).$$

Hence

$$q^H_T = \frac{c^H - c^L}{u^H - c^L} = \hat{q}.$$

We prove $L4.4$ by induction. By $L4.3$, $V^L_T = \alpha\rho^H_T\left(c^H - c^L\right) > 0$. Assume that $V^L_{k+1} > 0$ for some $k \leq T$. Since $r^L_k - c^L = \delta V^L_{k+1}$ by $DE.L$, then we have

$$\begin{aligned} V^L_k &= \alpha\left(\rho^H_k\left(c^H - c^L\right) + \rho^L_k\left(r^L_k - c^L\right)\right) + \left(1 - \alpha\left(\rho^H_k + \rho^L_k\right)\right)\delta V^L_{t+1} \\ &= \alpha\rho^H_k\left(c^H - c^L\right) + \left(1 - \alpha\rho^H_k\right)\delta V^L_{k+1} \\ &> 0. \end{aligned}$$

We prove $L4.5$. Suppose by way of contradiction that $\rho^L_t = 0$ for some t. Since $\rho^L_T > 0$ by $L4.3$, then $t < T$. Also $\rho^L_t = 0$ implies $\rho^H_t > 0$ by $L2.2$. Since $\rho^H_t < 1$ by $L4.1$, then buyers are indifferent at date t between offering c^H or less than r^L_t, i.e.,

$$q^H_t u^H + q^L_t u^L - c^H = \delta V^B_{t+1} < V^B_{t+1}.$$

We show that $\rho^H_{t+1} = 0$. Suppose that $\rho^H_{t+1} > 0$; then

$$V^B_{t+1} = \alpha \left(q^H_{t+1} u^H + q^L_{t+1} u^L - c^H \right) + (1-\alpha)\delta V^B_{t+2}.$$

Hence

$$q^H_t u^H + q^L_t u^L - c^H < \alpha \left(q^H_{t+1} u^H + q^L_{t+1} u^L - c^H \right) + (1-\alpha)\delta V^B_{t+2},$$

But $\rho^L_t = 0$ implies $q^H_{t+1} = q^H_t$, and therefore

$$q^H_{t+1} u^H + q^L_{t+1} u^L - c^H < \delta V^B_{t+2},$$

i.e., offering c^H yields a payoff smaller than offering less than r^L_{t+1}, which contradicts that $\rho^H_{t+1} > 0$.

Since $\rho^H_{t+1} = 0$, then $DE.L$ implies

$$\begin{aligned} V^L_{t+1} &= \alpha \rho^L_{t+1} \left(r^L_{t+1} - c^L \right) + \left(1 - \alpha \rho^L_{t+1} \right) \delta V^L_{t+2} \\ &= \delta V^L_{t+2}, \end{aligned}$$

and since $V^L_{t+1} > 0$ by $L4.4$, hence $V^L_{t+2} > 0$, and therefore $DE.L$ implies

$$r^L_t = c^L + \delta V^L_{t+1} = c^L + \delta^2 V^L_{t+2} < c^L + \delta V^L_{t+2} = r^L_{t+1}.$$

We show that these facts: $\rho^H_{t+1} = 0 < \rho^L_{t+1} < 1$ and $r^L_t < r^L_{t+1}$ are incompatible, thereby proving that $\rho^L_t > 0$.

The payoff to offering r^L_t at period t is

$$q^H_t \delta V^B_{t+1} + q^L_t (u^L - r^L_t) \leq \delta V^B_{t+1},$$

where the inequality follows since negligible price offers are optimal (because $\rho^L_t = 0 < \rho^H_t < 1$). Noticing that $q^H_t = 1 - q^L_t$, this inequality becomes

$$u^L - r^L_t \leq \delta V^B_{t+1}.$$

Since $0 < \rho^L_{t+1} < 1$, i.e., price offers of r^L_{t+1} and of less than r^L_{t+1} are optimal, we have

$$V^B_{t+1} = \alpha q^L_{t+1} (u^L - r^L_{t+1}) + \left(1 - \alpha q^L_{t+1} \right) \delta V^B_{t+2} = \delta V^B_{t+2}.$$

Hence

$$V^B_{t+1} = u^L - r^L_{t+1}.$$

Summing up

$$u^L - r_t^L \le \delta V_{t+1}^B < V_{t+1}^B = u^L - r_{t+1}^L;$$

i.e., $r_t^L \ge r_{t+1}^L$, which contradicts $r_t^L < r_{t+1}^L$.

