On Tacit Collusion among Asymmetric Firms in Bertrand Competition

Ichiro Obara
Department of Economics
UCLA

Federico Zincenko
Department of Economics
UCLA

November 11, 2011

Abstract

This paper studies a model of repeated Bertrand competition among asymmetric firms that produce a homogeneous product. The discounting rates and marginal costs may vary across firms. We identify the critical level of discount factor such that a collusive outcome can be sustained if and only if the average discount factor within the lowest cost firms is above the critical level. We also characterize the set of all efficient collusive equilibria when firms differ only in their discounting rates. Due to differential discounting, impatient firms gain a larger share of the market at an earlier stage of the game and patient firms gain a larger share at a later stage in efficient equilibrium. Although there are many efficient collusive equilibria, our model provides a unique prediction in the long run in the sense that every efficient collusive equilibrium converges to the unique efficient stationary collusive equilibrium within finite time.

JEL Classification: C72, C73, D43.
Keywords: Bertrand Competition, Collusion, Differential Discounting, Repeated Game, Subgame Perfect Equilibrium.

1 Introduction

The model of repeated Bertrand competition explains how firms may be able to collude and sustain a high price even when they produce identical goods. Thus it resolves so called “Bertrand paradox”, which would arise in one-shot interaction, that firms lose any monopoly power and make no profit as soon as two firms are present in the market. (Tirole [7]).\(^1\) Since it is a simple and very convenient model, it has been used in numerous applied works.

However, we still do not fully understand when and how collusion can be sustained except for the very special case where firms are symmetric. This assumption of symmetric firms is of course very strong and unrealistic; firms in general differ in various dimensions. What

\(^1\)There are many other ways to resolve “Bertrand paradox” such as introducing capacity constraints or differentiated demands etc.
we think is particularly strong is the assumption of equal discounting. There are at least two reasons to believe that future profit is discounted differently by different firms. First, some firms may be subject to a less favorable interest rate than others due to some kind of credit market imperfection. Second, even if the time preference is the same across firms, the time preferences of the managers who run those firms can be different. Some manager may discount future heavily if she expects to retire or be fired soon. Some manager’s preference may be more in line with the preference of the firm if she may own more stocks (and stock options) of the firm.

The goal of this paper is to understand the nature of collusion in the repeated Bertrand competition model when firms are asymmetric, especially when different firms discount future profits in different ways.

We have two main results. First we identify the critical level of discount factor such that a collusive outcome can be sustained if and only if the average discount factor within the lowest cost firms is above the critical level. More generally, we show that the necessary and sufficient condition for sustaining a collusion at a certain price (or more) is that the average discount factor of all the firms whose marginal cost is below the price must be larger than $\frac{n'-1}{n'}$, where $n'$ is the number of such firms. A more patient firm is willing to give up more market shares to more impatient firms, whose incentive constraints are then relaxed. So the distribution of discounting rates matters in general. In our simple setting with homogeneous good, the mean of discounting rates among colluding firms determines the possibility of collusion.

Our second result is a characterization of all efficient (profit-maximizing) collusive equilibria when firms differ only in their discounting rates. In efficient equilibria, more impatient firms gain a larger share of the market at an earlier stage and more patient firms gain a larger share at a later stage. Such an intertemporal substitution of the market share is subjective to the incentive constraint: we cannot assign 0% share forever even to the most impatient firm. Hence the equilibrium outcome is not the first best.

Our characterization provides a totally new picture of collusion, which is radically different from the one among symmetric firms. First, the equilibrium market share in any efficient collusive equilibrium changes over time. More specifically, the market share dynamics of each firm can be described by three phases. In the first phase, a firm has no share of the market, leaving the market to more impatient firms. In the second phase, the firm enters

\footnote{We assume that heterogeneous discounting rates are given exogenously. Of course, it would be interesting to think about a model in which they are endogenously determined for a variety of reasons. We think that our model with fixed heterogeneous discounting rates would open a possibility of building such a model.}
the market and gains all the rest after leaving more impatient firms the minimum amount of stationary market share, which correspond to the worst stationary collusive equilibrium market share for them. The final phase starts when a more patient firm enters the market. In the final phase, the firm’s marker share drops to the level that corresponds to its worst stationary collusive equilibrium market share and stays there forever.

Secondly, our results deliver the unique prediction in the long run. As described above, the equilibrium market share for each firm, except for the most patient firm, converges to its worst stationary collusive equilibrium market share in any efficient collusive equilibrium. More precisely, every efficient collusive equilibrium converges to the unique stationary collusive equilibrium within finite time.\(^3\)

We know that, with symmetric firms, there are many efficient stationary equilibria with different market shares because how to share the market is irrelevant for efficiency. With asymmetric discounting, however, efficiency imposes a sharp restriction on how the market should be allocated intertemporally. As a consequence, even though there are many efficient equilibria, the long run market share must be the same across all efficient equilibria.

From a more theoretical perspective, our results deliver new insights into the theory of repeated games with differential discounting. As reviewed briefly next, the major results for repeated games with differential discounting are restricted to asymptotic results (i.e. firms are infinitely patient) and the two-player case. In our setting, we characterize all the efficient equilibria with \(n\) players for a fixed discount factor, possibly due to some special structure of Bertrand competition game.

**Related Literature**

It is not without reason that previous works have focused on the symmetric model. First, there is the issue of equilibrium selection as mentioned. There are always many equilibria - hence there is always the issue of equilibrium selection - in repeated games. The model of dynamic Bertrand competition is no exception. For symmetric models, it might make sense to focus on the symmetric (and efficient) equilibrium, possibly as a focal point. However, it is not clear which equilibrium would be a focal point when firms are asymmetric. Secondly, the theory of repeated games with differential discounting is still at its development stage. For these reasons, there are not many works that study collusion among heterogeneous firms. In our view, this fact limits the scope of applications of the repeated Bertrand competition model.

\(^3\)The time to reach the efficient stationary collusive equilibrium is bounded across all efficient collusive equilibria for a given profile of discounting rates.
One notable exception is Harrington (1989) [4]. It shows that a stationary collusive equilibrium can be sustained with differential discounting if and only if the average discount factor exceeds some critical level. Our first result builds on and improves on this result. We provide a more complete characterization regarding the possibility of collusion by considering all equilibria including nonstationary ones. Clearly it is important to consider nonstationary equilibria because almost all stationary equilibria are not efficient with differential discounting as our second result shows. Another difference between our paper and [4] is that we obtain the unique equilibrium in the long run. To cope with the issue of multiple stationary equilibria, Harrington [4] uses a bargaining solution to select one equilibrium. On the other hand, we show that the long run equilibrium behavior is the same across all efficient equilibria. Thus we do not need to rely on any equilibrium selection criterion other than efficiency as long as we are concerned with the long-run outcome.

