Comments.  This assignment was hard.

On the first problem, some people forgot about the “trivial” equilibria in which both players demanded too much. More importantly, it appeared to be hard to write down the payoff function in the two parts where there was uncertainty about the size of the surplus. At least when $\epsilon$ is close to one, the best response can depend on $\epsilon$.

Two small comments on question 2. Ruling out non-degenerate mixed strategies requires a technical argument (outlined below). Few provided a convincing version of the argument. For pure strategy equilibria in part a, if player one bids less than 1, then player two has no best response (the contortions involving “will want to bid just a bit more” are intuitive, but it is enough to say that you cannot have an equilibrium in which player one bids less than 1 because if player one bids less than 1 player two cannot make a best response). Of course, bidding 1 can only be a best response if the other player is bidding 1.

The final three questions were unfair because you had to attempt them before I discussed the concepts. I graded them “seriously,” but you should not feel bad about getting a low score. I hope that the questions and answers make sense now. The difficulty of the assignment was in large part due to my bad timing. Sorry.

On the third question, most assumed the existence of continuation values. This allowed you to compute equilibrium values (and then, if you thought of it, the corresponding strategies), but it assumes uniqueness: if there are multiple equilibria, then the continuation value might depend on what happens in the first round. You can solve the problem by copying the steps of the Shaked-Sutton proof of uniqueness in the standard model. I do not do that below. Instead, I give a “shrinking” argument that is general enough to account for both the problem and the part of Monday’s class that I bungled.

Few made much progress on question 4. Please note that the Rubinstein model breaks down in the limit $\delta = 1$. I intended you to consider behavior as the discount factor approaches one, not the actual (indeterminate) behavior when there is no discounting at all.

On the fifth problem, some misunderstood the sunk cost nature of investment: If Bob invests, then when Alice suggests the price $p$ Bob has a choice between accepting and getting $V - p - c$ or rejecting and getting $-c$. Hence, in subgame perfect equilibrium, Alice will ask $V$ and Bob will accept. Assuming that Bob can invest after Alice’s offer trivializes the problem. When you understand this, the first part is straightforward (but only interesting if you restrict attention to subgame perfect equilibrium). Many “knew” the answer to the

1. In the first part, any pair of offers that sum to one is a Nash equilibrium. It is also a Nash equilibrium if both players bid “too much” (so that given one player’s strategy, the other gets zero no matter what he bids). In the second part, suppose that one player bids only \( y \) and \( 1 - y \) for \( y \in (0, 1) \). You can check that the other player’s best response must be supported on \( \{ y, 1 - y \} \). The resulting \( 2 \times 2 \) game has three Nash equilibria – two pure and one in which both players use nondegenerate mixed strategies. The third part has a unique pure-strategy Nash equilibrium, which is easy to compute (both players bid \( \frac{2}{3} \)) when you note that the best response to a bid of \( x_i \) is the \( x_j \) that maximizes:

\[
x_j(2 - x_i - x_j).
\]

The final part was harder. If opponent plays \( x_i \), then the payoff to playing \( x_j \) is:

\[
\epsilon x_j \frac{2 - x_i - x_j}{2} + (1 - \epsilon) x_j
\]

(if \( x_j + x_i \leq 1 \)) or to:

\[
\epsilon x_j \frac{2 - x_i - x_j}{2}
\]

otherwise.

The solution to the first order condition of the first part always violates the constraint. The solution to the first order condition of the second part is \( x_j = \frac{2 - x_i}{2} \). So either player \( j \) sets \( x_j = 1 - x_i \) or he sets \( x_j = \frac{2 - x_i}{2} \). (The choice depends on which payoff is better.) I guess that the interesting case is when \( \epsilon \) is small. In this case you can check to that equilibria in which \( x_i = 1 - x_j \) exist. You need, I think,

\[
(1 - \frac{\epsilon}{2})(1 - x_i) \geq \frac{\epsilon}{8}(2 - x_i)^2
\]

and

\[
(1 - \frac{\epsilon}{2})x_i \geq \frac{\epsilon}{8}(1 + x_i)^2.
\]

2. The only Nash equilibrium is for both players to bid one. Take a pair of equilibrium distributions. Let \( s_i \) be the supremum of player \( i \)’s bid distribution. In equilibrium, \( s_1 = s_2 \) or the bidder with the higher \( s_i \) can increase her payoff (by bidding just above the highest bid of her opponent). If the sup is strictly less than one, then both players have positive expected payoff in equilibrium; therefore, all bids must be strictly positive.
in equilibrium; but either player two’s lowest bid never wins (if player one never bids less), in which case player two is not best responding) or the inf of player one’s bid distribution is strictly less than that of player two, so player one’s lowest bids yield payoff zero and cannot be part of a best response. This demonstrates that the sup of both player’s equilibrium distribution is 1. Hence expected payoffs are zero. Hence players cannot bid less than 1 with positive probability.

Bidding 1 is weakly dominated. By the above, there are no Nash equilibria in weakly undominated strategies.

