Comments. 36 possible (12 points each problem); High: 36, Low: 21; Median: 32. Scores were high. There were some problems that led to minor deductions but could signal serious misunderstandings. Please review your quiz and these answers carefully. On question one: Several people apparently did not recall the definition of homogeneity or Euler’s Theorem. A few wrote the wrong equation for a plane. A few got confused with notation and used \( y_1 \) and \( y_2 \) to stand for both a particular point on the plane and a variable (see suggested answer). On question two: some, perhaps unwilling to believe that there would be no critical points, incorrectly asserted that there were solutions to \( Df = 0 \). On three, some forgot that eigenvectors must be nonzero. These mistakes are serious, but I hope that they are easy to recognize and correct.

1. (a) \( f'(x) = 3x^2 \) and \( Dg(y_1, y_2) = \begin{bmatrix} 1 + ye^{y_1}y_2 & ye^{y_1}y_2 \\ \end{bmatrix} \).

(b) \( D(f \circ g)(y_1, y_2) = Df(g(y_1, y_2)) = 3(y_1 + e^{y_1}y_2) \begin{bmatrix} 1 + ye^{y_1}y_2 & ye^{y_1}y_2 \\ \end{bmatrix} \)

(c) \( f(\cdot) \) is homogeneous of degree 3 because \( f(\lambda x) = \lambda^3 f(x) \). Euler’s Theorem states that in this case \( xf'(x) = 3f(x) \), which is obviously true. The other two functions are not homogeneous. In order to have \( g(\lambda y) = \lambda^n g(y) \) you would need \( \lambda y_1(1 - \lambda^{n-1}) = \lambda^n e^{y_1}y_2 - e^{y_1}y_2 \) for all \( \lambda, y_1, y_2 \). But if \( y_1 = 0 \), the equation becomes \( \lambda^n = 1 \), which obviously cannot hold globally. Similarly, the composite function cannot be homogeneous.

(d) The equation is:

\[
\begin{align*}
    z - g(y_1, y_2) &= Dg(y_1, y_2) \begin{bmatrix} u - y_1 \\ v - y_2 \\ \end{bmatrix} \\
    ((u, v, z) &\text{ is a point on the plane.})
\end{align*}
\]

2. (a) \( Df(x, y, z) = \begin{bmatrix} y & x & 1 \end{bmatrix} \). Hence \( Df(\cdot) \neq 0 \) and there are no critical points.

(b) \( Df(x, y) = \begin{bmatrix} e^{x-y^2} & -2ye^{x-y^2} \end{bmatrix} \). Since \( e^{x-y^2} \neq 0 \), \( Df(x, y) \neq 0 \) and again there are no critical points.

3. For all \( (x, y, z) \) (including \( (1, 1, 1) \)), \( D^2f(x, y, z) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \). This matrix is symmetric, so it must be diagonalizable. The matrix can be decomposed into the upper 2 \( \times \) 2 matrix and the lower 1 \( \times \) 1 matrix. It should be clear that one eigenvalue is 0 with associated eigenvector \( (0, 0, 1) \). The other eigenvalues are \( -1, 1 \) with associated eigenvectors \( \frac{1}{\sqrt{2}}(1, -1, 0) \) and \( \frac{1}{\sqrt{2}}(1, 1, 0) \). (I normalized to make them unit vectors.)

It follows that \( P^{-1}D^2f(1, 1, 1)P = D \), where \( P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \), and (since \( P \) is orthogonal) \( P^{-1} = P^t \).