The seminal contribution in the theory of repeated game with differential discounting is Lehrer and Pauzner (1999) [5]. It studies a general two-player repeated game with differential discounting and shows that the set of feasible payoffs is larger than the convex hull of the underlying stage game payoffs because players can mutually benefit from trading payoffs across time. They also characterize the limit equilibrium payoff set as discount factors go to 1 keeping their ratio fixed. In particular, they show that there is some individually rational and feasible payoff that cannot be sustained in equilibrium no matter how patient the players are.

There are some recent contributions in the theory of repeated games with differential discounting. Chen [1] and Gueron et. al [3] study stage games with one dimensional payoffs. Sugaya [6] proves a folk theorem for repeated games with imperfect monitoring and with differential discounting. Fong and Surti [2] study repeated prisoner’s dilemma games with differential discounting and with side payments. This paper seems to be particularly related to our paper because we use market share as a way to transfer utility.

This paper is organized as follows. We describe the model in detail in the next section. In section 3, we prove our first result regarding the critical average discount factor. In section 4, we characterize efficient equilibria. We conclude and discuss potential extensions of our results in the last section. Most of the proofs are relegated to the appendix.

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4Since a collusive outcome can be sustained by a stationary equilibrium when the average discount factor exceeds the critical level, the crucial step for our result is to show that no nonstationary collusive equilibrium exists when the average discount factor is below the same critical level.
2 Model of Repeated Bertrand Competition with Differential Discounting

This section describes the basic structure of our model, an infinitely repeated Bertrand game. In what follows, we first define the stage game, then construct the infinitely repeated game.

The main features of the stage game are the followings. The players are $n \geq 2$ firms represented by the numbers $I = \{1, 2, ..., n\}$. They offer a homogeneous product whose market demand is characterized by continuous function $D : \mathbb{R}^+ \to \mathbb{R}^+$. Each firm has a linear cost function $C_i : \mathbb{R}^+ \to \mathbb{R}^+$ given by $C_i(q_i) = c_i q_i$, where $i \in I$, $c_i \geq 0$ is the marginal cost, and $q_i$ indicates the quantity produced by firm $i$. We suppose that $c_1 \leq c_2 \leq ... \leq c_n$ without loss of generality and denote $I_* = \{i \in I : c_i = c_1\}$ and $n_* = \#(I_*)$. We assume that $n_* \geq 2$. Hence, in one-shot Bertrand competition, the market price would be $c_1$ and no firm would make any profit. It is assumed that the demand function satisfies the following regularity conditions: $D$ is decreasing on $(0, \infty)$; there exists the monopoly price for each firm: $p_i^m > c_i$ for firm $i$ that maximizes $p(D(p) - c_i)$. We assume that the marginal costs are not very different: even the highest cost $c_n$ is less than $p_1^m$. This implies that $p_i^m > c_j$ for any $i, j \in I$.

At the beginning of a stage game, firms make price decisions and suggest how to allocate output quotas in case of a draw in prices. If a firm charges a price that is higher than a price charged by another firm, then the firm’s market share is 0. The firm that charges the lowest price must produce enough output to satisfy the market demand. In case there are more than one firm that charges the lowest price, the market is allocated among those firms according to their suggestions. Formally, firm $i$’s pure action is given by a 2-tuple $a_i = (p_i, r_i) \in A_i$, where $p_i$ is the price choice, $r_i$ reflects firm $i$’s request of market share in case of tie. Hence $A_i = \mathbb{R}^+ \times [0, 1]$ is the set of pure actions available for player $i$. The set of pure action profiles is $A = \prod_{i \in I} A_i$. Firm $i$’s profit function $\pi_i : A \to \mathbb{R}$ can be written as

$$\pi_i[a] = \begin{cases} D(p_i)(p_i - c_i) & \text{if } p_i < p^*_i, \\ \frac{r_i}{|\hat{I}|} D(p_i)(p_i - c_i) & \text{if } p_i = p^*_i \text{ and } R^* \neq 0, \\ \frac{1}{|\hat{I}|} D(p_i)(p_i - c_i) & \text{if } p_i = p^*_i \text{ and } R^* = 0, \\ 0 & \text{if } p_i > p^*_i, \end{cases}$$

where $p^*_i = \min_{j \neq i} p_j$, $\hat{I} = \{i \in I : p_i = \min_{j \in I} p_j\}$, and $R^* = \sum_{j \in \hat{I}} r_j$.

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\footnote{If the marginal of some firm is too high, it is likely that the presence of such a firm is irrelevant for our analysis.}
Given the stage game described above, we now define the infinitely repeated game. Basically, we adopt a discrete time model in which the previous stage game is played in each of the periods \( t \in \mathbb{N} \). The distinguishing feature of our dynamic Bertrand competition model is that the players have different discount factors given by \( \delta_i \in (0, 1), i \in \mathcal{I} \).

The set of possible histories in period \( t \) is given by \( H^t = A^{t-1} \), where \( A^0 \) indicates the empty set, and \( A^t \) denotes the \( t \)-fold product of \( A \). A period \( t \)-history is thus a list of \( t - 1 \) action profiles. We suppose perfect monitoring throughout, i.e., at the end of each period, all players observe the action profile chosen in all the previous periods. Setting \( H = \bigcup_{t \in \mathbb{N}} H^t \), a pure strategy for firm \( i \) is defined as a mapping \( \sigma_i : H \rightarrow A_i \), and consequently, a strategy profile is given by \( \sigma = (\sigma_i)_{i \in \mathcal{I}} \).

Each strategy profile \( \sigma \) induces an infinite sequence of action profiles \( a(\sigma) = (a^t(\sigma))_{t \in \mathbb{N}} \in A^\infty \), where \( a^t(\sigma) \in A \) denotes the action profile induced by \( \sigma \) in period \( t \). We call the sequence \( a(\sigma) \) outcome path (or more simply, outcome) generated by a strategy profile \( \sigma \). Finally, for a given strategy profile \( \sigma \), and its corresponding outcome path \( a(\sigma) = (a^t(\sigma))_{t \in \mathbb{N}} \), the time-average repeated game payoff for firm \( i \) at time \( t \) is

\[
U_{i,t}[a(\sigma)] = (1 - \delta_i) \sum_{\tau=t}^{\infty} \delta_i^{\tau-t} \pi_i[a^\tau(\sigma)].
\]

In the following sections, we will just focus on subgame perfect equilibrium solutions, and we will limit our attention to pure strategy equilibria.

