Everyone had reasonable answers to Questions 1 and 2. Answers to Questions 5 and 6 were adequate, although not ambitious. Most answers to question 3 were almost right. In a couple of cases, I was not convinced that you really knew what was going on. Some people had Question 4 nearly correct, while others seemed to be bluffing.

Question 1 had subtle aspects. There is no doubt that (c) and (e) satisfy independence. For (a) and (b), every answer supplied counterexamples based on a statement of the independence axiom using strict preference. What if you used the form in which only weak preference was required? This axiom is satisfied. It is clear that something is “wrong” with the representation, but one could make the case that is a failure of a continuity assumption rather than the failure of independence. Similarly, (d) provides a representation that is clearly not (ordinally equivalent to) an expression that is linear in probabilities. Consequently, some assumption in the EU theorem must fail. One could argue that the assumption that fails in the (implicit) characterization of increasing risk. To see that the distributions have the same mean, note that differentiability, the derivative of the left-hand side of the inequalities must be less than or equal to zero when evaluated at $P = 0$. Consequently, it must be the case that $(3\pi - 1)u'(w)$ (from the first inequality) and $(2 - 3\pi)u'(w)$ (from the second inequality) must both be non positive, which is impossible if $u'(w) \neq 0$. (Strictly speaking the result is false otherwise. Someone who has constant preferences could be both (weakly) unwilling to take either bet and have smooth preferences.)

Some people were confused about what the rest of the question asked and consequently wrote things that I didn’t understand. I asked two more questions: First, show that you could satisfy expected utility and (Strictly speaking the result is false otherwise. Someone who has constant preferences could be both (weakly) unwilling to take either bet and have smooth preferences.)

In Question 4, you should be able to see that if $r > r'$, then the distribution $F(\cdot, r)$ has the same mean but is riskier than $F(\cdot, r')$. The conditions are just the continuous version of the integral condition in the characterization of increasing risk. To see that the distributions have the same mean, note that

$$\int_{0}^{1} F(\theta, r)d\theta = \theta F(\theta, r)|_{\theta=0}^{\theta=1} - \int_{0}^{1} \theta f(\theta, r)d\theta = 1 - \mu(r),$$

where $\mu(r)$ is the mean of the distribution $F(\cdot, r)$. The first condition in the problem says that the left-hand side is independent of $r$, no $\mu(r)$ must also be constant. The first question follows from standard results if you understand the meaning of the conditions. No one supplied the direct argument, which goes like this:

You want to investigate when $\int_{0}^{1} u(\theta)dF(\theta, r)$ is increasing in $r$. That is, you want to know when $\int_{0}^{1} u(\theta)dF_{r}(\theta, r) > 0$. Observe that

$$\int_{0}^{1} u(\theta)dF_{r}(\theta, r) = u(\theta)F_{r}(\theta, r)|_{\theta=0}^{\theta=1} - \int_{0}^{1} u'(\theta)F_{r}(\theta, r)d\theta = - \int_{0}^{1} u'(\theta)F_{r}(\theta, r)d\theta,$$

where the first equation comes from integration by parts and the second comes from the observations that $F(0, r) = 0$ for all $r$ and $F(1, r) = 1$ for all $r$. Let $G(y, r) = \int_{0}^{y} F(\theta, r)d\theta$. Integrating by parts again yields
Aumann’s letter: specifying the state space is an art rather than a science and suggests a specification that is consistent with Cambridge University Press, 1987, pages 76–81. Here is a quotation from Savage’s reply that points out that Savage and Savage’s reply appear in “Essays on Economic Decisions Under Uncertainty” by Jacques Drèze, he pointed out that by redefining states the problem disappears. The complete text of Aumann’s letter to there is another approach: the state space may be incomplete. Indeed, when Savage responded to Aumann, which of Savage’s assumptions might fail. While this response is intuitive and coherent, it is worth noting that there is another approach: the state space may be incomplete. Indeed, when Savage responded to Aumann, he pointed out that by redefining states the problem disappears. The complete text of Aumann’s letter to Savage and Savage’s reply appear in “Essays on Economic Decisions Under Uncertainty” by Jacques Drèze, Cambridge University Press, 1987, pages 76–81. Here is a quotation from Savage’s reply that points out that specifying the state space is an art rather than a science and suggests a specification that is consistent with Aumann’s letter:

\[- \int_0^1 u'(\theta)G_r(\theta, r)d\theta = - u'(\theta)G_r(\theta, r)|_{\theta=0}^{\theta=1} + \int_0^1 u''(\theta)G_r(\theta, r)d\theta = \int_0^1 u''(\theta)G_r(\theta, r)d\theta,\]

where the second equation follows because \( G(0, r) = 0 \) for all \( r \) and \( G(1, r) \) is constant (from the equal means condition). It follows that \( \int_0^1 u(\theta)dF_r(\theta, r) > 0 \) exactly when \( \int_0^1 u''(\theta)G_r(\theta, r)d\theta > 0 \). Since \( G_r(\theta, r) \geq 0 \) for all \( \theta \in [0, 1] \) by assumption, the result will hold provided that the agent is risk loving \( (u''(\cdot) > 0) \). Again, you could have concluded this from a theorem with the proper interpretation of the conditions.