We prove $L4.6$. Since $V_t^L \ge 0$, and $r_t^L - c^L = \delta V_{t+1}^L$ by $DE.L$, we have

$$\begin{aligned} V_t^L &= \alpha \left(\rho_t^H \left(c^H - c^L\right) + \rho_t^L \left(r_t^L - c^L\right)\right) + \left(1 - \alpha \left(\rho_t^H + \rho_t^L\right)\right) \delta V_{t+1}^L \\ &\ge \alpha \rho_t^H \left(c^H - c^L\right). \end{aligned}$$

Since $\rho_t^L > 0$ by $L4.5$ (i.e., price offers of r_t^L are optimal), and $V_t^B > 0$ by $L1.2$, then $u^L > r_t^L$. Hence

$$u^L - c^L > r_t^L - c^L = \delta V_{t+1}^L \ge \delta \alpha \rho_t^H \left(c^H - c^L\right),$$

i.e.,

$$\rho_t^H < \frac{u^L - c^L}{\delta \alpha \left(c^H - u^L\right)}. \square$$

Lemma 5. *Assume that $T > 1$ and frictions are small, and let $(\rho^H, \rho^L, r^H, r^L)$ be a market equilibrium. If $\delta^{T-1} \alpha \left(c^H - c^L\right) > u^L - c^L$, then $\rho_{t+1}^H > 0$ and $\rho_t^L + \rho_t^H < 1$ for all $t \in \{1, ..., T-1\}$.*

Proof: Let $t \in \{1, ..., T-1\}$. We proceed by showing that (i) $\rho_t^H > 0$ implies $\rho_t^L + \rho_t^H < 1$, and that (ii) $\rho_t^L + \rho_t^H < 1$ implies $\rho_{t+1}^H > 0$. Then Lemma 5 follows by induction: Since $\rho_1^H = 0$ by $L2.3$ and $\rho_1^L < 1$ by $L4.2$, then $\rho_1^H + \rho_1^L < 1$, and therefore $\rho_2^H > 0$ by (ii). Assume that the claim holds for $k \in \{1, ..., T-1\}$; we show that $\rho_{k+1}^H + \rho_{k+1}^L < 1$ and $\rho_{k+2}^H > 0$. Since $\rho_{k+1}^H > 0$, then $\rho_{k+1}^H + \rho_{k+1}^L < 1$ by (i), and therefore $\rho_{k+2}^H > 0$ by (ii).

We establish (i), i.e., $\rho_t^H > 0$ implies $\rho_t^L + \rho_t^H < 1$. Suppose not; let $t < T$ be the first date such that $\rho_t^H > 0$ and $\rho_t^L + \rho_t^H = 1$. Since $\rho_t^L + \rho_t^H = 1$ (i.e., all low quality seller who are matched trade) and $q_t^H \geq q_1^H = q^H$ by $L1.4$, then $L4.6$ implies

$$\begin{aligned}
q_{t+1}^H &= \frac{\left(1-\alpha\rho_t^H\right) q_t^H}{\left(1-\alpha\rho_t^H\right) q_t^H + (1-\alpha) q_t^L} \\
&> \frac{\left(1-\frac{u^L-c^L}{\delta\left(c^H-u^L\right)}\right) q_t^H}{\left(1-\frac{u^L-c^L}{\delta\left(c^H-u^L\right)}\right) q_t^H + (1-\alpha) q_t^L} \\
&> \frac{\left(1-\frac{u^L-c^L}{\delta\left(c^H-u^L\right)}\right) q^H}{\left(1-\frac{u^L-c^L}{\delta\left(c^H-u^L\right)}\right) q^H + (1-\alpha)(1-q^H)},
\end{aligned}$$

where the first and second inequality hold since q_{t+1}^H is decreasing in ρ_t^H and increasing in q_t^H (and $q^H > q_t^H$). Since

$$\delta\left[\alpha\left(1-q^H\right)\hat{q} - \hat{q} + q^H\right]\left(c^H-c^L\right) > q^H\left(1-\hat{q}\right)\left(u^L-c^L\right)$$

by $SF.2$, then we have

$$\begin{aligned}
\left(1-\frac{u^L-c^L}{\delta\left(c^H-u^L\right)}\right) q^H + (1-\alpha)(1-q^H) &= -\frac{u^L-c^L}{\delta\left(c^H-u^L\right)} q^H + 1 - \alpha(1-q^H) \\
&< -\frac{u^L-c^L}{\delta\left(c^H-u^L\right)} q^H + 1 - \left(1 - \left(1-(1-\hat{q})\frac{u^L-c^L}{\delta\left(c^H-u^L\right)}\right.\right. \\
&= \frac{q^H}{\hat{q}}\left(1-\frac{u^L-c^L}{\delta\left(c^H-u^L\right)}\right).
\end{aligned}$$

Hence

$$q_{t+1}^H > \frac{\left(1-\frac{u^L-c^L}{\delta\left(c^H-u^L\right)}\right) q^H}{\left(1-\frac{u^L-c^L}{\delta\left(c^H-u^L\right)}\right) \frac{q^H}{\hat{q}}} = \hat{q} = q_T,$$

which contradicts $L1.4$.