### 3 Critical Average Discount Factor for Collusion

In this section, we derive a necessary and sufficient condition to sustain a collusive equilibrium outcome. We say that the firms are colluding when there is at least one period in which the equilibrium outcome is not a competitive one, i.e. when there is at least one firm that makes positive profit in some period. We formalize this as follows.

**Definition 1.** An outcome \( a = (a^t)_{t \in \mathbb{N}} \) is considered a **collusive outcome** if and only if there exists \( t' \in \mathbb{N} \) such that \( \pi_i(a^{t'}) > 0 \) for some \( i \in \mathcal{I} \). A **collusive equilibrium** is a subgame perfect equilibrium that generates a collusive outcome.

Then we can obtain the following sharp characterization, which says that a collusive outcome can be sustained if and only if the average discount factor among the lowest cost firms is above some threshold.
Theorem 3.1. There exists a collusive equilibrium if and only if

\[ \frac{\sum_{i \in I} \delta_i}{n^*} \geq \frac{n^* - 1}{n^*}. \]

Proof. See the appendix.

When the firms are symmetric, there exists a collusive equilibrium if and only if \( \delta \geq \frac{n-1}{n} \). Thus our result is a substantial generalization of this well-known result to the case with heterogeneous discounting and costs.

It follows from the result in [4] that \( \frac{n^*-1}{n^*} \) is the critical threshold to support a collusive outcome by a stationary collusive equilibrium, i.e., an equilibrium in which each firm keeps a certain level of market share every period and the price is always the same. Take any price \( p \) strictly between the minimum cost \( c^* = \min_{i \in I} c_i \) and the next smallest cost. There exists a stationary collusive equilibrium by the lowest cost firms in which the market price is always \( p \) and firm \( i(\in I^*) \) gains share \( \alpha_i \in [0,1] \) of the joint profit \( \pi \) in every period if the following inequalities are satisfied for all \( i \in I^* \).

\[ (1 - \delta_i)\pi \leq \alpha_i\pi \]

By dividing both sides by \( \pi \) and summing up these inequalities across the firms, it can be shown that such \( \alpha_i, i \in I^* \) exists if and only if the average discount factor among the lowest cost firms is larger than or equal to \( \frac{n^*-1}{n^*} \).

A more difficult part of the proof is to show that collusion is impossible when the average discount factor is less than \( \frac{n^*-1}{n^*} \), even if nonstationary equilibria are considered. In nonstationary equilibrium, it is possible to transfer market shares over time to generate larger continuation profits in the future, which may enable the firms to sustain collusion. It turns out that such transfer does not work. To improve efficiency, it is necessary to let less patient firms to gain more market shares first and let more patient firms to gain more shares later. Intuitively, such an arrangement is in conflict with less patient firms’ incentive constraints in later periods.

Here is a sketch of our formal proof. We assume that the marginal cost is the same across all firms to simplify our exposition. Firm \( i \)'s incentive constraint in period \( t \) is given by the equality

\[ U_{i,t} = (1 - \delta_i)\pi_i(a_t) + \delta_iU_{i,t+1} = (1 - \delta_i)\pi^*(a_t) + \eta_{i,t} \]
where $a^t$ is the action profile in period $t$, $\pi^*(a^t) = \sum_i \pi_i(a^t)$ is the joint profit in period $t$, $U_{i,t+1}$ is firm $i$'s continuation profit from period $t+1$ on, and $\eta_{i,t} \geq 0$ is a slack variable (firm $i$’s incentive constraint is binding in period $t$ if and only if $\eta_{i,t} = 0$). Note that each firm gains the same equilibrium join profit by price-cutting because the cost is assumed to be the same. Since this equality holds in every period, we can replace $U_{i,t+1}$ with $(1 - \delta_i) \pi^*(a^{t+1}) + \eta_{i,t+1}$ and divide both sides by $1 - \delta_i$ to obtain

$$\pi_i(a^t) + \delta_i \pi^*(a^{t+1}) = \pi^*(a^t) + \frac{\eta_{i,t} - \delta_i \eta_{i,t+1}}{1 - \delta_i}.$$

Summing up these equalities across the firms, we have the following equation regarding $\pi^*(a^t)$:

$$\pi^*(a^{t+1}) = \frac{n - 1}{\sum_{i \in I} \delta_i} \pi^*(a^t) + \frac{1}{\sum_{i \in I} \delta_i} \sum_{i \in I} u_{i,t},$$

where $u_{i,t} = \frac{\eta_{i,t} - \delta_i \eta_{i,t+1}}{1 - \delta_i}$.

The coefficient of $\pi^*(a^t)$ is larger than 1 if and only if the average discount factor is less than $\frac{n - 1}{n}$. In fact, we can show that, when the joint profit is strictly positive in some period, the sequence $\pi^*(a^t), t = 1, 2, ..$ must diverge to infinity, which is a contradiction. To prove this formally, however, we need to examine carefully the behavior of $\sum_{i \in I} u_{i,t}, t = 1, 2, 3, ..$

A collusive equilibrium we construct uses a price between the lowest cost and the second lowest cost, so it is not very profitable when this difference between them is small. In such a case, the lowest cost firms would prefer to include the second lowest cost firm(s) in their coalition to raise the equilibrium price. Our result can be easily generalized to accommodate such possibility. Let $p$ be any price. Let $I(p)$ be the set of firms such that $c_i \leq p$ if and only if $i \in I(p)$ and $|I(p)| = n_s(p)$. Call a subgame perfect equilibrium $p$-collusive equilibrium if the equilibrium price is always at least as large as $p$. We can prove the following generalization of the above result.

**Theorem 3.2.** For any $0 < p \leq p^m$, there exists a $p$-collusive equilibrium if and only if

$$\frac{\sum_{i \in I(p)} \delta_i}{n_s(p)} \geq \frac{n_s(p) - 1}{n_s(p)}.$$

The proof is almost the same, hence omitted.
When $I_\ast = 1$, i.e. there is the unique lowest cost firm, Theorem 2 still holds. But we need to rely on a less natural punishment. The assumption $I_\ast \geq 2$ guarantees that any deviation from a collusive outcome is punished by Nash reversion with 0 profit forever. If $I_\ast = 1$, the 0 profit equilibrium requires that there are at least two firms charging $c_1$, but firm 1 serves the whole market ($r_1 = 1, r_i = 0$ for all $i \neq 1$).