The next part is asks a question about a derivative. A complete answer would follow the answer above and integrate by parts twice. \( a^*(r) \) is define as the solution to: \( \max_a \int_0^1 u(\theta,a)dF(\theta, r) \). Hence \( a^*(r) \) is characterized by the first-order condition:

\[ \int_0^1 u_a(\theta,a)dF(\theta, r) = 0. \]

Differentiation of this equation gives an expression for \( (a^*)'(\cdot) \):

\[ \int_0^1 u_{aa}(\theta,a^*(r))(a^*)'(r)dF(\theta, r) + \int_0^1 u_a(\theta,a)dF_r(\theta, r) = 0. \]

You can solve for \( (a^*)'(\cdot) \) because (a) \( (a^*)'(r) \) does not depend on \( \theta \) so it can be factored out of the first integral and (b) \( u_{aa} < 0 \) so you can divide by \( \int_0^1 u_{aa}(\theta,a^*(r))dF(\theta, r) \). It follows that

\[ (a^*)'(r) = - \frac{\int_0^1 u_a(\theta,a)dF_r(\theta, r)}{\int_0^1 u_{aa}(\theta,a^*(r))dF(\theta, r)}. \]

Further, since the denominator on the right-hand side is negative, the sign of \( (a^*)'(\cdot) \) is the same as the sign of the integral in the numerator on the right. So the question asks when is this integral positive. Integrating by parts twice yields:

\[ \int_0^1 u_a(\theta,a)dF_r(\theta, r) = - \int_0^1 u_{a\theta}(\theta,a)F_r(\theta, r)d\theta = \int_0^1 u_{a\theta}(\theta,a)G_r(\theta, r)d\theta. \]

(I leave it to you to confirm that the boundary terms disappear; \( G(\cdot) \) was defined above.) It follows that \( a^*(r) \) is increasing in \( r \) if \( u_{a\theta}(\theta,a) > 0 \) (that is, that \( u_a \) is convex in \( \theta \). You should be able to come up with an interpretation of this condition.

Philosopher Robert Novick received a Ph.D. and a job at Harvard for his work on this problem. In my opinion, the paradox disappears when one carefully defines the state space (although depending on how you define the states, the answer to the question changes). The obvious (to an economist) specification points out that selecting both boxes dominates choosing just one. This formulation is the one suggested in the problem statement. Alternatively, the states could be “genie guesses correctly” and “genie guesses incorrectly,” which have self-referential properties, but (for suitable beliefs) rationalize picking only one box. For the record, I believe neither in genies nor hypothetical decision problems, but to the extent that I can imagine being offered the choice described, I’d take just one box.

Everyone responded to Question 6 by stating that preferences are state contingent. No one identified which of Savage’s assumptions might fail. While this response is intuitive and coherent, it is worth noting that there is another approach: the state space may be incomplete. Indeed, when Savage responded to Aumann, he pointed out that by redefining states the problem disappears. The complete text of Aumann’s letter to Savage and Savage’s reply appear in “Essays on Economic Decisions Under Uncertainty” by Jacques Drèze, Cambridge University Press, 1987, pages 76–81. Here is a quotation from Savage’s reply that points out that specifying the state space is an art rather than a science and suggests a specification that is consistent with Aumann’s letter:
I believe, and examples have confirmed, that decision situations can be usefully structured in terms of consequences, states, and acts in such a way that the postulates of F. of S. [Foundations of Statistics] are satisfied. Just how to do that seems to be an art for which I can give no prescription and for which it is perhaps unreasonable to expect one — as we know from other postulate systems for application. Thus ..., I would be glad to pay a nickel to rent an umbrella for a fall football match but given that it will not rain, I would prefer the nickel. I analyze this in terms of several consequences: the status quo, being miserably drenched, and being undrenched but out a nickel. These ‘consequences’ seem to enable me to describe the situation in terms of, and consistent with, the postulates in F. of S. Of course, they are not ultimate. A nickel is itself a lottery ticket, and one objection to getting miserably drenched is that it seems conducive to illness. If the problem were concerned with illness or the possibility of accidentally buying poisoned food, then of course the notion of consequence would have to be further analyzed. An ultimate analysis might seem desirable, but probably it does not exist and certainly threatens to be cumbersome.