Next we prove (ii), i.e., $\rho_t^L + \rho_t^H < 1$ implies $\rho_{t+1}^H > 0$. Suppose not; let $t < T$ be such that $\rho_t^L + \rho_t^H < 1$ and $\rho_{t+1}^H = 0$. Since $\rho_t^L > 0$ by $L4.5$, then offers of r_t^L and of less than r_t^L are optimal at date t, and we have

$$u^L - r_t^L = \delta V_{t+1}^B.$$

Since $\rho_{t+1}^H = 0$ we have

$$V_{t+1}^L = \delta V_{t+2}^L.$$

Therefore $L4.4$ implies

$$r_{t+1}^L = c^L + \delta V_{t+2}^L = c^L + V_{t+1}^L > c^L + \delta V_{t+1}^L = r_t^L.$$

Since $0 < \rho_{t+1}^L < 1$ by $L4.2$ and $L4.5$, then offers of r_{t+1}^L and of less than r_{t+1}^L are optimal at $t+1$, i.e.,

$$u^L - r_{t+1}^L = \delta V_{t+2}^B,$$

and

$$V_{t+1}^B = \delta V_{t+2}^B.$$

Summing up

$$u^L - r_t^L = \delta V_{t+1}^B < V_{t+1}^B = \delta V_{t+2}^B = u^L - r_{t+1}^L,$$

i.e.,

$$r_t^L > r_{t+1}^L,$$

which is a contradiction. □

Proof of propositions 2 and 3. (2.1) follows from $L2.3$ and $L4.2$. (2.2) follows from $L4.5$ and Lemma 5. (2.3) follows from $L2.1$ and $L4.3$.???

Proof of Proposition 4. We have $\tilde{\rho}_1^H = \lim_{\alpha,\delta\to 1} \rho_1^H = 0$, and for $1 < t < T$, using (9) above we have

$$\tilde{\rho}_t^H = \lim_{\alpha,\delta\to 1} \rho_t^H = \lim_{\alpha,\delta\to 1} \frac{1-\delta}{\delta\alpha} \frac{u^L - c^L}{c^H - u^L + \delta^{T-t}\alpha(1-\hat{q})(u^L - c^L)} = 0.$$

Also (10) yields

$$\tilde{\rho}_T^H = \lim_{\alpha,\delta\to 1} \rho_T^H = \frac{u^L - c^L}{c^H - u^L}\hat{q}.$$

Since

$$\lim_{\alpha,\delta\to 1} q_t^H = \lim_{\alpha,\delta\to 1} \frac{c^H - u^L + \delta^{T-t}\alpha(1-\hat{q})(u^L - c^L)}{u^H - u^L} = \hat{q},$$

for $t > 1$, then (11) yields

$$\tilde{\rho}_t^L = \lim_{\alpha,\delta\to 1} \rho_t^L = \lim_{\alpha,\delta\to 1} (1 - \alpha\rho_t^H) \frac{q_{t+1}^H - q_t^H}{\alpha q_{t+1}^H (1 - q_t^H)} = 0.$$

for $1 < t < T$. Also

$$\tilde{\rho}_1^L = \lim_{\alpha,\delta\to 1} \rho_1^L = \frac{\hat{q} - q^H}{\hat{q} - \hat{q}q^H}.$$

And (12) yields

$$\tilde{\rho}_T^L = \lim_{\alpha,\delta\to 1} \rho_T^L = 1 - \frac{u^L - c^L}{c^H - u^L}\hat{q}.$$

Note that the limiting values $(\tilde{\rho}^H, \tilde{\rho}^L)$ form a sequence probability distributions, i.e., $\tilde{\rho}_t^H, \tilde{\rho}_t^L < 0$ and $\tilde{\rho}_t^H + \tilde{\rho}_t^L < 1$ for all $t \in \{1, ..., T\}$. (However, one can show that for $\alpha = \delta = 1$ there are multiple decentralized market equilibria.)

As for traders' expected utilities, (4) implies

$$\tilde{V}_t^B = \lim_{\alpha,\delta\to 1} \delta^{T-t}\alpha\left(1-\hat{q}\right)\left(u^L - c^L\right) = \left(1-\hat{q}\right)\left(u^L - c^L\right),$$

(7) implies

$$\tilde{V}_1^L = \lim_{\alpha,\delta\to 1} \left(1 - \delta^{T-1}\alpha\left(1-\hat{q}\right)\right)\left(u^L - c^L\right) = \hat{q}\left(u^L - c^L\right),$$

and $\tilde{V}_t^H = \lim_{\alpha,\delta\to 1} V_t^H = 0$.