4 Characterization of Efficient Collusive Equilibria

In this section, we characterize efficient collusive equilibria with differential discounting rates. We assume that the marginal cost is the same across firms and normalize it to 0. Then the monopoly price can be determined without any ambiguity. Let $p^m$ be the monopoly price and $\pi^m$ be the monopoly profit. We also assume that $0 < \delta_1 < \delta_2 < \ldots < \delta_{n-1} < \delta_n < 1$ for the sake of simplicity. The result can be easily extended to the case where the discounting factors of some firms are the same.

Let $\pi_{i,t}, i = 1, \ldots, n, t \in \mathbb{N}$ be a sequence of profits associated with any collusive equilibrium. By definition, they satisfy the following incentive compatibility condition in every period:

$$(1 - \delta_i) \pi_t \leq U_{i,t}$$

where $U_{i,t}$ is firm $i$’s equilibrium continuation profit in the beginning of period $t$ and $\pi_t = \sum_i \pi_{i,t}$. On the other hand, it is clear that any sequence of profit profiles that satisfy those conditions are generated by a collusive equilibrium. Hence we use such a sequence of profit profiles to describe any collusive equilibrium.

A collusive equilibrium is efficient if there is no subgame perfect equilibrium that makes every firm better off weakly and some strictly. Observe that $\pi_t$ is always in $(0, \pi^m]$ for any efficient collusive equilibrium. $\pi_t$ cannot exceed the monopoly profit by definition. If $\pi_t < 0$, then we can construct a more efficient equilibrium by just dropping period $t$.

We know that there exists a stationary collusive equilibrium with monopoly price if and only if $\frac{\sum_{i=1}^{n-1} \delta_i}{n} \geq \frac{n-1}{n}$. When the average discount factor is strictly larger than $\frac{n-1}{n}$, there is a range of market shares that can be supported by stationary collusive equilibrium. Let $\hat{\pi}_i$ be firm $i$’s per period profit in the worst stationary collusive equilibrium profit for firm $i$. Note that $\hat{\pi}_i = (1 - \delta_i) \pi^m$ by the incentive compatibility condition. We assume $\frac{\sum_{i=1}^{n} \delta_i}{n} > \frac{n-1}{n}$ for the rest of this section.

We first prove that, in any efficient collusive equilibrium, the joint profit must be strictly increasing until it reaches the monopoly profit and stays there forever.
We start with the following lemma.

**Lemma 4.1.** Consider any efficient collusive equilibrium where, for some \( t \geq 1 \), \( \pi_{t+1} < \pi^m \) and there is a firm \( i' \) such that \( U_{i',t+1} > (1 - \delta) \pi_{t+1} \) and \( \pi_{t+1} > 0 \). Then \( \pi_{t+1} \geq \frac{\pi}{\delta} n \).

**Proof.** Define \( \tilde{I}_{t+1} = \{ i \in I : U_{i,t+1} = (1 - \delta) \pi_{t+1} \} \), which is not empty (otherwise the joint profit can be increased to improve efficiency). Suppose that \( \pi_{t+1} < \frac{\pi}{\delta} n \). Then \( \pi_{i,t} \geq \pi_t - \delta \pi_{t+1} \geq \pi_t - \delta_n \pi_{t+1} > 0 \) for all \( i \in \tilde{I}_{t+1} \). Consequently, the profits can be perturbed as follows: \( \pi_{i,t}' = \pi_{i,t} - \delta \varepsilon \) and \( \pi_{i,t+1}' = \pi_{i,t+1} + \varepsilon \), for \( i \in \tilde{I}_{t+1} \); whereas \( \pi_{i',t}' = \pi_{i',t} + \sum_{i \in \tilde{I}_{t+1}} \delta_i \varepsilon \) and \( \pi_{i',t+1}' = \pi_{i',t+1} - (|\tilde{I}_{t+1}| - 1) \varepsilon \). Since \( \pi_{t+1} < \pi^m \) and \( \pi_{i',t+1} > 0 \), this new allocation is feasible and incentive compatible for \( \varepsilon > 0 \) small enough. Moreover, as \( \sum_{i \in \tilde{I}_{t+1}} \delta_i > |\tilde{I}_{t+1}| - 1 \), it also Pareto-dominates the initial one. This is a contradiction. \( \Box \)

The next theorem proves a strong monotonicity property for efficient collusive equilibria.

**Theorem 4.1.** For any efficient collusive equilibrium, there exists \( T \) such that \( \pi_t < \pi_{t+1} \) for \( t = 1, ..., T - 1 \) and \( \pi_T = \pi^m \) for any \( t \geq T \). Furthermore, this \( T \) is bounded across all efficient collusive equilibria.

**Proof.** Take any efficient collusive equilibrium. Let \( \pi_t \in (0, \pi_m) \) be a joint profit in any period \( t \). We assume that \( \pi_t > \delta_n \pi_{t+1} \) and \( \pi_{t+1} < \pi^m \), and derive a contradiction. If those two conditions are satisfied, then it must be the case that \( \pi_{t+1} = \sum_{i \in \tilde{I}_{t+1}} \pi_{i,t+1} \) by Lemma 4.1. Therefore, there is \( j \in \tilde{I}_{t+1} \) such that \( \pi_{j,t+1} > (1 - \delta_j) \pi_{t+1} \), otherwise,

\[
\pi_{t+1} = \sum_{i \in \tilde{I}_{t+1}} \pi_{i,t+1} \leq \pi_{t+1} \sum_{i \in \tilde{I}_{t+1}} (1 - \delta_i) = \pi_{t+1} (|\tilde{I}_{t+1}| - \sum_{i \in \tilde{I}_{t+1}} \delta_i),
\]

but \( \sum_{i \in \tilde{I}_{t+1}} \delta_i > |\tilde{I}_{t+1}| - 1 \).

As a result, \( (1 - \delta_j) \pi_{t+2} \leq U_{j,t+2} < (1 - \delta_j) \pi_{t+1} \). The first inequality is derived from the incentive constraint in period \( t + 2 \), whereas the second one from the fact that \( \pi_{j,t+1} > (1 - \delta_j) \pi_{t+1} \) and \( U_{j,t+1} = (1 - \delta_j) \pi_{t+1} \). Then, \( \pi_{t+1} > \pi_{t+2} \). We can proceed in a similar manner to obtain \( \pi_{t+k} > \pi_{t+k+1} \) for every \( k \geq 1 \), which contradicts the efficiency assumption. Hence it must be the case that either \( \pi_t \leq \delta_n \pi_{t+1} \) or and \( \pi_{t+1} = \pi^m \). Clearly this implies that there is \( T \) such that \( \pi_t < \pi_{t+1} \) for \( t = 1, ..., T - 1 \) and \( \pi_T = \pi^m \) for any \( t \geq T \).