Essentially the same argument as above implies that equilibrium strategies are supported on 1 and the next lower bid. Player one bids one unit less than 1 and player two bids at least as much as player 2. The only strategy profile that survives iterative deletion of weakly dominated strategies is for each player to bid one unit less than one. (This conclusion depends on the order of deleting strategies. 1 is weakly dominated from player 2 as long as there are strategies in which player one bids two or more units less than 1.)

You need a more sophisticated definition of subgame perfection in games with continuous strategy spaces (if you want existence). Intuitively, deleting weakly dominated strategies creates open set problems, which in turn may lead to non-existence of equilibrium.

3. Intuition: When it is player one’s turn, player two must wait twice to get a counter offer. It is therefore as if his discount factor is $\delta^2$. Indeed, the algebra shows that the model behaves exactly as a standard model with the second player’s discount factor squared. You can see this using the argument I presented in class. (It is important, for completeness, to assume that the continuation value can vary. That is, that in any subgame beginning with an offer from player 2 the players’ values are in an interval. Most people assumed stationarity in values – that the value starting from the third period is the same as from the initial position. This need not be true.)

Anyway, here is a “shrinkage” argument. Let $L_t$ be the amount of the surplus that is not allocated at time $t$. $L_0 = 1$ and future $L_t$ will be defined inductively. Suppose that in period $t$ player $i$ makes offers. This player must be able to extract at least $(1 - \delta_i)L_t$ from the other player. Let $v_t(i)$ be the amount that player $i$ can extract in period $t$. By my previous comment, $v_t(i) = (1 - \delta_i)L_t$ if $i$ moves in period $t$ and is zero otherwise. Now let $L_{t+1} = L_t - v_t(1) - v_t(2)$. I claim that the unique subgame perfect value of the game for player $i$ is $\sum_{t=0}^{\infty} v_t(i)$.

Let us look at this in the two situations we have studied. In the standard Rubinstein game, player 1 moves in even periods, so that $v_{2t}(1) = (1 - \delta_2)L_{2t}$ (and $v_{2t+1}(1) = 0$), while $v_{2t+1}(2) = (1 - \delta_1)L_{2t+1}$ (and $v_{2t}(2) = 0$.) Check that this implies that $L_{t+2} = \delta_1\delta_2L_t(t)$ so that $L_{2t} = (\delta_1\delta_2)^t$ and
\[ L_{2t+1} = (\delta_1 \delta_2)^t (1 - \delta_2). \] If you add up the sums, you’ll get the standard Rubinstein values.

This is the argument that I screwed up in class on Monday. It probably is more transparent without the notation, but you should be able to write down the two columns and correct my mistakes.

In the homework we have \( v_{3t+a}(1) = (1 - \delta_2)L_{3t+a} \) for \( a = 0, 1 \) (and \( v_{3t+2}(1) = 0 \)). Also \( v_{3t+2}(2) = 0 \) and \( v_{3t+2}(2) = (1 - \delta_1)L_{3t+2} \). Also \( L_{t+3} = \delta_1 \delta_2^2 \), which allows you to sum the series and get the appropriate values (player one gets \( \frac{1 - \delta_2}{1 - \delta_1 \delta_2} \)). (Oops. This discussion assumes that player one makes the first two offers. As the problem is written, player two goes first in period \( t = 0 \). Player two’s value would be \( 1 - \delta_1 \delta_2^2 \) and player one would get the rest.)

Please note that this argument is heuristic: It does not clearly indicate how players can extract the rent as I indicated. It is, I think, a useful way to get insight into where the solution comes from. Another shortcoming is that it just generates equilibrium values. Given the values one must go back and construct the equilibrium strategies that go along with the values. This is straightforward, but it needs to be done. You can create a complete argument by mimicking the standard proof.

4. Simple intuition: suppose that \( \delta = .999 \) and you bargain in pennies. Consider a game in which player one offers, player two offers, and then player one offers, but then bargaining stops. In the final period, player one can ask for everything and get it. If so, then in the penultimate period, player 2 still gets nothing; if she asks for a penny, player one will prefer to say no and get value .999. And so, player one in the first period can demand everything. This argument does not depend on the length of the horizon (as long as player one moves last). On the other hand, the three-offer game also has an outcome in which player two rejects the offer of nothing, so player one offers her one cent. If you continued using backward induction, you could generate equilibria with smaller and smaller initial offers for player one.