Finally, using (13) we get

$$S = \lim_{\alpha,\delta\to 1} \left[q^L(u^L - c^L) + q^H\delta^{T-1}\alpha(1-\hat{q})(u^L - c^L)\right] = \left[q^L + q^H\left(1-\hat{q}\right)\right](u^L - c^L). \square$$

Proof of Proposition 5. Assume that $T = \infty$, and frictions are small. We show that the strategy distribution $(\rho^H, \rho^L, r^H, r^L)$ given by $r_t^H = c^H$, $r_t^L = u^L$ for all t, $\rho_1^H = 0$,

$$\rho_1^L = \frac{\bar{q} - q^H}{\alpha\left(1 - q^H\right)\bar{q}},$$

and $\rho_t^L = 0$,

$$\rho_t^H = \frac{1-\delta}{\delta\alpha}\frac{u^L - c^L}{c^H - u^L}$$

for $t > 1$ forms a decentralized market equilibrium.

Since $\hat{q} > \bar{q}$, $SF.2$ implies

$$\begin{aligned}
\alpha\left(1 - q^H\right) &> 1 - \left(1 - \left(1-\hat{q}\right)\frac{u^L - c^L}{\delta\left(c^H - u^L\right)}\right)\frac{q^H}{\hat{q}} \\
&> 1 - \frac{q^H}{\hat{q}} \\
&> 1 - \frac{q^H}{\bar{q}}.
\end{aligned}$$

Then $0 < \rho_1^L < 1$. As $\delta\alpha\left(c^H - u^L\right) > u^L - c^L$ by $SF.1$, we have $0 < \rho_t^H < 1$ for all $t > 1$ – recall that $\delta < 1$ by assumption. Since $r_t^H = c^H$, $r_t^L = u^L$, then the expected utilities buyers and high quality sellers are $V_t^B = V_t^H = 0$. For $t > 1$ low quality sellers expected utility is $V_t^L = (u^L - c^L)/\delta$. Then $r_t^H = c^H$ and $r_t^L = u^L$ satisfy $DE.H$ and $DE.L$, respectively. Using ρ_1^H and ρ_1^L we have

$$q_2^H = \frac{q^H}{q^H + (1 - \alpha\rho_1^L)(1 - q^H)} = \bar{q}.$$

And since $\rho_t^L = 0$ for $t > 1$, then $q_t^H = q_1^H = \bar{q}$. Hence

$$q_t^H \left(u^H - c^H\right) + (1 - q_t^H) \left(u^L - c^H\right) = 0.$$

Since $r_t^L = u^L$, then the payoff to a low price offer is also zero. Then high, low and negligible price offers are optimal at date $t > 1$. Moreover, since $V_t^B = 0$, then low and negligible price offers are optimal and date 1. Hence any distribution of price offers λ such that ρ_t^H and ρ_t^L have the values defined above satisfies $DE.B$. Therefore the strategy distribution defined is a decentralized market equilibrium. □

In lemmas 5 and 6 we establish some basic properties of dynamic competitive equilibria.

Lemma 6. *In every CE, (p, u, s^H, s^L, d), we have*

$$\sum_{\{t|s_t^H > 0\}} s_t^L < q^L.$$

Proof. Let (p, u, s^H, s^L, d) be a CE. For all t such that $s_t^H > 0$ we have

$$\delta^{t-1}(p_t - c^H) = v^H(p) \geq 0$$

by $(S.2)$. Hence $p_t \geq c^H$. Also $d_t > 0$ by $CE.1$, and therefore

$$v^B(p) = \delta^{t-1}(u_t - p_t) \geq 0$$

implies

$$0 \leq u_t - p_t \leq u_t - c^H;$$

i.e., $u_t \geq c^H = u(\bar{q})$. Thus

$$\frac{s_t^H}{s_t^H + s_t^L} \geq \bar{q}.$$

Hence

$$(1-\bar{q})\sum_{\{t|s_t^H>0\}} s_t^H \geq \bar{q}\sum_{\{t|s_t^H>0\}} s_t^L.$$

Since

$$\sum_{\{t|s_t^H>0\}} s_t^H \leq q^H < \bar{q},$$

then

$$(1-\bar{q})q^H \geq (1-\bar{q})\sum_{\{t|s_t^H>0\}} s_t^H \geq \bar{q}\sum_{\{t|s_t^H>0\}} s_t^L \geq q^H \sum_{\{t|s_t^H>0\}} s_t^L;$$

i.e.,

$$\sum_{\{t|s_t^H>0\}} s_t^L \leq 1-\bar{q} < 1-q^H = q^L. \ \square$$

Lemma 7. *Let (p, u, m^H, m^L, m^B) be a CE. If $s_{\bar{t}}^H > 0$ for some $\bar{t}$, then there is $t < \bar{t}$ such that $s_t^L > 0 = s_t^H$, and*