Finally we prove that this \( T \) is bounded across all efficient equilibria. For any given \( T \), each firm’s profit per period is at most \( \delta_n^{T-1} \pi^m \) for the first \( T - T' \) periods for any \( T' \leq T \). If \( T \) is large, then firm \( i' \)’s payoff is less than \( \tilde{\pi} \). Such payoff profile is Pareto-dominated by any stationary collusive equilibrium. \( \Box \)
Next we provide an (almost) complete characterization of efficient collusive equilibria. Consider any efficient collusive equilibrium where firm $i$’s equilibrium profit exceeds $\hat{\pi}_i$. Then every firm’s incentive constraint is not binding in the first period, hence the equilibrium joint profit must be $\pi^m$ in the first period. Given our monotonicity result, this implies that the equilibrium price is always $p^m$ for this class of efficient collusive equilibria. We call such collusive equilibrium $p^m$-efficient collusive equilibrium.

The next theorem characterizes the structure of $p^m$-efficient collusive equilibrium. Observe that this characterization is a complete characterization of the asymptotic behavior of all efficient collusive equilibria, because every efficient collusive equilibrium converges to some $p^m$-efficient collusive equilibrium eventually within finite time by our previous result.

In $p^m$-efficient collusive equilibrium, more patient firms “lend” the market share initially to more impatient firms. However, the ability of impatient firms to “pay back” the market share is limited by the requirement that each firm’s profit cannot be lower than its worst stationary equilibrium profit $\hat{\pi}_i$.

**Theorem 4.2.** Every $p^m$-efficient collusive equilibrium has the following structure: there exists $t_1 \leq t_2 \leq \ldots \leq t_{n-1}$ such that, for every $i$,

1. $\pi_{i,t} = 0$ for every $t < t_{i-1}$
2. $\pi_{i,t} \in [0, \pi^m - \sum_{h=1}^{i-1} \hat{\pi}_h]$ for $t = t_{i-1}$
3. $\pi_{i,t} = \pi^m - \sum_{h=1}^{i-1} \hat{\pi}_h$ for $t = t_{i-1} + 1, \ldots, t_i - 1$
4. $\pi_{i,t} \in [\hat{\pi}_i, \pi^m - \sum_{h=1}^{i-1} \hat{\pi}_h]$ for $t = t_i$
5. $\pi_{i,t} = \hat{\pi}_i$ for $t > t_i$

6. **Incentive Constraints in the first period**

$$\delta^{t_{i-1}-1}_i \left[ (1 - \delta_i) \pi_{i,t_{i-1}} + \left( \delta_i - \delta^{t_{i-1} - t_{i-1}-1}_i \right) \left\{ \pi^m - \sum_{h=1}^{i-1} \hat{\pi}_h \right\} \right]$$

$$+ \delta^{t_{i-1}-1}_i \left[ (1 - \delta_i) \delta^{t_{i-1} - t_{i-1}+1}_i \hat{\pi}_{i,t_i} + \delta^{t_{i-1} - t_{i-1}+1}_i \hat{\pi}_i \right]$$

$$\geq (1 - \delta_i) \pi^m$$

Furthermore, if there exist $(t_1, t_2, \ldots, t_{n-1})$ and a sequence of profit profiles $\pi_{i,t}$ that satisfy the above conditions, then there exists a corresponding $p^m$-efficient collusive equilibrium that generates them.
Proof. See the appendix.

In words, every $p^m$-efficient collusive equilibrium has the following properties.

- From period 1 to period $t_1-1$, firm 1 gets the whole share.
- In period $t_1$, firm 1 and 2 shares the market where $\pi_{i,t_1} \geq \hat{\pi}_1$. After this period, firm 1’s share is going to be always $\hat{\pi}_1$.
- From period $t_1+1$ to period $t_2-1$, firm 2 gets $\pi^m - \hat{\pi}_1$.
- In period $t_2$, firm 2 and 3 shares the market where $\pi_{i,t_2} \geq \hat{\pi}_2$. After this period, firm 2’s share is going to be always $\hat{\pi}_2$.
- From period $t_2+1$ to period $t_3-1$, firm 3 gets $\pi^m - \hat{\pi}_1 - \hat{\pi}_2$.
- ... 
- After period $t_{n-1}$, firm $n$ gets $\pi^m - \sum_{h=1}^{n-1} \hat{\pi}_h$ and firm $h < n$ gets $\hat{\pi}_h$ forever.

There are two critical periods for firm $i$: $t_{i-1}$ and $t_i$. Up to $t_{i-1}$, firm $i$’s market share is 0. The periods between $t_{i-1}$ and $t_i$ is the pay back time when firm $i$ gets all the market share subject to the constraint that each less patient firm $h < i$ gains $\hat{\pi}_h$. After $t_i$, firm $i$’s profit is reduced to $\hat{\pi}_i$ and stay there forever. It may be the case that there is some overlap: $t_k = t_{k+1} = ... = t_m = t'$ for some $m > k$. Note that $\pi_{i,t'} \geq \pi_i$ for $i = t_k, t_{k+1}, ..., t_{m-1}$ in such a case.

Comment

- One implication of our theorem is that there exists the unique efficient stationary collusive equilibrium, to which every efficient collusive equilibrium converges. This is the stationary collusive equilibrium where the price is $p^m$, firm $i$’s market share is $\hat{\pi}_i$ for $i = 1, ..., n - 1$ and firm $n$’s market share is $\pi^m - \sum_{i=1,...,n-1} \pi_i$ in every period, which corresponds to the worst stationary collusive equilibrium for firm $i = 1, ..., n - 1$ (and the best one for firm $n$). All the other efficient collusive equilibria must be nonstationary.
- Our result delivers the unique prediction in the long run without any equilibrium selection. This is not the case if we focus on stationary collusive equilibria.
When \( \delta_i = \delta_{i+1} \) for some \( i \), their market share is characterized by similar conditions: their market share is 0 before \( t_{i-1} \), \( \pi_{i,t} = \pi_{i-1,t} = \pi_i(= \pi_{i+1}) \) after \( t_i \), and can be somewhat arbitrary between \( t_i \) and \( t_{i-1} \) (but we can assume that their market shares are constant within these periods without loss of generality).

