Everyone had reasonable answers to Questions 1, 2, and 3. Most people efficiently exploited the loophole in problem 5 and exhibited a lottery that stochastically dominates the sure thing. (Problem Set 3 has a problem that closes the loophole.) Two people constructed a counter example for the first part of the fourth question; no one really completed the answer to the second part. Here are some notes on Problem 4.

First, to construct a counterexample, imagine a piecewise linear function that satisfies

\[ u(x) = \begin{cases} x & \text{if } x < 2, \\ \frac{19}{30} x + \frac{22}{30} & \text{if } x \geq 0. \end{cases} \]  

(1)

The formula is less mysterious than it looks. I just set \( u(0) = 0 \), \( u(2) = 2 \), and \( u(5) = 3.9 \) so that the given inequality held, and then assumed that \( u(\cdot) \) was linear between 0 and 2 and again (with a different slope) between 2 and 5. Now a simple computation shows that \( u(2) < .25u(-2) + .5u(3) + .25u(8) \) (the right hand side is the utility of the lottery you get when you play the game twice). The example demonstrates that if you repeat the lottery, you may (depending on your preferences) prefer the gamble to the sure thing.

As an exercise, you can confirm that if the utility function satisfies, for all \( x \),

\[ u(x) > .5u(x-2) + .5u(x+3) \]

then you will always prefer the sure thing to an \( N \)-fold repetition of the lottery.

Now for the second part. If you repeat the (scaled) lottery \( N \) times, then when you “win” precisely \( k \) times, your payoff is \( 2 + 3k/N - 2(N-k)/N = 2 + (5k - 2N)/N \). Consequently, the expected utility of \( N \) repetitions of the lottery is equal to:

\[ \sum_{k=0}^{N} 2^{-N} \binom{N}{k} u(2 + \frac{5k - 2N}{N}), \] 

(2)

where \( 2^{-N} \binom{N}{k} \) is the probability of precisely \( k \) wins out of \( N \) tries. The goal is to show that (2) is greater than \( u(2) \) when \( N \) is sufficiently large.

Divide the sum into two parts, depending on whether \( k > .48N \). That is, rewrite (2) as:

\[ \sum_{0 \leq k \leq .48N} 2^{-N} \binom{N}{k} u(2 + \frac{5k - 2N}{N}) + \sum_{N \geq k > .48N} 2^{-N} \binom{N}{k} u(2 + \frac{5k - 2N}{N}). \] 

(3)

The utility in the first term of (3) is at least as great as \( u(2.4) \). The utility in the second term is at least as great as \( u(0) \). Let \( P_N = \sum_{0 \leq k \leq .48N} 2^{-N} \binom{N}{k} \) be the probability that \( k > .48N \). It follows that in order to complete the proof it suffices to show that

\[ u(2.4)P_N + u(0)(1 - P_N) > u(2). \] 

(4)

The inequality in (4) follows because \( u(2.4) > u(2) \) and \( P_N \) converges to one by the law of large numbers.

Note that there is nothing special about .48 here, what you need is something less than one half (so that the law of large numbers kicks in) and something large enough so that \( 3k > 2(N-k) \), so that you are a net winner. The reason that scaling bets is important is that it guarantees that losses are bounded (here by \( u(0) \)).