$$\delta^{t-1}(u^L - c^L) \geq \delta^{\bar{t}-1}(c^H - c^L).$$

Proof. Let (p, u, s^H, s^L, d) be a CE, and assume that $s_{\bar{t}}^H > 0$. Then $\delta^{\bar{t}-1}(p_{\bar{t}} - c^H) = v^H(p) \geq 0$ by $S.2$, and therefore $p_{\bar{t}} \geq c^H$. Hence $v^L(p) \geq \delta^{\bar{t}-1}(p_{\bar{t}} - c^L) \geq \delta^{\bar{t}-1}(c^H - c^L) > 0$, and therefore

$$\sum_{t=1}^{T} s_t^L = q^L$$

by $(S.3)$. Since $\sum_{\{t|s_t^H>0\}} s_t^L < q^L$ by Lemma 6, there there is $\hat{t}$ such that $s_{\hat{t}}^L > 0 = s_{\hat{t}}^H$. Hence $d_{\hat{t}} > 0$ by $CE.1$ implies $u_{\hat{t}} = u^L$ by $CE.2$, and $p_{\hat{t}} \leq u^L$ by $D.2$. Also $s_{\hat{t}}^L > 0$ implies $v^L(p) = \delta^{\hat{t}-1}(p_{\hat{t}} - c^L) \geq \delta^{\bar{t}-1}(p_{\bar{t}} - c^L)$ by $S.2$. Thus

$$\delta^{\hat{t}-1}(u^L - c^L) \geq \delta^{\hat{t}-1}(p_{\hat{t}} - c^L) \geq \delta^{\bar{t}-1}(p_{\bar{t}} - c^L) \geq \delta^{\bar{t}-1}(c^H - c^L).$$

Since $u^L < c^H$ this inequality implies $\hat{t} < \bar{t}$. $\square$

Proof of Proposition 6. Let (p, u, s^H, s^L, d) be a CE, and assume that $\delta^{T-1}(u^H - c^L) > u^L - c^L$.

We show that $s_t^H = 0$ for all $t \in \{1, \ldots, T\}$. Suppose that $s_t^H > 0$ for some t. Then Lemma 7 implies that there is $t' < t$ such that

$$u^L - c^L \geq \delta^{t'-1}(u^L - c^L) \geq \delta^{t-1}(c^H - c^L) \geq \delta^{T-1}(c^H - c^L),$$

which is a contradiction.

We show that $p_t \geq u^L$ for all t. If $p_t < u^L$ for some t, then

$$v^B(p, u) = \max_{t \in \{1,\ldots,T\}} (0, \delta^{t-1}(u_t - p_t)) > 0,$$

and therefore $\sum_{t=1}^T d_t = 1$. Since $s_t^H = 0$ for all t, then $CE.1$ implies

$$q^L = \sum_{t=1}^T s_t^L = \sum_{t=1}^T (s_t^H + s_t^L) = \sum_{t=1}^T d_t = 1,$$

which contradicts $q^L = 1 - q^H < 1$. Hence $p_t \geq u^L$ for all t.

We now show that $p_1 = u^L$ and $s_1^L = d_1 = q^L$. Suppose $s_t^L > 0$. Then $s_t^H = 0$ implies $u_t = u^L$. By $CE.1$ we have $d_t > 0$ and thus

$$\delta^{t-1}(u_t - p_t) = \delta^{t-1}(u^L - p_t) \geq 0$$

by $D.2$. This inequality, and $p_t \geq u^L$ for all t, imply that $p_t = u^L$. If $t > 1$, then $p_1 \geq u^L$ implies $p_1 - c^L > \delta^{t-1}(p_t - c^L)$, which contracts $S.2$. Hence $s_t^L = 0$ for $t > 1$. Since $p_1 - c^L > 0$, then $v^L(p) > 0$ and thus $\sum_{t=1}^T s_t^L - q^L = 0$ by $S.3$, which implies that $s_1^L = q^L$. $CE.1$ and $s_1^H = 0$ then implies $d_1 = q^L$. $\square$

Proof of Proposition 7. Assume that $\delta^{T-1}\left(u^H - c^L\right) < \left(u^L - c^L\right)$. We show that the profile (p, u, s^H, s^L, d) given by $p_t = u_t = u^L$ for $t < T$, and $p_T = u_T = u^H$, $s_1^H = 0$, $s_1^L = q^L = d_1$, $s_t^L = s_t^H = d_t = 0$ for $1 < t < T$, $s_T^H = d_T = q^H$, $s_T^L = 0$ is a CE.