5 Conclusion and Discussion

In the context of Bertrand price competition in an infinitely repeated game, this paper studies collusive behavior among \( n \) firms with asymmetric discount factors and asymmetric marginal costs.

We find that it is possible to sustain a collusion if and only if the average discount factor of the lowest cost firm is at least as large as \((n^* - 1)/n^*\), where \( n^* \) is the number of the lowest cost firms.

This paper also studies efficient collusive equilibria among \( n \) firms with differential discounting when the marginal cost is the same across firms. We succeed in characterizing the structure of efficient collusive equilibria. More specifically, we show the followings.

- In any efficient collusive equilibrium, the joint profit must be strictly increasing over time until it reaches the monopoly profit level within finite time and stay there forever.

- Every efficient collusive equilibrium where no firm’s payoff is not too low must be a collusive equilibrium with the monopoly price in every period (“\( p^m \)-efficient collusive equilibrium”). In every \( p^m \)-efficient collusive equilibrium, a firm’s market share is 0 initially, reaches a peak, then converges to the market share that corresponds to the worst stationary collusive equilibrium with the monopoly price (except for the most patient firm).

- Every efficient collusive equilibrium converges to the unique efficient stationary collusive equilibrium in the long run, where the equilibrium price is \( p^m \), firm \( i \)’s profit per period is \( \hat{\pi}_i \) for \( i = 1, \ldots, n - 1 \) and \( 1 - \sum_{i=1}^{n-1} \hat{\pi}_i \) for \( i = n \) in every period.
References


Appendix

Proof of Theorem 3.1

Proof. We already discussed that there exists a collusive stationary subgame perfect equilibrium when the inequality is satisfied. Thus we just need to show that there is no collusive subgame perfect equilibrium when \( \sum_{i \in I^*} \delta_i n^* < n^* - \frac{1}{n^*} \).

By contradiction, begin by assuming that \( \tilde{a} = (\tilde{a}_t)_{t \in \mathbb{N}} \) is a collusive equilibrium outcome, and without loss of generality, assume that \( \pi^*(\tilde{a}_1) = \sum_{i \in I} \pi(\tilde{a}_t) > 0 \).

Note first that for each \( i \in I^* \), there exists a bounded nonnegative sequence \( \{\eta_{i,t} : t \in \mathbb{N}\} \) defined by \( \eta_{i,t} = U_{i,t}(\tilde{a}) - (1 - \delta_i)\pi^*(\tilde{a}_t) \). Moreover, since \( U_{i,t}(\tilde{a}) = (1 - \delta_i)\pi_i(\tilde{a}_t) + \delta_i U_{i,t}(\tilde{a}) \), we have that

\[
(1 - \delta_i)\pi_i(\tilde{a}_t) + \delta_i U_{i,t}(\tilde{a}) = (1 - \delta_i)\pi^*(\tilde{a}_t) + \eta_{i,t},
\]

and therefore

\[
(1 - \delta_i)\pi_i(\tilde{a}_t) + \delta_i [(1 - \delta_i)\pi^*(\tilde{a}_t + 1) + \eta_{i,t + 1}] = (1 - \delta_i)\pi^*(\tilde{a}_t) + \eta_{i,t},
\]

or equivalently,

\[
\pi_i(\tilde{a}_t) = \left[ \pi^*(\tilde{a}_t) + \frac{\eta_{i,t}}{1 - \delta_i} \right] - \delta_i \left[ \pi^*(\tilde{a}_t + 1) + \frac{\eta_{i,t + 1}}{1 - \delta_i} \right] .
\]

Adding up this inequality across \( i \in I^* \) and denoting \( s_* = \sum_{i \in I^*} \delta_i \), we obtain

\[
\pi^*(\tilde{a}_t) = n_* \pi^*(\tilde{a}_t) - s_* \pi^*(\tilde{a}_{t + 1}) + \sum_{i \in I^*} \frac{\eta_{i,t} - \delta_i \eta_{i,t + 1}}{1 - \delta_i},
\]

or more shortly,

\[
\pi^*(\tilde{a}_{t + 1}) = \gamma \pi^*(\tilde{a}_t) + \frac{1}{s_*} \sum_{i \in I^*} u_{i,t},
\]

where \( \gamma = (n_* - 1)/s_* \) and \( u_{i,t} = (\eta_{i,t} - \delta_i \eta_{i,t + 1})/(1 - \delta_i) \).

Before proceeding, it is useful to note that \( \gamma > 1 \) and therefore

\[
\pi^*(\tilde{a}_{t + 1}) \geq \pi^*(\tilde{a}_t) + \frac{1}{s_*} \sum_{i \in I^*} u_{i,t} \geq \pi^*(\tilde{a}_t) + \frac{1}{s_*} \sum_{i \in I^*} \sum_{j=1}^t u_{i,j},
\]

for every \( t \in \mathbb{N} \).
Now consider the series $\sum_{j=1}^{t} u_{i,j}$ for $i \in \mathcal{I}_s$, and observe that it maybe written as

$$\sum_{j=1}^{t} u_{i,j} = \frac{\eta_i^{(1)}}{(1 - \delta_i)} + \sum_{j=2}^{t} \eta_i^{(j)} - \frac{\delta_i \eta_i^{(t+1)}}{(1 - \delta_i)}.$$ 

Since the equilibrium profit is bounded above for each firm by assumption, the equilibrium profit is bounded below as well for each firm; otherwise the average discounted profit is negative. This implies that there exists $M^*$ such that $0 \leq \eta_i^{(j)} \leq M^*$ for all $j \in \mathbb{N}$ and $i \in \mathcal{I}_s$. Observe that this implies that the series $\sum_{j=2}^{t} \eta_i^{(j)}$ must be either unbounded above or converging to a finite (nonnegative) real number.

Suppose first that $\sum_{j=2}^{\infty} \eta_i^{(j)}$ is unbounded above for some $i \in \mathcal{I}_s$. On the one hand, we know that $\sum_{i \in \mathcal{I}_s} (\sum_{j=1}^{t} u_{i,j})$ is unbounded above, too. On the other hand, we have that

$$\pi_s(\tilde{a}^{t+1}) \geq \pi_s(\tilde{a}^1) + (1/s^*) \sum_{i \in \mathcal{I}_s} \sum_{t=1}^{t} u_{i,t},$$

which is a contradiction because the sequence \{\pi_s(\tilde{a}^i) : i \in \mathbb{N}\} is bounded above.