One way to approach the problem is to assume that the unit size is given as, say, \( g \) and a feasible allocation is of the form \((n_1 g, n_2 g)\) with \( n_1 g + n_2 g \leq 1 \) (that is, player \( i \) gets \( n_i \) pennies). To vary the discount factor you could write \( \delta(\Delta) = e^{-r\Delta} \). (In this formulation, \( \delta^* = \delta(1) = e^{-r} \) is the basic discount factor corresponding to a period of length one and shrinking the period length leads to a corresponding change in the discount factor.) I claim that if the discount factor is close to one relative to the size of the monetary unit, then you get many subgame perfect equilibria. Specifically, assume that \( 1 - g \leq \delta \). Now consider the strategy: always propose an efficient, feasible allocation \( x = (x_1, x_2) = (n_1 g, n_2 g) \); accept proposals if and only if they give you as much as \( x \). These strategies lead to \( x \) being proposed and accepted immediately. To show that these strategies are
really subgame perfect, we need to check that it does not pay to deviate.
Given the acceptance strategy, proposers can’t gain by offering more than
the equilibrium amount. What if they offer less? Suppose player $i$ is
offered $x_i - g$ instead of $x_i$. If he rejects, then he gets $\delta x_i$ in the next
period. But if $1 - g \leq \delta$ then $x_i - g < \delta x_i$, so it is better to accept
$x_i - g$ (and any lower offer). It follows that the proposed strategy is an
equilibrium. The idea is that if money comes in discrete units, then when
the pie shrinks more slowly than a unit of money, your opponent need not
be able to “hold you up” and gain some surplus when making an offer.
Once you have the equilibrium that I constructed, you can show that
virtually any outcome (including delay) can be supported as a subgame
perfect equilibrium. If you hold $\delta$ fixed and send $g$ to zero, then you can
recover Rubinstein’s uniqueness result.
When time is discrete, but units are small relative to the discount factor,
the standard argument works (except that you need to be careful about
rounding – demands must be feasible). Specifically, instead of getting $\frac{1}{1+\delta}$
in equilibrium, player one will this rounded up to the nearest “penny.”
More formally, if $\pi (v)$ is the sup (inf) over all subgame perfect Nash equi-
librium payoffs (for the player who makes the first offer) then “familiar”
arguments establish that $\pi \leq 1 - \delta v + g$ (player 1 must offer player 2 at
least $\delta v$) and similarly, $v \geq 1 - \delta \pi - g$ (player one can’t do worse than
offering offering $2 \delta v$). The extra $g$ terms enter to correct for the possi-
bility of rounding. Combining the inequalities yields $\pi - v \leq \frac{2g}{1-\delta}$, which
demonstrates that the values converge when $g$ goes to zero.

5. Here I wanted you to concentrate on subgame perfect equilibria. In part a,
Alice knows Bob’s valuation when she makes her offer and since Bob will
accept anything that leaves him with positive value, she’ll necessarily offer
the valuation in a subgame perfect equilibrium. Investment cost is sunk,
so Bob ends up with $0$ if he doesn’t invest (Alice gets $v$) and $-c$ if he does
invest. So the equilibrium involves Bob not investing, Alice demanding $v$ if
Bob does not invest and $V$ otherwise, and Bob agreeing to pay anything
less than $V$ if he invests and anything less than $v$ if he doesn’t.
The second part has the same outcome. The argument that I gave in class
on Monday guarantees that in the infinite bargaining game where Alice is
completely informed and makes all the offers, she’ll never set a price less
than Bob’s valuation. Proof: let $p$ be the smallest price Alice offers in
any subgame perfect equilibrium. If Bob’s valuation is $v$ and $p < v$, then
he must accept any price less than $\delta p + (1 - \delta)v$ instead of waiting for
$\delta (v - p)$ (the best possible continuation). Consequently Alice will never
ask less than $\delta p + (1 - \delta)v$, which contradicts the definition of $p$ (or the
assumption $p < v$).
Suppose that Alice does not observe investment. In a one shot bargain-
ing game, she can offer just one price $p$. Bob will pay $V$ if he invested
and $v$ otherwise. If Alice believes that Bob invested with probability $a$,
then Alice will charge $V$ if $aV > v$ and charge $v$ if $v > aV$. In the first case, Bob won’t invest (because it would lead to positive surplus). Hence, in equilibrium, $a = 0$. So the first case is impossible. The second case is similarly impossible. This means that in equilibrium $a = \frac{1}{V}$ (Bob randomizes). When he does, Alice is indifferent between charging a high or a low price. The price she charges must make Bob indifferent between investing or not. Investment leads to payoff $Ep - c$; not investing leads to payoff 0. So Bob invests with probability $\frac{1}{V}$. Alice changes $v$ with probability $\frac{V-v-c}{V-v}$ (which should leave Bob indifferent between investing and not investing). A feature of this solution is that there is positive investment, although Bob and Alice have the same utility they do in the earlier parts of the problem.

The simplest two-period case is when $c = \delta(V - v)$ so that if Bob invests and buys in the second period at $v$ he breaks even. Properly selecting the probability of investment can induce a game in which Alice charges $(1 - \delta)V + \delta v$ in the first period, selling with positive probability, and then lowering the price to $v$ if no sale is made. Bob will still break even, investment will go up, and Alice will earn strictly more than $v$. When the bargaining is dynamic, Alice loses bargaining power when period length does to zero. It is possible to show that as period length goes to zero, Bob’s investment goes to one, and Alice loses her bargaining power.