For high quality sellers we have $v^H(p) = \delta^{T-1}(p_T - c^H) = \delta^{T-1}(u^H - c^H) > 0 > \delta^{t-1}(p_t - c^H)$ for $t < T$, and hence $S^H(p) = \{(0, ..., 0, q^H)\}$. For low quality sellers we have $v^L(p) = p_1 - c^L = u^L - c^L > \delta^{t-1}(p_t - c^H)$ for $t > 1$. (In particular, $u^L - c^L > \delta^{T-1}(p_T - c^H) = \delta^{T-1}(u^H - c^H)$.) Hence $S^H(p) = \{(q^L, 0, ..., 0)\}$. For buyers, $v^B(p, u) = 0 = \delta^{t-1}(u_t - p_t)$ for all t. Hence $D(p, u) = \{d \in \mathbb{R}_+^T \mid \sum_{t=1}^T d_t \leq 1\}$, and $(q^L, 0, ..., 0, q^H) \in D(p, u)$. Finally, note that buyers' value expectations at dates 1 and T, u_1 and u_T, are correct. Thus, the profile defined is a CE.

Assume that $\delta^{T-1}\left(u^H - c^L\right) \geq \left(u^L - c^L\right)$. We show that the profile (p, u, s^H, s^L, d) given by $p_t = u_t = u^L$ for $t < T$,and $p_T = u_T = \delta^{1-T}\left(u^L - c^L\right) + c^L$, $s_1^H = 0$, $s_1^L = q^L - q = d_1$, $s_t^L = s_t^H = d_t = 0$ for $1 < t < T$, and $s_T^H = q^H$, $s_T^L = q$, and $d_T = q + q_H$, where

$$q = \frac{\left(\left(u^L - c^L\right) - \delta^{T-1}(u^H - c^L)\right) q^H}{\left(\delta^{T-1}\left(u^L - c^L\right) - \left(u^L - c^L\right)\right)},$$

is a CE. Note that since $q^H < \bar{q}$ (i.e. $u(q^H) = q^H u^H + (1 - q^H)u^L < c^H$), and $u^L - c^L - \delta^{T-1}\left(c^H - c^L\right) \geq 0$ by assumption, then

$$\begin{aligned}
& 1 - q^H - \frac{\left(\left(u^L - c^L\right) - \delta^{T-1}(u^H - c^L)\right) q^H}{\left(\delta^{T-1}\left(u^L - c^L\right) - \left(u^L - c^L\right)\right)} \\
= & \frac{\left(u^L - c^L\right) - \delta^{T-1}\left(q^H u^H + (1 - q^H)u^L - c^L\right)}{\left(\delta^{T-1} - 1\right)\left(c^L - u^L\right)} \\
> & \frac{\left(u^L - c^L\right) - \delta^{T-1}\left(c^H - c^L\right)}{\left(\delta^{T-1} - 1\right)\left(c^L - u^L\right)} \\
\geq & \ 0,
\end{aligned}$$

and therefore $q < q^L$.

For high quality sellers we have

$$\begin{aligned}
v^H(p) &= \delta^{T-1}(p_T - c^H) \\
&= \delta^{T-1}(\delta^{1-T}\left(u^L - c^L\right) - \left(c^H - c^L\right)) \\
&= \left(u^L - c^L\right) - \delta^{T-1}\left(c^H - c^L\right)) \\
&> 0 \\
&> \delta^{t-1}(p_t - c^H)
\end{aligned}$$

for all $t < T$. Hence $S^H(p) = \{(0, ..., 0, q^H)\}$. For low quality sellers we have

$$\begin{aligned}
v^L(p) &= p_1 - c^L \\
&= u^L - c^L \\
&= \delta^{T-1}(\delta^{1-T}\left(u^L - c^L\right) + c^L - c^L) \\
&= \delta^{T-1}(p_T - c^L) \\
&> \delta^{t-1}(p_t - c^H) \\
&> 0.
\end{aligned}$$

for all $1 < t < T$. Hence $S^H(p) = \{(s_1^L, 0, ..., s_T^L) \mid s_1^L + s_T^L = q^L\}$. For buyers, $v^B(p,u) = 0 = \delta^{t-1}(u_t - p_t)$ for all t. Hence $D(p,u) = \{d \in \mathbb{R}_+^T \mid \sum_{t=1}^T d_t \leq 1\}$, and $(q^L, 0, ..., 0, q^H) \in D(p,u)$. Finally, we show that buyers value expectations are correct at dates 1 and T. Clearly $u_1 = u^L$ is the correct expectation as only low sellers supply at date 1. As for u_T we have

$$\begin{aligned} u_t &= \frac{s_T^H}{s_T^H + s_T^L} u^H + \frac{s_T^L}{s_T^H + s_T^L} u^L \\ &= \frac{q^H}{q^H + q} u^H + \frac{q}{q^H + q} u^L \\ &= \delta^{1-T}\left(u^L - c^L\right) + c^{L-1}. \ \square \end{aligned}$$