Suppose now that $\sum_{j=2}^{\infty} \eta_i^{(j)}$ is finite for all $i \in \mathcal{I}_s$. Then we have that $\eta_i^{(t)}$ as well as $u_{i,t}$ converge to zero for all $i \in \mathcal{I}_s$. If $\eta_i^{(j)} = 0$ for all $i \in \mathcal{I}_s$ and $j \in \mathbb{N}$, it follows immediately that $\sum_{i \in \mathcal{I}_s} (\sum_{j=1}^{t} u_{i,t}) \geq 0$. On the other hand, if $\eta_i^{(t)} = c_i > 0$ for some $i \in \mathcal{I}_s$ and $t_i \in \mathbb{N}$, there exists $T_i > t_i$ such that $|\eta_i^{(t)}| < c_i(1 - \delta_i)/(2\delta_i)$ for all $t \geq T_i$. As a result, we have that

$$\sum_{j=1}^{t} u_{i,j} \geq \frac{\eta_i^{(1)}}{(1 - \delta_i)} + c_i - \frac{c_i}{2},$$

when $t > T_i$. As $\mathcal{I}_s$ is a finite set, there is $T \in \mathbb{N}$ (independent of $i$) such that $\sum_{i \in \mathcal{I}_s} (\sum_{j=1}^{t} u_{i,t}) \geq 0$ for all $t \geq T$, and consequently, $\pi_s(\tilde{a}^t) \geq \pi_s(\tilde{a}^1)$ as long as $t \geq T$.

Before proceeding, observe first that there is $\tilde{t} \in \mathbb{N}$ such that $\gamma^\tilde{t} \pi_s(a^1) > 2M$. Secondly, since $\mathcal{I}_s$ is a finite set and $u_{i,t}$ converge to zero for each $i \in \mathcal{I}_s$, there exists $\tilde{T} \in \mathbb{N}$ (independent of $i$) such that $\tilde{T} > T$ and $|u_{i,t}| < (s^* M^*)/(n^* T\gamma)$ for all $i \in \mathcal{I}_s$ and $t \geq \tilde{T}$.

The following inequality is a straightforward implication:

$$\pi_s(\tilde{a}^{\tilde{T}+t}) = \gamma \pi_s(\tilde{a}^{\tilde{T}+t-1}) + \frac{1}{s^*} \sum_{i \in \mathcal{I}_s} u_{i,\tilde{T}+t-1} \geq \gamma \pi_s(\tilde{a}^{\tilde{T}+t-1}) - \frac{M}{\tilde{T}\gamma},$$

and by induction, we can prove that

$$\pi_s(\tilde{a}^{\tilde{T}+t}) \geq \gamma^t \pi_s(\tilde{a}^{\tilde{T}}) - \sum_{j=0}^{t-1} \frac{M^*}{\tilde{T}\gamma^{t-j}},$$
for every $t \in \mathbb{N}$. Finally, after replacing $t = \tilde{t}$ in the previous inequality, we obtain the desired result:

$$\pi_\star(\tilde{a}^{\tilde{t}+\tilde{t}}) \geq \gamma^{\tilde{t}}\pi_\star(\tilde{a}^{\tilde{t}}) - \sum_{j=0}^{\tilde{t}-1} \frac{M^*}{t_{\tilde{t}-j}}$$

$$> \gamma^{\tilde{t}}\pi_\star(\tilde{a}^{\tilde{t}}) - \sum_{j=0}^{\tilde{t}-1} \frac{M^*}{t}$$

$$> 2M^* - M^*.$$

The second inequality follows by $\pi_\star(\tilde{a}^{\tilde{t}}) \geq \pi_\star(\tilde{a}^{\tilde{t}})$ and $\gamma > 1$, whereas the last one by $\gamma^{\tilde{t}}\pi_\star(\tilde{a}^{\tilde{t}}) > 2M^*$. And obviously, this is a contradiction because $\pi_\star(\tilde{a}^{\tilde{t}+\tilde{t}}) \leq M^*$.

\[\square\]

**Proof of Theorem 4.2**

We prove the theorem through a series of lemmata.

**Lemma 5.1.** For any efficient subgame perfect equilibrium, if firm $i$’s incentive constraint is not binding in period $t > 1$, then $\pi_{j,t-1} = 0$ for every $j > i$.

**Proof.** Suppose not, i.e. there exists some monopoly-price efficient SPE where firm $i$’s incentive constraint is not binding in period $t > 1$ and $\pi_{j,t-1} > 0$ for some $j > i$. Then there is a period $t' > t$ such that firm $i$’s incentive constraint is not binding for $t, t+1, \ldots, t'$ and $\pi_{i,t'} > 0$. We can find such $t'$, otherwise $\pi_{i,t+1} = \pi_{i,t+2} = \ldots = 0$ (if $\pi_{i,t+1} = 0$, then $U_{i,t+2} > \hat{\pi}_i$ hence $i$’s incentive constraint in period $t + 2$ is not binding. If $\pi_{i,t+2} = 0$, then $U_{i,t+2} = \ldots$). Such a path is not sustainable.

Now perturb the profit of firm $i$ and $j$ as follows.

$$\pi'_{i,t} = \pi_{i,t} + \varepsilon,$$

$$\pi'_{j,t} = \pi_{j,t} - \varepsilon,$$

$$\pi'_{i,t'} = \pi_{i,t'} - \varepsilon',$$

$$\pi'_{j,t'} = \pi_{j,t'} + \varepsilon'.$$

We are basically exchanging firm $j$’s market share in period $t$ with firm $i$’s market share in period $t'$, keeping every other firm’s profit at the same level. Since $\delta_i < \delta_j$, $\pi_{i,t} > 0$ and $\pi_{i,t'} > 0$, we can pick $\varepsilon, \varepsilon' > 0$ so that firm $j$’s continuation payoff in every period from $t$ to $t'$ increases and firm $i$’s continuation payoff in period $t$ increases. So this allocation Pareto-dominates the original SPE allocation. Firm $j$’s incentive constraints are not affected at all. Firm $i$’s incentive constraints in period $t$ is satisfied by construction. Finally, we can
take $\varepsilon, \varepsilon' > 0$ small enough so that firm $i$’s incentive constraint from period $t + 1$ to $t'$ is still not binding. So we can construct a more efficient SPE in this case, a contradiction. □

Lemma 5.2. For any monopoly-price efficient subgame perfect equilibrium, if $\pi_{i,t} < \hat{\pi}_i$, then $\pi_{j,t} = 0$ for every $j > i$.

Proof. If $\pi_{i,t} < \hat{\pi}_i$, then $U_{i,t+1} > \hat{\pi}_i$. Hence firm $i$’s incentive constraint is not binding in period $t + 1$. Then $\pi_{j,t} = 0$ for every $j > i$ by Lemma 1. □

Lemma 5.3. For any efficient subgame perfect equilibrium with $\pi > \hat{\pi}$, (1) $\pi_{1,t} \geq \hat{\pi}_1$ for every $t \geq 1$ and (2) $\pi_{1,t'+k} = \hat{\pi}_1$ for any $k = 0, 1, \ldots$ if firm 1’s incentive constraint is binding in period $t'$.

Proof. If $\pi_{1,t} < \hat{\pi}_1$ for any $t$, then $\pi_{j,t} = 0$ for every $j = 2, 3, \ldots, n$ by Lemma 2. This contradicts to $\sum_i \pi_{i,t} = \pi^m$.

Firm 1’s incentive constraint is binding in period $t'$ if and only if $U_{1,t'} = \hat{\pi}_1$. Clearly this holds if and only if $\pi_{1,t'+k} = \hat{\pi}_1$ for $k = 0, 1, 2, \ldots$.

By induction, a similar property holds for every firm.

Lemma 5.4. For any efficient subgame perfect equilibrium with $\pi > \hat{\pi}$, suppose that $\pi_{h,t+k} = \hat{\pi}_h$ for every $k = 0, 1, 2, \ldots$ and every $h = 1, 2, \ldots, i$ for some $t$ and some $i \in I$. Then (1) $\pi_{i+1,t+k} \geq \hat{\pi}_{i+1}$ for every $k = 0, 1, 2, \ldots$ and (2) $\pi_{i+1,t'+k} = \hat{\pi}_{i+1}$ for every $k = 0, 1, 2, \ldots$ if firm $i + 1$’s incentive constraint is binding in period $t' \geq t$.

Proof. Suppose that $\pi_{i+1,t+k} < \hat{\pi}_{i+1}$ for any $k$. Then $U_{i+1,t+k+1} > \hat{\pi}_{i+1}$. Hence firm $i + 1$’s incentive constraint is not binding in period $t + k + 1$. Then $\pi_{j,t+k+1} = 0$ for every $j > i + 1$ by Lemma 1. However, $\sum_h \pi_{h,t+k+1} = \sum_{h=1}^{i+1} \pi_{h,t+k+1} = \sum_{h=1}^{i} \pi_h + \pi_{i+1,t+k+1} < \sum_{h=1}^{i+1} \pi_h \leq \pi^*$, which is a contradiction. This proves (1).

As for (2), firm $i + 1$’s incentive constraint is binding in period $t' \geq t$ if and only if $U_{i,t'} = \hat{\pi}_i$. By Lemma 4, this holds if and only if $\pi_{i+1,t'+k} = \hat{\pi}_{i+1}$ for every $k = 0, 1, 2, \ldots$ □

Now we can prove Theorem 4.2.
Proof. In period 1, we have \( \pi_1 \) such that (1) \( \sum_{h=1}^{n} \pi_{h,1} = \pi^{m} \) and (2) \( \pi_{h,1} \geq \hat{\pi}_h \) for \( h = 1, 2, ..., h_1 - 1 \), \( \pi_{h_1,1} \in \left[ 0, \pi^m - \sum_{h=1}^{h_1-1} \hat{\pi}_h \right] \), and \( \pi_{h,1} = 0 \) for \( h > h_1 \) for some \( h_1 \geq 1 \) by Lemma 2.

By Lemma 1, the incentive constraint must be binding for \( h = 1, 2, ..., h_1 - 1 \) in period 2. By Lemma 3 and Lemma 4, \( \pi_{h,1+k} = \hat{\pi}_h \) for \( h = 1, 2, ..., h_1 - 1 \) for the rest of the game \( (k = 1, 2, ...) \).

In period 2, we have \( \pi_2 \) such that (1) \( \sum_{h=1}^{n} \pi_{h,2} = \pi^{m} \), (2) \( \pi_{h,2} = \hat{\pi}_h \) for \( h = 1, 2, ..., h_1 - 1 \) (by the previous step), (3) \( \pi_{h,2} \geq \hat{\pi}_h \) for \( h = h_1, h_1 + 1, ..., h_2 - 1 \), \( \pi_{h_2,2} \in \left[ 0, \pi^m - \sum_{h=1}^{h_2-1} \hat{\pi}_h \right] \), and \( \pi_{h,2} = 0 \) for \( h > h_2 \) for some \( h_2 \geq h_1 \) by Lemma 4.

By Lemma 1, the incentive constraint must be binding for \( h = h_1, ..., h_2 - 1 \) in period 3. By Lemma 3 and Lemma 4, \( \pi_{h,2+k} = \hat{\pi}_h \) for \( h = h_1, ..., h_2 - 1 \) for the rest of the game \( (k = 1, 2, ...) \) and so on... This proves 1-6 in the statement of the theorem.

On the other hand, suppose that there exist \( (t_1, t_2, ..., t_{n-1}) \) and a sequence of profit profiles \( \pi_{i,t} \) that satisfy 1-6. It is clear that this corresponds to some monopoly-price SPE. So we just show that it is an efficient equilibrium. Suppose not. Then there exists a Pareto-superior monopoly-price efficient SPE, which of course satisfies 1-6. Let \( (\tilde{t}_1, \tilde{t}_2, ..., \tilde{t}_{n-1}) \) be the corresponding critical periods and \( \tilde{\pi}_{i,t} \) be the associated sequence of profit profiles. Since this equilibrium is more efficient than the former one, it must be the case that either (1) \( t_1 < \tilde{t}_1 \) or (2) \( t_1 = \tilde{t}_1 \) and \( \pi_{1,t_1} \leq \tilde{\pi}_{1,t_1} \). In either case, it must be the case that, for firm 2, either (1) \( t_2 < \tilde{t}_2 \) or (2) \( t_2 = \tilde{t}_2 \) and \( \pi_{1,t_2} \leq \tilde{\pi}_{1,t_2} \). By induction, either (1) or (2) holds up to firm \( n - 1 \). Then firm \( n \)'s average profit given \( \pi_{n,t} \) is higher than firm \( n \)'s average profit given \( \tilde{\pi}_{n,t} \). This is a contradiction to the assumption that the latter equilibrium is more efficient. \( \square \)