Proof of Proposition 8. Assume that $T = \infty$. Consider the profile (p, u, s^H, s^L, d) given by $p_t = u_t = u^L$ for $t < \bar{T}$, and $p_t = u_t = u^H$ for $t > \bar{T}$, $s_1^H = 0$, $s_1^L = d_1 = q^L$, $s_{\bar{T}}^H = d_{\bar{T}} = q^H$, $s_{\bar{T}}^L = 0$, and $s_t^H = s_t^L = d_t = 0$ for $t \notin \{1, \bar{T}\}$, where $\bar{T}$ is the unique date satisfying

$$\delta^{\bar{T}-2}\left(u^H - c^L\right) > u^L - c^L \geq \delta^{\bar{T}-1}\left(u^H - c^L\right)$$

By following the steps of the fist part of the proof of Proposition 7, it is easy to see that this profile is a CE. The surplus is readily calculated as

$$S^{CE} = \sum_{\tau \in \{H,L\}} \sum_{t=1}^{T} s_t^\tau \delta^{t-1}(u^\tau - c^\tau) = q^L(u^L - c^L) + q^H \delta^{\bar{T}-1}(u^H - c^H).$$

We show that the surplus in this equilibrium approaches the surplus generated in a decentralized market equilibrium with finite T as friction vanish, $\tilde{S}^{DE}$, we establishes the proposition. In order to calculate the surplus as δ approaches, note that

$$\delta^{\bar{T}-2} > \frac{u^L - c^L}{u^H - c^L} \geq \delta^{\bar{T}-1}$$

i.e.,

$$\frac{1}{\ln \delta} \ln\left(\frac{u^L - c^L}{u^H - c^L}\right) + 1 \leq \bar{T} < \frac{1}{\ln \delta} \ln\left(\frac{u^L - c^L}{u^H - c^L}\right) + 2.$$

Since

$$\lim_{\delta \to 1}\left(\delta^{\frac{1}{\ln \delta}\ln\left(\frac{u^L - c^L}{u^H - c^L}\right) + 1}\right) = \lim_{\delta \to 1}\left(\delta^{\frac{1}{\ln \delta}\ln\left(\frac{u^L - c^L}{u^H - c^L}\right) + 2}\right) = \frac{u^L - c^L}{u^H - c^L}$$

then

$$\lim_{\delta \to 1} \delta^{\bar{T}-1} = \frac{u^L - c^L}{u^H - c^L}.$$

Substituting, we have

$$\begin{aligned}
\lim_{\delta \to 1} S^{CE} &= q^L(u^L - c^L) + q^H(u^H - c^H)\frac{u^L - c^L}{u^H - c^L} \\
&= \left[q^L + q^H(1 - \hat{q})\right](u^L - c^L) \\
&= \tilde{S}^{DE}.
\end{aligned}$$

References

[1] Akerlof, G., The Market for 'Lemons': Quality Uncertainty and the Market Mechanism, *Quarterly Journal of Economics* (1970) **84**, 488-500.

[2] Binmore, K. and M. Herrero (1988), Matching and Bargaining in Dynamic Markets, *Review of Economic Studies* (1988) **55**, 17-31.

[3] Bilancini, E., and L. Boncinelli, Dynamic adverse selection and the size of the informed side of the market, manuscript (2011).

[4] Blouin, M. , Equilibrium in a Decentralized Market with Adverse Selection, *Economic Theory* (2003) **22**, 245-262.

[5] Blouin, M., and R. Serrano , A Decentralized Market with Common Values Uncertainty: Non-Steady States," *Review of Economic Studies* (2001) **68**, 323-346.

[6] Camargo, B., and B. Lester, Trading dynamics in decentralized markets with adverse selection, manuscript (2011).

[7] Gale, D.,Limit Theorems for Markets with Sequential Bargaining, *Journal of Economic Theory* (1987) **43**, 20-54.

[8] Gale, D., Equilibria and Pareto Optima of Markets with Adverse Selection, *Economic Theory* **7**, (1996) 207-235.

[9] Jackson, M., and T. Palfrey, Efficiency and voluntary implementation in markets with repeated pairwise bargaining," *Econometrica* (1999) **66**, 1353-1388.

[10] Janssen, M., and S. Roy, Dynamic Trading in a Durable Good Market with Asymmetric information," *International Economic Theory* (2002) **43**, 257 - 282.

[11] Kim, K., Information about sellers' past behavior in the market for lemons, manuscript. 2011.

[12] Moreno, D., and J. Wooders, Prices, Delay and the Dynamics of Trade, *Journal of Economic Theory* (2002) **104**, 304-339.

[13] Moreno, D., and J. Wooders, Decentralized trade mitigates the lemons problem, *International Economic Review* (2010) **51**: 383-399.

[14] Moreno, D., and J. Wooders, The Efficiency of Centralized and Decentralized Markets for Lemons, University of Arizona Working Paper (2001), 01-03 – http://econ.arizona.edu/docs/Working_Papers/Misc%20Years/quality_y2.pdf.

[15] Morris, S., and H.S. Shin, Contagious adverse selection, manuscript 2011.

[16] Osborne, M., and A. Rubinstein, *Bargaining and Markets*, Academic Press, New York, 1990.

[17] Rubinstein, A., and A. Wolinsky, Equilibrium in a Market with Sequential Bargaining," *Econometrica* (1985) **53**, 1133-1150.

[18] Rubinstein, A., and A. Wolinsky, Decentralized Trading, Strategic Behavior and the Walrasian Outcome, *Review of Economic Studies* (1990) **57**, 63-78.

[19] Serrano, R., and O. Yosha, Decentralized information and the walrasian outcome: a pairwise meetings market with private values," manuscript, 1996.

[20] Wilson, C., The Nature of Equilibrium in Markets with Adverse Selection, *The Bell Journal of Economics* (1980) **11**, 108-130.

[21] Wolinsky, A., Information Revelation in a Market with Pairwise Meetings, *Econometrica* (1990) **58**, 1-23.

[22] Wooders, J., Walrasian Equilibrium in Matching Models," *Mathematical Social Sciences* (1998) **35**, 245-259.

Figure 1
Competitive Equilibria

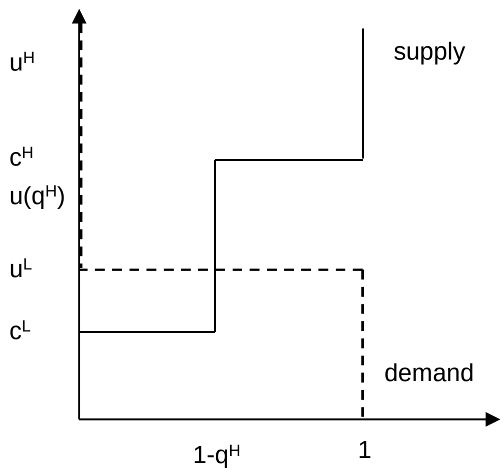

Figure 1: $c^H > u(q^H) > u^L$

Figure 2(a): High Quality Stocks

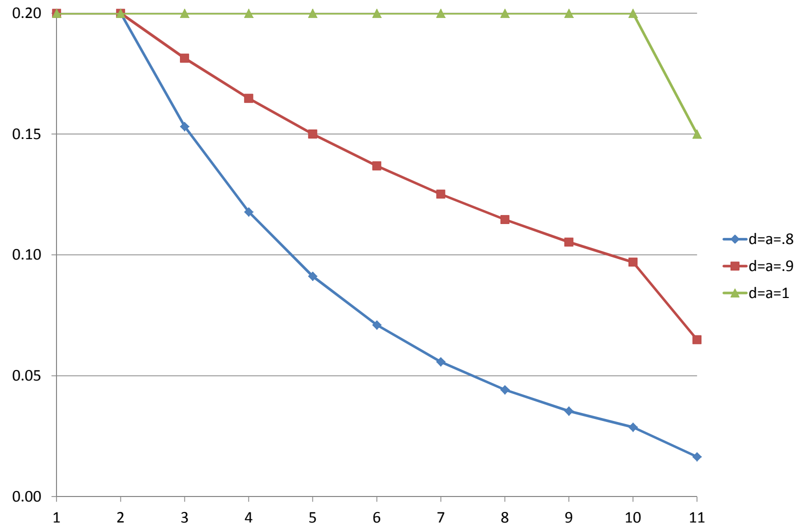

Figure 2(b): Low Quality Stocks

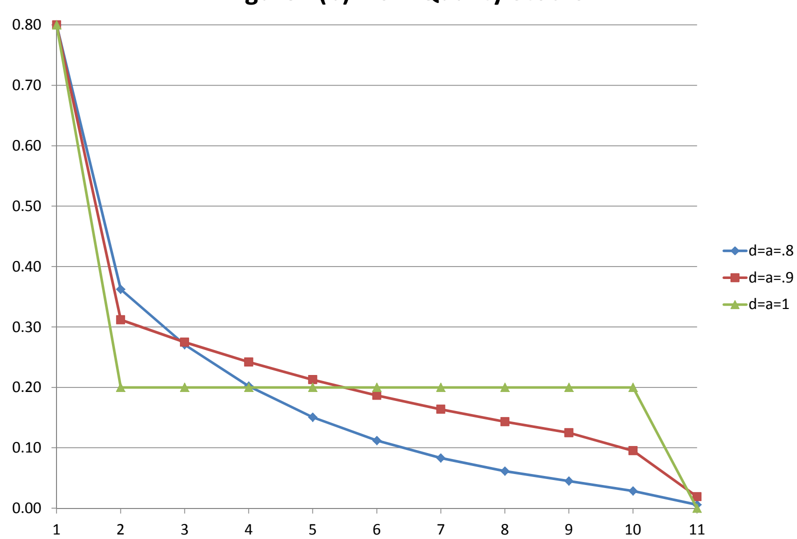

Figure 2(c): Proportion of High Quality in the Market